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On the Roughness of Quasinilpotency Property of One-parameter Semigroups

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Abstract. Let $\mathbf{S} := \{S(t)\}_{t\geq 0}$ be a C_0-semigroup of quasinilpotent operators (*i.e.*, $\sigma(S(t)) = \{0\}$ for each $t > 0$). In dynamical systems theory the above quasinilpotency property is equivalent to a very strong concept of stability for the solutions of autonomous systems. This concept is frequently called superstability and weakens the classical finite time extinction property (roughly speaking, disappearing solutions). We show that under some assumptions, the quasinilpotency, or equivalently, the superstability property of a C_0-semigroup is preserved under the perturbations of its infinitesimal generator.

1 Introduction

Let $\mathbf{S} := \{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup generated by the linear densely defined and closed operator A on a Banach space $\mathcal{X}$. Assume that we can find a linear operator T such that $A+T$ generates a strongly continuous semigroup $\widetilde{\mathbf{S}} := \{\widetilde{S}(t)\}_{t\geq 0}$. The following question has been posed by many authors, starting with R. S. Phillips back in 1951 (see, *e.g.*, [10,11]):

> Under which conditions is the asymptotic behavior of the trajectories of $\mathbf{S}$ preserved by the trajectories of the perturbed semigroup $\widetilde{\mathbf{S}}$?

This problem is regarded as quite difficult, since Phillips shows that even by bounded perturbations we can wash out important properties as eventual norm continuity or the Spectral Mapping Theorem. As far as we know, the above question was considered most recently by S. Brendle, R. Nagel, and J. Poland in an interesting paper from 2000 (see [4]), where they determined the spectrum $\sigma(\widetilde{S}(t))$ from the spectrum $\sigma(A+T)$ and the spectrum $\sigma(S(t))$ of the unperturbed semigroup operators.

The aim of this paper is to show that under some assumptions, the quasinilpotency, or equivalently, the superstability property of a C_0-semigroup is preserved under the perturbations of its infinitesimal generator. The problem is strongly motivated, taking into account that Brendle, Nagel, and Poland [4] point out an example of a nilpotent (and hence superstable) C_0-semigroup that does not even stay exponentially stable under bounded perturbation of its generator. Our approach employs spectral techniques (first used by J. Voigt [13,14], and later by Brendle, Nagel, and Poland [3,4,8]) and is based on concepts like critical spectrum, essential spectrum, and Weyl's theorem.

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2 Superstable Semigroups

The concept of superstability originally appears in the most quoted monograph of E. Hille and R. S. Philips [6] who were concerned primarily with its mathematical aspects and related properties, particularly the relationship between the spectrum of the infinitesimal generator and the stability of the semigroup. Later work refined and extended this notion, applying more complicated machinery [9, 12]. More recently, A. V. Balakrishnan [1] has become interested in the superstability phenomena arising in the control theory of physical systems (*i.e.*, boundary-value problems for partial differential equations).

A family $\mathbf{S} := \{S(t)\}_{t\ge 0}$ of bounded linear operators on a Banach space $\mathcal{X}$ is called a *strongly continuous semigroup* (or $\mathbf{C_0}$*-semigroup*) when $S(0) = I$ (the identity operator on $\mathcal{X}$), $S(t+s) = S(t)S(s)$ for all $t, s \ge 0$, and $\lim_{t\downarrow 0} S(t) = I$ in the strong operator topology (*i.e.*, $S(t)x \to x$ as $t \downarrow 0$ for all $x \in \mathcal{X}$). As is well known, this implies that the map $t \mapsto S(t)$ is strongly continuous.

For a strongly continuous semigroup $\mathbf{S}$, define $\mathcal{D}(A)$ as the set of all $x \in \mathcal{X}$ such that $\lim_{t\downarrow 0} t^{-1}(T(t)x - x)$ exists. The *infinitesimal generator* of the semigroup $\mathbf{S}$ is the operator $A\colon \mathcal{D}(A) \to \mathcal{X}$, given by

$$Ax = \lim_{t\downarrow 0} \frac{S(t)x - x}{t}.$$

It can be proved that $\mathcal{D}(A)$ is a dense subspace of $\mathcal{X}$, A is a closed operator and that the pair $(A, \mathcal{D}(A))$ and the semigroup $\{S(t)\}_{t\ge 0}$ uniquely determine one another [5, Theorem II.1.4].

