THE JOURNAL OF SYMBOLIC LOGIC
Volume 88, Number 1, March 2023

MOST(?) THEORIES HAVE BOREL COMPLETE REDUCTS

MICHAEL C. LASKOWSKI AND DOUGLAS S. ULRICH

Abstract. We prove that many seemingly simple theories have Borel complete reducts. Specifically, if a countable theory has uncountably many complete one-types, then it has a Borel complete reduct. Similarly, if $Th(M)$ is not small, then M^{eq} has a Borel complete reduct, and if a theory T is not ω-stable, then the elementary diagram of some countable model of T has a Borel complete reduct.

§1. Introduction. In their seminal paper [1], Friedman and Stanley define and develop a notion of *Borel reducibility* among classes of structures with universe ω in a fixed, countable language L that are Borel and invariant under permutations of ω. It is well known (see, e.g., [3] or [2]) that such classes are of the form $\mathrm{Mod}(\Phi)$, the set of models of Φ whose universe is precisely ω for some sentence $\Phi \in L_{\omega_1,\omega}$, but here we concentrate on first-order, countable theories T. For countable theories T, S in possibly different language, a *Borel reduction* is a Borel function $f : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ that satisfies $M \cong N$ if and only if $f(M) \cong f(N)$. One says that T is *Borel reducible* to S if there is a Borel reduction $f : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$. As Borel reducibility is transitive, this induces a quasi-order on the class of all countable theories, where we say T and S are *Borel equivalent* if there are Borel reductions in both directions. In [1], Friedman and Stanley show that among Borel invariant classes (hence among countable first-order theories) there is a maximal class with respect to $\leq_B$. We say Φ is *Borel complete* if it is in this maximal class. Examples include the theories of graphs, linear orders, groups, and fields.

The intuition is that Borel complexity of a theory T is related to the complexity of invariants that describe the isomorphism types of countable models of T. Given an L-structure M, one naturally thinks of the reducts M_0 of M to be 'simpler objects', hence the invariants for a reduct 'should' be no more complicated than for the original M, but we will see that this intuition is incorrect. As a paradigm, let T be the theory of 'independent unary predicates' i.e., $T = Th(2^\omega, U_n)$, where each U_n is a unary predicate interpreted as $U_n = \{\eta \in 2^\omega : \eta(n) = 1\}$. The countable models of T are rather easy to describe. The isomorphism type of a model is specified by which countable, dense subset of 'branches' is realized, and how many elements realize each of those branches. However, with Theorem 3.2, we will see that T has a Borel complete reduct.

To be precise about reducts, we have the following definition.

DEFINITION 1.1. Given an L-structure M, a *reduct M' of M* is an L'-structure with the same universe as M, and for which the interpretation in every atomic L'-formula $\alpha(x_1, \dots, x_k)$ is an L-definable subset of M^k (without parameters). An L'-theory

Received March 17, 2021.
2020 *Mathematics Subject Classification.* 03C50, 03E15.
Key words and phrases. Borel complexity, countable models, reducts.

0022-4812/23/8801-0015
DOI:10.1017/jsl.2021.78

T' is a *reduct of an L-theory T* if $T' = Th(M')$ for some reduct M' of some model M of T.

In the above definition, it would be equivalent to require that the interpretation in M' of every L'-formula $\theta(x_1, \dots, x_k)$ is a 0-definable subset of M^k.

§2. An engine for Borel completeness results. This section is devoted to proving Borel completeness for a specific family of theories. All of the theories T_h are in the same language $L = \{E_n : n \in \omega\}$ and are indexed by strictly increasing functions $h : \omega \to \omega \setminus \{0\}$. For a specific choice of h, the theory T_h asserts that

- Each E_n is an equivalence relation with exactly $h(n)$ classes; and
- The E_n's cross-cut, i.e., for all nonempty, finite $F \subseteq \omega$, $E_F(x, y) := \bigwedge_{n\in F} E_n(x, y)$ is an equivalence relation with precisely $\Pi_{n\in F} h(n)$ classes.

