WEAK COMPACTNESS AND SEPARATION

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The purpose of this paper is to develop characterizations of weakly compact subsets of a Banach space in terms of separation properties. The sets A and B are said to be *separated* by a hyperplane H if A is contained in one of the two closed half-spaces determined by H, and B is contained in the other; A and B are *strictly separated* by H if A is contained in one of the two open half-spaces determined by H, and B is contained in the other. The following are known to be true for locally convex topological linear spaces.

(A) Disjoint convex subsets can be *separated* by a hyperplane if A has an interior point or if A is weakly compact (see **4**, pp. 456–457 and **5**), but every non-reflexive Banach space contains a pair of disjoint bounded closed convex sets that cannot be separated by a hyperplane (**4**, p. 881).

(B) Disjoint closed convex subsets A and B can be *strictly separated* by a hyperplane if A is compact (**1**, p. 73).

(C) If A and B are disjoint closed convex subsets and A is weakly compact, then there is a continuous linear functional f such that

$$\inf\{f(x) : x \in A\} > \sup\{f(x) : x \in B\}$$

(**4**, p. 457), so that $d(A, B) > 0$ if the space is normed.

If an element x of a locally convex linear topological space does not belong to a closed convex set C, then there is a continuous linear functional f such that $f(x) > \sup\{f(y) : y \in C\}$ (see **2**, Theorem 5, p. 22). Therefore all closed convex sets are weakly closed, and the assumption in the following lemma that B is weakly closed could be replaced by the assumption that B is closed and convex.

LEMMA. *If A and B are disjoint weakly closed subsets of a normed linear space and A is weakly compact, then $d(A, B) > 0$.*

Proof. If $d(A, B) = 0$ and A is weakly compact, then there is a sequence of ordered pairs (a_i, b_i) for which each $a_i \in A$, each $b_i \in B$, $d(a_i, b_i) \to 0$, and $\{a_i\}$ converges weakly to a member α of A. Then $\alpha \notin B$, but $\{b_i\}$ converges weakly to α. This implies that B is not weakly closed.

THEOREM 1. *A necessary and sufficient condition that a weakly closed subset A of a Banach space be weakly compact is that $d(A, B) > 0$ for all weakly closed sets B such that $A \cap B$ is empty.*

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Proof. If A is not weakly compact, then there is a continuous linear functional f that does not attain its supremum on A **(3)**. Let $c = \sup\{f(x): x \in A\}$ and $B = \{x: f(x) = c\}$. Then A and B are disjoint and B is closed, convex, and weakly closed, but $d(A, B) = 0$. Now suppose that A is weakly compact, A and B are disjoint, and B is weakly closed. Then it follows from the lemma that $d(A, B) > 0$.

In case A is bounded as well as weakly closed, Theorem 1 can be modified to state that A is weakly compact if and only if $d(A, B) > 0$ for all bounded, weakly closed sets B such that $A \cap B$ is empty. Also, it should be clear from the proof that the property of weak closure for the set B could be replaced by closure and convexity. If we assume that A also is closed and convex, we obtain part (a) of the following theorem.

THEOREM 2. *Each of the following is a necessary and sufficient condition that a closed convex subset A of a Banach space be weakly compact:*

(a) *For each closed convex subset B such that $A \cap B$ is empty, $d(A, B) > 0$.*

(b) *For each closed convex subset B such that $A \cap B$ is empty, there is a hyperplane that strictly separates A and B.*

Proof. To show the sufficiency of (b), we assume that A is not weakly compact. Then there is a continuous linear functional f that does not attain its supremum on A **(3)**. Let $c = \sup\{f(x): x \in A\}$ and $B = \{x: f(x) = c\}$. Then A and B are disjoint and B is closed and convex. Suppose there is a continuous linear functional g and a number θ such that

$$g(x) < \theta \quad \text{if} \quad x \in A, \qquad g(x) > \theta \quad \text{if } x \in B.$$

Also choose ξ and x as elements of the Banach space for which $f(\xi) = 0$ and $f(x) = c$. Then for all k we have $f(x + k\xi) = c$. Therefore $x + k\xi \in B$ and $g(x + k\xi) > \theta$ for all k. This is impossible unless $g(\xi) = 0$. Therefore the null spaces of f and g are the same, f and g are proportional, and there is a number ϕ such that

$$g(x) = \frac{\phi\theta}{c} f(x) \qquad \text{for all } x.$$

When $x \in B$, we have $f(x) = c$ and $g(x) > \theta$. Therefore $\phi > 1$. Since $g(x) < \theta$ if $x \in A$, we have

$$f(x) = \frac{c}{\theta\phi} g(x) < \frac{c}{\phi} \qquad \text{for all } x \in A.$$

This is impossible, since $c = \sup\{f(x): x \in A\}$ and $\phi > 1$. Now suppose that A is weakly compact and B is closed and convex. Then it follows from (C) that there is a hyperplane which strictly separates A and B.

The following theorems are related to results of Tukey **(5)** and Klee **(4,** p. 881**)** that can be combined to give the following theorem: *A necessary and sufficient condition that a Banach space be reflexive is that each pair of disjoint bounded closed convex sets can be separated by a hyperplane.*

THEOREM 3. *A necessary and sufficient condition that a Banach space be reflexive is that* $d(A, B) > 0$ *for all disjoint pairs* (A, B) *of weakly closed subsets at least one of which is bounded.*

Proof. If the space is not reflexive, then the unit sphere A is weakly closed but not weakly compact (**2**, p. 52). It follows from Theorem 1 that there is a weakly closed set B such that $A \cap B$ is empty and $d(A, B) = 0$. If the space is reflexive and A is bounded and weakly closed, then A is weakly compact and it follows from Theorem 1 that $d(A, B) > 0$ for each weakly closed set B such that $A \cap B$ is empty.

THEOREM 4. *Each of the following is a necessary and sufficient condition that a Banach space be reflexive:*

(a) *For each disjoint pair* (A, B) *of closed convex subsets at least one of which is bounded,* $d(A, B) > 0$.

(b) *For each disjoint pair* (A, B) *of closed convex subsets at least one of which is bounded, there is a hyperplane that strictly separates* A *and* B.

Proof. If the space is not reflexive, then the unit sphere is not weakly compact. With A the unit sphere, it follows from Theorem 2 that neither (a) nor (b) is satisfied. Now suppose that the space is reflexive and A and B are as stated, with A bounded. Then A is weakly closed, since A is convex and closed. Therefore A is weakly compact and it follows from Theorem 2 that $d(A, B) > 0$ and that there is a hyperplane which strictly separates A and B.

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