A strongly continuous semigroup $\mathbf{S} := \{S(t)\}_{t\ge 0}$ is said to be *exponentially stable* (or just *stable*) when the *stability index*, defined by

$$\nu(\mathbf{S}) := \sup\{\omega : \text{there exists } M > 0 \text{ such that } \|T(t)\| \le Me^{-\omega t} \text{ for all } t \ge 0\},$$

is positive; that is, there exists an exponentially decreasing function that bounds every trajectory. The *growth characteristic* of $\mathbf{S}$ is

$$\omega_0(\mathbf{S}) := \lim_{t\downarrow 0} \frac{\log \|S(t)\|}{t},$$

which, as is well known, equals $-\nu(\mathbf{S})$. The *essential growth bound* of $\mathbf{S}$ is defined by

$$\omega_{\mathrm{ess}}(\mathbf{S}) := \inf\{\omega \in \mathbb{R} : \text{there exists } M_\omega \text{ such that } \|S(t)\|_{\mathrm{ess}} \le M_\omega e^{\omega t}\},$$

where $\|T\|_{\mathrm{ess}} := \inf\{\|T - K\| : K \text{ is compact on } \mathcal{X}\}$, for any bounded operator T. Clearly, $\omega_{\mathrm{ess}}(\mathbf{S}) \le \omega_0(\mathbf{S})$.

The semigroup $\mathbf{S}$ is said to be *superstable* if for every $\nu > 0$, there exists M_ν such that $\|S(t)\| \le M_\nu e^{-\nu t}$ for every $t \ge 0$. Thus, superstability is equivalent to the condition that the growth characteristic $\omega_0(\mathbf{S})$ is $-\infty$. In particular, superstability can be defined as the equivalent condition that $S(t)$ be *quasinilpotent* for every $t > 0$ (recall that an operator T is quasinilpotent when $\sigma(T) = \{0\}$). "Superstability is a truly infinite-dimensional phenomenon – at least for linear systems – *i.e.*, it can occur only in systems where the state space is not finite-dimensional." (Balakrishnan, [1, Section 3])

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A simpler type of superstability is the following: **S** is said to have *finite time extinction* if there exists $t_0 \geq 0$ such that $S(t)x = 0$ for all $t \geq t_0$ and $x \in \mathcal{X}$ with $\|x\| \leq 1$. The smallest possible choice t_0 is called the *extinction time* of **S** and does not depend on x. Further, **S** is *nilpotent* when there exists t_0 such that $S(t_0) = 0$. Similarly, the smallest possible choice t_0 such that $S(t) = 0$ for all $t > t_0$ is called the *index of nilpotency* for **S**.

It is remarkable that one can find quite simple physical systems that have finite time extinction, hence that are superstable. Also, examples of superstable physical systems that do not exhibit extinction in finite time can be pointed out.

3 A Short Review of Spectral Theory

We assume in this section that the reader is familiar with the following results of spectral theory, and with one of its most important tools, the *spectral decomposition*. For instance, take A to be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then the spectral set $\sigma(A)$ is composed of two disjoint parts, namely, $\sigma(A) = \sigma_d(A) \sqcup \sigma_{\text{ess}}(A)$. The *essential spectrum* of A, $\sigma_{\text{ess}}(A)$ is the closed subset of $\sigma(A)$ that is invariant under compact perturbations (first introduced by H. Weyl in 1910 for a certain differential operator) as the part of the spectrum independent of boundary conditions. Later, it was proved that the complement part of the spectrum, the so-called *discrete spectrum*, consists of all isolated eigenvalues of A with finite algebraic multiplicity. Using Fredholm's theory, several definitions for the essential spectrum were given, leading to slightly different closed subsets of $\sigma(A)$. Here, we use the definitions given in [5,7,13].

Formally, the notion of essential spectrum can be given for a densely defined and closed operator A on a Banach space $\mathcal{X}$. Thus, A is said to be a *Fredholm operator* if its kernel, $\ker A$, is finite-dimensional and the range, $\operatorname{ran} A$, is closed and of finite codimension (*i.e.*, $\dim(\mathcal{X}/\operatorname{ran} A) < \infty$). The Fredholm index of A is the difference between the dimension of the kernel and the codimension of the range. A complex number λ is in the *essential spectrum* of A if $\lambda I - A$ is not Fredholm.