It is well known that each of these theories T_h is complete and admits elimination of quantifiers. Thus, in any model of T_h, there is a unique one-type. However, the strong type structure is complicated.[1] So much so, that the whole of this section is devoted to the proof of:

THEOREM 2.1. *For any strictly increasing $h : \omega \to \omega \setminus \{0\}$, T_h is Borel complete.*

PROOF. Fix a strictly increasing function $h : \omega \to \omega \setminus \{0\}$. We begin by describing representatives $\mathcal{B}$ of the strong types and a group G that acts faithfully and transitively on $\mathcal{B}$. As notation, for each n, let $[h(n)]$ denote the $h(n)$-element set $\{1, \dots, h(n)\}$ and let $Sym([h(n])$ be the (finite) group of permutations of $[h(n)]$. Let

$$\mathcal{B} = \{f : \omega \to \omega : f(n) \in [h(n)] \text{ for all } n \in \omega\},$$

and let $G = \Pi_{n\in\omega} Sym([h(n)])$ be the direct product. As notation, for each $n \in \omega$, let $\pi_n : G \to Sym([h(n)])$ be the natural projection map. Note that G acts coordinatewise on $\mathcal{B}$ by: For $g \in G$ and $f \in \mathcal{B}$, $g \cdot f$ is the element of $\mathcal{B}$ satisfying $g \cdot f(n) = \pi_n(g)(f(n))$.

Define an equivalence relation $\sim$ on $\mathcal{B}$ by:

$$f \sim f' \quad \text{if and only if} \quad \{n \in \omega : f(n) \neq f'(n)\} \text{ is finite.}$$

For $f \in \mathcal{B}$, let $[f]$ denote the $\sim$-class of f and, abusing notation somewhat, for $W \subseteq \mathcal{B}$

$$[W] := \bigcup\{[f] : f \in W\}.$$

Observe that for every $g \in G$, the permutation of $\mathcal{B}$ induced by the action of g maps $\sim$-classes onto $\sim$-classes, i.e., G also acts transitively on $\mathcal{B}/\sim$.

We first identify a countable family of $\sim$-classes that are 'sufficiently indiscernible'. Our first lemma is where we use the fact that the function h defining T_h is strictly increasing.

LEMMA 2.2. *There is a countable set $Y = \{f_i : i \in \omega\} \subseteq \mathcal{B}$ such that whenever $i \neq j$, $\{n \in \omega : f_i(n) = f_j(n)\}$ is finite.*

[1] Recall that in any structure M, two elements a, b have the same *strong type*, $\mathrm{stp}(a) = \mathrm{stp}(b)$, if $M \models E(a, b)$ for every 0-definable equivalence relation. Because of the quantifier elimination, in any model $M \models T_h$, $\mathrm{stp}(a) = \mathrm{stp}(b)$ if and only if $M \models E_n(a, b)$ for every $n \in \omega$.

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

PROOF. We recursively construct Y in ω steps. Suppose $\{f_i : i < k\}$ have been chosen. Choose an integer N large enough so that $h(N) > k$ (hence $h(n) > k$ for all $n \geq N$). Now, construct $f_k \in \mathcal{B}$ to satisfy $f_k(n) \neq f_i(n)$ for all $n \geq N$ and all $i < k$. ⊣

Fix an enumeration $\langle f_i : i \in \omega \rangle$ of Y for the whole of the argument. The 'indiscernibility' of Y alluded to above is formalized by the following definition and lemma.

DEFINITION 2.3. Given a permutation $\sigma \in Sym(\omega)$, a group element $g \in G$ *respects* σ if $g \cdot [f_i] = [f_{\sigma(i)}]$ for every $i \in \omega$.

LEMMA 2.4. *For every permutation $\sigma \in Sym(\omega)$, there is some $g \in G$ respecting σ.*

PROOF. Note that since h is increasing, $h(n) \geq n$ for every $n \in \omega$. Fix a permutation $\sigma \in Sym(\omega)$ and we will define some $g \in G$ respecting σ coordinatewise. Using Lemma 2.2, choose a sequence

$$0 = N_0 \ll N_1 \ll N_2 \ll \cdots$$

of integers such that for all $i \in \omega$, both $f_i(n) \neq f_j(n)$ and $f_{\sigma(i)}(n) \neq f_{\sigma(j)}(n)$ hold for all $n \geq N_i$ and all $j < i$.

Since $\{N_i\}$ are increasing, it follows that for each $i \in \omega$ and all $n \geq N_i$, the subsets $\{f_j(n) : j \leq i\}$ and $\{f_{\sigma(j)}(n) : j \leq i\}$ of $[h(n)]$ each have precisely $(i+1)$ elements. Thus, for each $i < \omega$ and for each $n \geq N_i$, there is a permutation $\delta_n \in Sym([h(n)])$ satisfying

$$\bigwedge_{j \leq i} \delta_n(f_j(n)) = f_{\sigma(j)}(n).$$

(Simply begin defining δ_n to meet these constraints, and then complete δ_n to a permutation of $[h(n)]$ arbitrarily.) Using this, define $g := \langle \delta_n : n \in \omega \rangle$, where each $\delta_n \in Sym([h(n)])$ is constructed as above. To see that g respects σ, note that for every $i \in \omega$, $(g \cdot f_i)(n) = f_{\sigma(i)}(n)$ for all $n \geq N_i$, so $(g \cdot f_i) \sim f_{\sigma(i)}$. ⊣

DEFINITION 2.5. For distinct integers $i \neq j$, let $d_{i,j} \in \mathcal{B}$ be defined by:

$$d_{i,j}(n) := \begin{cases} f_i(n) & \text{if } n \text{ even;} \\ f_j(n) & \text{if } n \text{ odd.} \end{cases}$$

Let $Z := \{d_{i,j} : i \neq j\}$.

Note that $d_{i,j} \not\sim f_k$ for all distinct i, j and all $k \in \omega$, hence $\{[f_i] : i \in \omega\}$ and $\{[d_{i,j}] : i \neq j\}$ are disjoint.

LEMMA 2.6. *For all $\sigma \in Sym(\omega)$, if $g \in G$ respects σ, then $g \cdot [d_{i,j}] = [d_{\sigma(i),\sigma(j)}]$ for all $i \neq j$.*

PROOF. Choose $\sigma \in Sym(\omega)$, g respecting σ, and $i \neq j$. Choose N such that $(g \cdot [f_i])(n) = [f_{\sigma(i)}](n)$ and $(g \cdot [f_j])(n) = [f_{\sigma(j)}](n)$ for every $n \geq N$. Since $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even,

$$(g \cdot d_{i,j})(n) = \pi_n(g)(d_{i,j}(n)) = \pi_n(g)(f_i(n)) = (g \cdot f_i)(n) = f_{\sigma(i)}(n).$$

Dually, $(g \cdot d_{i,j})(n) = f_{\sigma(j)}(n)$ when $n \geq N$ is odd, so $(g \cdot d_{i,j}) \sim d_{\sigma(i),\sigma(j)}$. ⊣

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

With the combinatorial preliminaries out of the way, we now prove that T_h is Borel complete. We form a highly homogeneous model $M^* \models T_h$ and thereafter, all models we consider will be countable, elementary substructures of M^*. Let $A = \{a_f : f \in \mathcal{B}\}$ and $B = \{b_f : f \in \mathcal{B}\}$ be disjoint sets and let M^* be the L-structure with universe $A \cup B$ and each E_n interpreted by the rules:

- For all $f \in \mathcal{B}$ and $n \in \omega$, $E_n(a_f, b_f)$; and
- For all $f, f' \in \mathcal{B}$ and $n \in \omega$, $E_n(a_f, a_{f'})$ iff $f(n) = f'(n)$,

with the other instances of E_n following by symmetry and transitivity. For any finite $F \subseteq \omega$, $\{f\restriction_F : f \in \mathcal{B}\}$ has exactly $\Pi_{n\in F} h(n)$ elements, hence $E_F(x, y) := \bigwedge_{n\in F} E_n(x, y)$ has $\Pi_{n\in F} h(n)$ classes in M^*. Thus, the $\{E_n : n \in \omega\}$ cross cut and $M^* \models T_h$.