By $s(A)$ we denote the *spectral bound* of A, that is, $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ (with $s(A) := -\infty$ when $\sigma(A) = \varnothing$). We set $\rho(A)$ to be the complement of $\sigma(A)$ in $\mathbb{C}$, while $R(\cdot\,; A)$ denotes the *resolvent function* of A. If we assume that A is a bounded linear operator, it is known that the *spectral radius* $r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$ is finite, satisfying $r(A) \leq \|A\|$.

A sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ is said to be a Weyl sequence for A and $\lambda \in \mathbb{C}$ if $\|\psi_n\| = 1$, $n \in \mathbb{N}$, $\psi_n \to 0$ weakly as $n \to \infty$ and $(\lambda I - A)\psi_n \to 0$ for $n \to \infty$. We say that $\lambda \in \mathbb{C}$ is in the *Weyl spectrum* of A if there is a Weyl sequence for A and λ. If A is self-adjoint, we have the *Weyl's criterion*, which states that $W(A) = \sigma_{\text{ess}}(A)$ (see, for instance, [7, Chapter 7]). In general, we only have the inclusion $W(A) \subset \sigma_{\text{ess}}(A)$.

Lemma 3.1 ([7, Theorem 10.10, p. 105]) *Let A be a closed operator on the Hilbert space $\mathcal{H}$ with $\rho(A) \neq \varnothing$. Then $W(A) \subset \sigma_{\text{ess}}(A)$ and the boundary of $\sigma_{\text{ess}}(A)$ is contained in $W(A)$. If, in addition, each connected component of the complement of $W(A)$ in $\mathbb{C}$ contains a point of $\rho(A)$, then $W(A) = \sigma_{\text{ess}}(A)$. The converse of this last statement also holds.*

Let A and T be two linear operators such that $\mathcal{D}(A) \subset \mathcal{D}(T)$. We will say that T is (relatively) *A-bounded* if there exist $\alpha, \beta \geq 0$ such that $\|Tx\| \leq \alpha\|Ax\| + \beta\|x\|$, for all $x \in \mathcal{D}(A)$. Clearly, if T is bounded, then it is also A-bounded. Further, the following is a stronger notion than relative boundedness: T is (relatively) *A-compact* if for any sequence $(x_n)_n \subset \mathcal{D}(A)$ with both $(x_n)_n$ and $(Ax_n)_n$ bounded, $(Tx_n)_n$ contains a convergent subsequence. If $\rho(A) \neq \varnothing$ (as is the case of an infinitesimal generator), then T is A-compact if and only if $TR(\lambda; A)$ is compact for some (and hence for all) $\lambda \in \rho(A)$ (see [7, Section 14.1]).

The next result is known as *Weyl's Theorem*, and states that the essential spectrum of a closed operator in a Hilbert space is invariant under relatively compact operators.

Lemma 3.2 ([7, Theorem 18.8, p. 193]) *Let A be a closed operator on the Hilbert space $\mathcal{H}$ and T be an A-compact operator. Then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A+T)$.*

3.1 The Critical Spectrum of a Strongly Continuous Semigroup

For a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathcal{X}$, we consider the Banach space $\ell^\infty_{\mathcal{X}}$ of all bounded sequences in $\mathcal{X}$, and we define the semigroup $\mathbf{S}_\infty = \{S_\infty(t)\}_{t\geq 0}$ on $\ell^\infty_{\mathcal{X}}$ by

$$\big(S_\infty(t)\big)(x_n)_{n\in\mathbb{N}} = \big(S(t)x_n\big)_{n\in\mathbb{N}}.$$

For this semigroup, consider its space of strong continuity

$$\mathcal{X}_\infty := \Big\{(x_n)_{n\in\mathbb{N}} : \lim_{h\downarrow 0}\Big(\sup_{n\in\mathbb{N}}\|S_\infty(h)x_n - x_n\|\Big) = 0\Big\},$$

which is closed and $\{S_\infty(t)\}_{t\geq 0}$-invariant. Therefore, on the quotient space $\widehat{\mathcal{X}} = \ell^\infty_{\mathcal{X}}/\mathcal{X}_\infty$, the semigroup $\mathbf{S}_\infty$ induces a quotient semigroup $\widehat{\mathbf{S}} = \{\widehat{S}(t)\}_{t\geq 0}$.