Let $E_\infty(x, y)$ denote the (type definable) equivalence relation $\bigwedge_{n\in\omega} E_n(x, y)$. Then, in M^*, E_∞ partitions M^* into two-element classes $\{a_f, b_f\}$, indexed by $f \in \mathcal{B}$. Note also that every $g \in G$ induces an L-automorphism $g^* \in Aut(M^*)$ by

$$g^*(x) := \begin{cases} a_{(g\cdot f)} & \text{if } x = a_f \text{ for some } f \in \mathcal{B}; \\ b_{(g\cdot f)} & \text{if } x = b_f \text{ for some } f \in \mathcal{B}. \end{cases}$$

Recall the set $Y = \{f_i : i \in \mathcal{B}\}$ from Lemma 2.2, so $[Y] = \{[f_i] : i \in \omega\}$. Let $M_0 \subseteq M^*$ be the substructure with universe $\{a_f : f \in [Y]\}$. As T_h admits elimination of quantifiers and as $[Y]$ is dense in $\mathcal{B}$, $M_0 \preceq M^*$. Moreover, every substructure M of M^* with universe containing M_0 will also be an elementary substructure of M^*, hence a model of T_h.

To show that $Mod(T_h)$ is Borel complete, we define a Borel mapping from $\{$irreflexive graphs $\mathcal{G} = (\omega, R)\}$ to $Mod(T_h)$ as follows: Given $\mathcal{G}$, let $Z(R) := \{d_{i,j} \in Z : \mathcal{G} \models R(i, j)\}$, so $[Z(R)] = \bigcup\{[d_{i,j}] : d_{i,j} \in Z(R)\}$. Let $M_G \preceq M^*$ be the substructure with universe

$$M_0 \cup \{a_d, b_d : d \in [Z(R)]\}.$$

That the map $\mathcal{G} \mapsto M_G$ is Borel is routine, given that Y and Z are fixed throughout.

Note that in M_G, every E_∞-class has either one or two elements. Specifically, for each $d \in [Z(R)]$, the E_∞-class $[a_d]_\infty = \{a_d, b_d\}$, while the E_∞-class $[a_f]_\infty = \{a_f\}$ for every $f \in [Y]$.

We must show that for any two graphs $\mathcal{G} = (\omega, R)$ and $\mathcal{H} = (\omega, S)$, $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if and only if the L-structures M_G and M_H are isomorphic.

To verify this, first choose a graph isomorphism $\sigma : (\omega, R) \to (\omega, S)$. Then $\sigma \in Sym(\omega)$ and, for distinct integers $i \neq j$, $d_{i,j} \in Z(R)$ if and only if $d_{\sigma(i),\sigma(j)} \in Z(S)$. Apply Lemma 2.4 to get $g \in G$ respecting σ and let $g^* \in Aut(M^*)$ be the L-automorphism induced by g. By Lemma 2.6 and Definition 2.3, it is easily checked that the restriction of g^* to M_G is an L-isomorphism between M_G and M_H.

Conversely, assume that $\Psi : M_G \to M_H$ is an L-isomorphism. Clearly, Ψ maps E_∞-classes in M_G to E_∞-classes in M_H. In particular, Ψ permutes the one-element E_∞-classes $\{\{a_f\} : f \in [Y]\}$ of both M_G and M_H, and maps the two-element E_∞-classes $\{\{a_d, b_d\} : d \in [Z(R)]\}$ of M_G onto the two-element E_∞-classes $\{\{a_d, b_d\} : d \in [Z(S)]\}$ of M_H. That is, Ψ induces a bijection $F : [Y \sqcup Z(R)] \to [Y \sqcup Z(S)]$ that permutes $[Y]$.

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

As well, by the interpretations of the E_n's, for $f, f' \in [Y \sqcup Z(R)]$ and $n \in \omega$,

$$f(n) = f'(n) \quad \text{if and only if} \quad F(f)(n) = F(f')(n).$$

From this it follows that F maps $\sim$-classes onto $\sim$-classes. As F permutes $[Y]$ and as $[Y] = \bigcup\{[f_i] : i \in \omega\}$, F induces a permutation $\sigma \in Sym(\omega)$ given by $\sigma(i)$ is the unique $i^* \in \omega$ such that $F([f_i]) = [f_{i^*}]$.

We claim that this σ induces a graph isomorphism between $\mathcal{G} = (\omega, R)$ and $\mathcal{H} = (\omega, S)$. Indeed, choose any $(i, j) \in R$. Thus, $d_{i,j} \in Z(R)$. As F is $\sim$-preserving, choose N large enough so that $F(f_i)(n) = F(f_{\sigma(i)})(n)$ and $F(f_j)(n) = F(f_{\sigma(j)})(n)$ for every $n \geq N$. By definition of $d_{i,j}$, $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even, so $F(d_{i,j})(n) = F(f_i)(n) = f_{\sigma(i)}(n)$ for such n. Dually, for $n \geq N$ odd, $F(d_{i,j})(n) = F(f_j)(n) = f_{\sigma(j)}(n)$. Hence, $F(d_{i,j}) \sim d_{\sigma(i),\sigma(j)} \in [Z(S)]$. Thus, $(\sigma(i), \sigma(j)) \in S$. The converse direction is symmetric (i.e., use Ψ^{-1} in place of Ψ and run the same argument). $\dashv$

Remark 2.7. If we relax the assumption that $h : \omega \to \omega \setminus \{0\}$ is strictly increasing, there are two cases. If h is unbounded, then the proof given above can easily be modified to show that the associated T_h is also Borel complete. Conversely, with Theorem 6.2 of [6] the authors prove that if $h : \omega \to \omega \setminus \{0\}$ is bounded, then T_h is not Borel complete. The salient distinction between the two cases is that when h is bounded, the associated group G has bounded exponent. However, even in the bounded case T_h has a Borel complete reduct by Lemma 3.1 below.

§3. Applications to reducts.

We begin with one easy lemma that, when considering reducts, obviates the need for the number of classes to be strictly increasing.

Lemma 3.1. *Let $L = \{E_n : n \in \omega\}$ and let $f : \omega \to \omega \setminus \{0, 1\}$ be any function. Then every model M of T_f, the complete theory asserting that each E_n is an equivalence relation with $f(n)$ classes, and that the $\{E_n\}$ cross-cut, has a Borel complete reduct.*

Proof. Given any function $f : \omega \to \omega \setminus \{0, 1\}$, choose a partition $\omega = \bigsqcup\{F_n : n \in \omega\}$ into non-empty finite sets for which $\Pi_{k \in F_n} f(k) < \Pi_{k \in F_m} f(k)$ whenever $n < m < \omega$. For each n, let $h(n) := \Pi_{k \in F_n} f(k)$ and let $E_n^*(x, y) := \bigwedge_{k \in F_n} E_k(x, y)$. Then, as h is strictly increasing and $\{E_n^*\}$ is a cross-cutting set of equivalence relations with each E_n^* having $h(n)$ classes.

Now let $M \models T_f$ be arbitrary and let $L' = \{E_n^* : n \in \omega\}$. As each E_n^* described above is 0-definable in M, there is an L'-reduct M' of M. It follows from Theorem 2.1 that $T' = Th(M')$ is Borel complete, so T_f has a Borel complete reduct. $\dashv$

Theorem 3.2. *Suppose T is a complete theory in a countable language with uncountably many one-types. Then every model M of T has a Borel complete reduct.*

Proof. Let $M \models T$ be arbitrary. As usual, by the Cantor–Bendixon analysis of the compact, Hausdorff–Stone space $S_1(T)$ of complete one-types, choose a set $\{\varphi_\eta(x) : \eta \in 2^{<\omega}\}$ of 0-definable formulas, indexed by the tree $(2^{<\omega}, \trianglelefteq)$ ordered by initial segment, satisfying:

1. $M \models \exists x \varphi_\eta(x)$ for each $\eta \in 2^{<\omega}$;
2. For $\nu \trianglelefteq \eta$, $M \models \forall x(\varphi_\eta(x) \to \varphi_\nu(x))$;
3. For each $n \in \omega$, $\{\varphi_\eta(x) : \eta \in 2^n\}$ are pairwise contradictory.