The *critical spectrum* of $S(t)$ is then defined as $\sigma_{\mathrm{crit}}(S(t)) := \sigma(\widehat{S}(t))$, while its *critical spectral radius* is defined as $r_{\mathrm{crit}}(S(t)) := r(\widehat{S}(t))$. Moreover, the *critical growth bound* is defined as $\omega_{\mathrm{crit}}(\mathbf{S}) := \omega_0(\widehat{\mathbf{S}})$. For the proofs of the following results we refer to the paper of Nagel and Poland [8].

Lemma 3.3 ([4, Theorem 2.1]) *For a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ the following statements hold:*

(i) $\sigma_{\mathrm{crit}}(S(t)) \subset \sigma(S(t))$;
(ii) $r_{\mathrm{crit}}(S(t)) = e^{t\omega_{\mathrm{crit}}(\mathbf{S})}$;
(iii) $\sigma(S(t)) \setminus \{0\} = e^{t\sigma(A)} \cup \sigma_{\mathrm{crit}}(S(t)) \setminus \{0\}$;
(iv) $\omega_0(\mathbf{S}) = \max\{s(A), \omega_{\mathrm{crit}}(\mathbf{S})\}$.

For a strongly continuous semigroup $\mathbf{S} = \{S(t)\}_{t\geq 0}$ its *growth bound of non-norm continuity* (which was firstly introduced by Blake [2, Definition 2.2]) is given by

$$\delta(\mathbf{S}) := \inf\Big\{\nu \in \mathbb{R} : \text{there is } M_\nu > 0 \text{ such that } \limsup_{h\to 0}\|S(t+h) - S(t)\| \leq M_\nu e^{\nu t}, t \geq 0\Big\}.$$

Nagel and Poland [8] gave a formula to compute the critical growth bound of a strongly continuous semigroup.

Lemma 3.4 ([8, Prop. 4.6]) *For a strongly continuous semigroup* $\mathbf{S} := \{S(t)\}_{t\geq 0}$, *we have that*

$$\delta(\mathbf{S}) = \omega_{\mathrm{crit}}(\mathbf{S}). \tag{3.1}$$

4 The Perturbed Semigroup

In order to ensure that $A+T$ (with the domain $\mathcal{D}(A+T) = \mathcal{D}(A)$) generates a strongly continuous semigroup, we make the following assumption. Clearly, it is satisfied for any bounded operator T.

Assumption A The operator T is A-bounded operator on $\mathcal{X}$, and there exists a function $q\colon [0,\infty) \to [0,\infty)$ with $\lim_{t\downarrow 0} q(t) = 0$ such that $\int_0^h \|TS(\xi)x\| \mathrm{d}\xi \leq q(h)\|x\|$ for each $x \in \mathcal{D}(A)$ and $h \geq 0$.

The Myiadera–Voigt perturbation theorem (see [5, Theorem III.3.14]) assures us that $A+T$ generates a strongly continuous semigroup $\widetilde{\mathbf{S}} := \{\widetilde{S}(t)\}_{t\geq 0}$, which is given by the *Dyson–Phillips series*

$$\widetilde{S}(t) = \sum_{k=0}^{\infty} S_k(t),$$

where the operators $S_k(t)$ are defined inductively as

$$S_0(t)x := S(t)x \quad \text{and} \quad S_k(t)x := \int_0^t S_{k-1}(t-\xi)TS(\xi)x\mathrm{d}\xi,$$

for each $x \in \mathcal{D}(A)$ and $t \geq 0$.

Assumption B There exists some $k \in \mathbb{N}$ such that $t \mapsto S_k(t)$ is norm continuous on $[0,\infty)$.

In the special case of a bounded linear operator T, a characterization of norm continuity of the term $t \mapsto S_k(t)$ is given in [3]. Other equivalent formulations of Assumption B are given in [4, Proposition 4.7].

Lemma 4.1 ([3, Theorem 3.1]) *For a bounded operator T on the Hilbert space $\mathcal{H}$, consider the following assertions for every $\lambda > \omega_0(\mathbf{S})$ and $k \in \mathbb{N}$:*

(a_k) *$t \mapsto S_k(t)$ is norm continuous on $[0,\infty)$;*
(b_k) $\lim_{|\mu|\to\infty} \|P_k(\lambda + i\mu)\| = 0$ *(where $P_k(z) := R(z;A)(TR(z;A))^k$).*

Then (a_k) implies (b_k), and (b_k) implies (a_{k+2}).