By increasing these formulas slightly, we can additionally require

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

4. For each $n \in \omega$, $M \models \forall x(\bigvee_{\eta \in 2^n} \varphi_\eta(x))$.

Given such a tree of formulas, for each $n \in \omega$, define

$$\delta_n^0(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \to \varphi_{\eta^\frown 0}(x)] \quad \text{and} \quad \delta_n^1(x) := \bigwedge_{\eta \in 2^n} [\varphi_\eta(x) \to \varphi_{\eta^\frown 1}(x)].$$

Because of (4) above, $M \models \forall x(\delta_n^0(x) \vee \delta_n^1(x))$ for each n. Also, for each n, let

$$E_n(x, y) := [\delta_n^0(x) \leftrightarrow \delta_n^0(y)].$$

From the above, each E_n is a 0-definable equivalence relation with precisely two classes.

Claim: *The equivalence relations $\{E_n : n \in \omega\}$ are cross-cutting.*

Proof. It suffices to prove that for every $m > 0$, the equivalence relation $E_m^*(x, y) := \bigwedge_{n<m} E_n(x, y)$ has 2^m classes. So fix m and choose a subset $A_m = \{a_\eta : \eta \in 2^m\} \subseteq M$ forming a set of representatives for the formulas $\{\varphi_\eta(x) : \eta \in 2^m\}$. It suffices to show that $M \models \neg E_m^*(a_\eta, a_\nu)$ whenever $\eta \neq \nu$ are from 2^m. But this is clear. Fix distinct $\eta \neq \nu$ and choose any $k < m$ such that $\eta(k) \neq \nu(k)$. Then $M \models \neg E_k(a_\eta, a_\nu)$, hence $M \models \neg E_m^*(a_\eta, a_\nu)$. ⊣

Thus, taking the 0-definable relations $\{E_n\}$, M has a reduct that is a model of T_f (where f is the constant function 2). As reducts of reducts are reducts, it follows from Lemma 3.1 and Theorem 2.1 that M has a Borel complete reduct.

We highlight how unexpected Theorem 3.2 is with two examples. First, the theory of 'Independent unary predicates' mentioned in the Introduction has a Borel complete reduct.

Next, we explore the assumption that a countable, complete theory T is not small, i.e., for some k there are uncountably many k-types. We conjecture that some model of T has a Borel complete reduct. If $k = 1$, then by Theorem 3.2, every model of T has a Borel complete reduct. If $k > 1$ is least, then it is easily seen that there is some complete $(k-1)$-type $p(x_1, \dots, x_{k-1})$ with uncountably many complete $q(x_1, \dots, x_k)$ extending p. Thus, if M is any model of T realizing p, say by $\bar{a} = (a_1, \dots, a_{k-1})$, the expansion $(M, a_1, \dots, a_{k-1})$ has a Borel complete reduct, also by Theorem 3.2. Similarly, we have the following result.

Corollary 3.3. *Suppose T is a complete theory in a countable language that is not small. Then for any model M of T, M^{eq} has a Borel complete reduct.*

Proof. Let M be any model of T and choose k least such that T has uncountably many complete k-types consistent with it. In the language L^{eq}, there is a sort U_k and a definable bijection $f : M^k \to U_k$. Hence $Th(M^{eq})$ has uncountably many one-types consistent with it, each extending U_k. Thus, M^{eq} has a Borel complete reduct by Theorem 3.2. ⊣

Finally, recall that a countable, complete theory is not ω-stable if, for some countable model M of T, the Stone space $S_1(M)$ is uncountable. From this, we immediately obtain our final corollary.

Corollary 3.4. *If a countable, complete T is not ω-stable, then for some countable model M of T, the elementary diagram of M in the language $L(M) = L \cup \{c_m : m \in M\}$ has a Borel complete reduct.*

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

PROOF. Choose a countable M so that $S_1(M)$ is uncountable. Then, in the language $L(M)$, the theory of the expanded structure M_M in the language $L(M)$ has uncountably many one-types, hence it has a Borel complete reduct by Theorem 3.2. ⊣

The results above are by no means characterizations. Indeed, there are many Borel complete ω-stable theories. In [5], the first author and Shelah prove that any ω-stable theory that has eni-DOP or is eni-deep is not only Borel complete, but also λ-Borel complete for all λ.[2] As well, there are ω-stable theories with only countably many countable models that have Borel complete reducts. To illustrate this, we introduce three interrelated theories. The first, T_0, in the language $L_0 = \{U, V, W, R\}$ is the paradigmatic DOP theory. T_0 asserts that:

- U, V, W partition the universe;
- $R \subseteq U \times V \times W$;
- $T_0 \models \forall x \forall y \exists^\infty z R(x, y, z)$ [more formally, for each n, $T_0 \models \forall x \forall y \exists^{\geq n} z R(x, y, z)$]; and
- $T_0 \models \forall x \forall x' \forall y \forall y' \forall z [R(x, y, z) \wedge R(x', y', z) \rightarrow (x = x' \wedge y = y')]$.