It turns out that $P_k(\cdot)$ is the Laplace transform of $S_k(\cdot)$ [3, Proposition 2.2]; that is, for all $x \in \mathcal{X}$ and $\operatorname{Re} z > \omega_0(\mathbf{S})$,

$$P_k(z)x = \int_0^{\infty} e^{-zt} S_k(t)x\mathrm{d}t\,.$$

Remark 4.2 Notice that the term $S_0(t)$ coincides with the semigroup operator $S(t)$, while the operator $P_0(\lambda)$ coincides with the resolvent $R(\lambda; A)$. It is also known that (b_0) implies (a_0) [5, Theorem II.4.20].

Assumption C There exists some $k \in \mathbb{N}$ such that $S_k(t)$ is compact for each $t \geq 0$.

Lemma 4.3 *For a perturbation operator T that fulfills Assumptions A and B, we have that $\omega_{\text{crit}}(\widetilde{\mathbf{S}}) = \omega_{\text{crit}}(\mathbf{S})$, and therefore $\omega_0(\widetilde{\mathbf{S}}) = \max\{s(A+T), \omega_{\text{crit}}(\mathbf{S})\}$. Similarly, if T satisfies Assumptions A and C, then $\omega_{\text{ess}}(\widetilde{\mathbf{S}}) = \omega_{\text{ess}}(\mathbf{S})$ and $\omega_0(\widetilde{\mathbf{S}}) = \max\{s(A+T), \omega_{\text{ess}}(\mathbf{S})\}$*

For a complete proof of the above facts we refer the reader to [3] and Lemma 3.3(iv).

5 Results

Theorem 5.1 *Let $\mathbf{S} := \{S(t)\}_{t\geq 0}$ be a C_0-semigroup on the Hilbert space $\mathcal{H}$, let A be its infinitesimal generator, and let T be a linear operator. If $\sigma(S(t)) = \{0\}$ for each $t > 0$ (or equivalently, $\mathbf{S}$ is superstable) and assume that:*

(i) *T satisfies Assumption A,*
(ii) *T satisfies Assumption B or Assumption C,*
(iii) *T is A-compact.*

Then the perturbed C_0-semigroup $\widetilde{\mathbf{S}}$ generated by $A+T$ is superstable if and only if $A+T$ has an empty discrete spectrum.

Proof From the superstability hypothesis, we have that for each $\nu \in \mathbb{R}$ there exists $M_\nu > 0$ such that $\|S(t)\| \leq M_\nu e^{\nu t}$ for all $t \geq 0$. It follows that for any $h > 0$, we have

$$\|S(t+h) - S(t)\| = \|(S(h) - I)S(t)\| \leq \Big(1 + \sup_{h\geq 0}\|S(h)\|\Big) M_\nu e^{\nu t}. \tag{5.1}$$

By (3.1) and (5.1) we get that $\omega_{\text{crit}}(\mathbf{S}) = \delta(\mathbf{S}) = -\infty$. Therefore, using Lemma 4.3, we have that $\omega_0(\widetilde{\mathbf{S}}) = -\infty$ if and only if $s(A+T) = -\infty$. Thus, we have

$$\sigma(A+T) = \sigma_d(A+T) \sqcup \sigma_{\text{ess}}(A+T) = \varnothing. \tag{5.2}$$

We claim that (5.2) holds, subject to the condition that $A+T$ has an empty discrete spectrum. The claim follows easily, taking into account that $\{S(t)\}_{t\geq 0}$ is superstable, and therefore $W(A) = \sigma(A) = \varnothing$. By assumption (ii) and Lemma 3.2, it follows that $W(A+T) = \varnothing$. Hence, applying Lemma 3.1, we have $\sigma_{\text{ess}}(A+T) = W(A+T) = \varnothing$. ∎

Proposition 5.2 *Let $\mathbf{S} := \{S(t)\}$ be a C_0-semigroup on the Hilbert space $\mathcal{H}$ with generator A. If $\sigma(S(t)) = \{0\}$ for each $t > 0$ (or equivalently, $\mathbf{S}$ is superstable) and the generator A is perturbed by a linear and bounded operator T that satisfies*

$$\lim_{|\mu|\to\infty} \|R(\lambda + i\mu; A) T R(\lambda + i\mu; A)\| = 0 \tag{5.3}$$

for some $\lambda > \omega_0(\mathbf{S})$, then T obeys Assumptions A and B.

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Proof By the hypothesis, **S** will be uniformly stable; that is, $\|S(t)\| \le C$, $t \ge 0$ for some constant $C > 0$. Hence, it is easy to see that Assumption A holds for $q(t) = C\|T\|t$.

Furthermore, it follows from Lemma 4.1 that a Dyson–Phillips term, namely $t \mapsto S_3(t)$, is norm continuous on $[0, \infty)$ provided that (5.3) holds. ∎

Corollary 5.3 *Let* $\mathbf{S} := \{S(t)\}_{t\ge0}$ *be a* C_0*-semigroup on the Hilbert space* $\mathcal{H}$ *with generator A. If the bounded operator T fulfills the following conditions:*

(i) $\lim_{|\mu|\to\infty} \|R(\lambda + i\mu; A)TR(\lambda + i\mu; A)\| = 0$, *for some* $\lambda > \omega_0(\mathbf{S})$;

(ii) *T is A-compact,*

then the semigroup generated by $A + T$ *is superstable if and only if* $A + T$ *has an empty discrete spectrum.*

Proof This follows easily from Theorem 5.1 and Proposition 5.2. ∎

6 Applications

We point out below a class of bounded perturbation operators of $-\frac{d}{dx}$ that preserve the superstability of the translation (to the right) semigroup on the space of all square-integrable functions defined on $[0,2]$, $\mathcal{L}^2_{[0,2]}$.

Let $\mathbf{S} := \{S(t)\}_{t\ge0}$ be the translation semigroup, defined by

$$(S(t)f)(u) = \begin{cases} f(u-t) & u \ge t, \\ 0 & u < t, \end{cases}$$

on $\mathcal{H} := \mathcal{L}^2_{[0,2]}$. It can be easily checked that its infinitesimal generator is $-\frac{d}{dx}$ with the domain $\{f \in H^1([0,2]) : f(0) = 0\}$. We can see that $S(t) = 0$ for any $t > 2$, hence **S** is superstable.

We will show that $R(\lambda; -\frac{d}{dx})$ is compact for $\lambda = 0$ and therefore for $\lambda \in \rho(-\frac{d}{dx}) = \mathbb{C}$. Taking $\lambda \in \rho(-\frac{d}{dx})$, we notice that

$$\left(R\left(\lambda, -\frac{d}{dx}\right)f\right)(u) = \int_0^u e^{-\lambda(u-\xi)} f(\xi)\mathrm{d}\xi, \quad f \in \mathcal{L}^2_{[0,2]}$$

Now let $(g_n)_{n\in\mathbb{N}^*}$ be a bounded sequence and consider

$$G_n(u) := \left(\left(-\frac{d}{dx}\right)^{-1} g_n\right)(u) = \int_0^u g_n(\xi)\mathrm{d}\xi.$$

Clearly, we have that G_n is continuous, hence it belongs to $\mathcal{L}^2_{[0,2]}$, for every $n \in \mathbb{N}^*$. Using the Arzelà-Ascoli Theorem, we have that $(-\frac{d}{dx})^{-1}$ is compact. Thus, every bounded operator T is $(-\frac{d}{dx})$-compact.

Applying Corollary 5.3, we obtain that for each bounded operator T satisfying (5.3), the operator $-\frac{d}{dx} + T$ generates a superstable semigroup if and only if $\sigma_d(-\frac{d}{dx} + T) = \varnothing$.

Remark 6.1 The application illustrated above is not at all trivial, since if we perturbe the generator of the translation to the right semigroup on $\mathcal{L}^2_{[0,2]}$ with the simple left-shift defined as

$$(Lf)(u) = \begin{cases} f(u+1) & u \in [0,1], \\ 0 & u > 1, \end{cases}$$

the perturbed semigroup generated by $-\frac{d}{dx} + L$ does not remain even exponentially stable, since its growth bound is zero (for details see [4, Example 5.1]).

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