T_0 is both ω-stable and ω-categorical and its unique countable model is rather tame. The complexity of T_0 is only witnessed with uncountable models, where one can code arbitrary bipartite graphs in an uncountable model M by choosing the cardinalities of the sets $R(a, b, M)$ among $(a, b) \in U \times V$ to be either $\aleph_0$ or $|M|$.

To get bad behavior of countable models, we expand T_0 to an $L = L_0 \cup \{f_n : n \in \omega\}$-theory $T \supseteq T_0$ that additionally asserts:

- Each $f_n : U \times V \rightarrow W$;
- $\forall x \forall y R(x, y, f_n(x, y))$ for each n; and
- for distinct $n \neq m$, $\forall x \forall y (f_n(x, y) \neq f_m(x, y))$.

This T is ω-stable with eni-DOP and hence is Borel complete by Theorem 4.12 of [5].

However, T has an expansion T^* in a language $L^* := L \cup \{c, d, g, h\}$ whose models are much better behaved. Let T^* additionally assert:

- $U(c) \wedge V(d)$;
- $g : U \rightarrow V$ is a bijection with $g(c) = d$;
- Letting $W^* := \{z : R(c, d, z)\}$, $h : U \times V \times W^* \rightarrow W$ is an injective map that is the identity on W^* and, for each $(x, y) \in U \times V$, maps W^* onto $\{z \in W : R(x, y, z)\}$; and moreover
- h commutes with each f_n, i.e., $\forall x \forall y (h(x, y, f_n(c, d)) = f_n(x, y))$.

Then T^* is ω-stable and two-dimensional (the dimensions being $|U|$ and $|W^* \setminus \{f_n(c, d) : n \in \omega\}|$), hence T^* has only countably many countable models. However, T^* visibly has a Borel complete reduct, namely T.

§4. Observations about the theories T_h.

In addition to their utility in proving Borel complete reducts, the theories T_h in Section 2 illustrate some novel behaviors. First off, model theoretically, these theories are extremely simple. More precisely,

[2] Definitions of eni-DOP and eni-deep are given in Definitions 2.3 and 6.2, respectively, of [5], and the definition of λ-Borel complete is recalled in Section 4 of this paper.

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

each theory T_h is weakly minimal with the geometry of every strong type trivial (such theories are known as mutually algebraic in [4]).

Additionally, the theories T_h are the simplest known examples of theories that are Borel complete, but not λ-Borel complete for all cardinals λ. For λ any infinite cardinal, λ-Borel completeness was introduced in [5]. Instead of looking at L-structures with universe ω, we consider X_L^λ, the set of L-structures with universe λ. We topologize X_L^λ analogously; namely a basis consists of all sets

$$U_{\varphi(\alpha_1,\dots,\alpha_n)} := \{M \in X_L^\lambda : M \models \varphi(\alpha_1,\dots,\alpha_n)\}$$

for all L-formulas $\varphi(x_1,\dots,x_n)$ and all $(\alpha_1,\dots,\alpha_n) \in \lambda^n$. Define a subset of X_L^λ to be λ*-Borel* if it is the smallest λ^+-algebra containing the basic open sets, and call a function $f : X_{L_1}^\lambda \to X_{L_2}^\lambda$ to be λ*-Borel* if the inverse image of every basic open set is λ-Borel. For T, S theories in languages L_1, L_2, respectively, we say that $\mathrm{Mod}_\lambda(T)$ is λ*-Borel reducible* to $\mathrm{Mod}_\lambda(S)$ if there is a λ-Borel $f : \mathrm{Mod}_\lambda(T) \to \mathrm{Mod}_\lambda(S)$ preserving back-and-forth equivalence in both directions (i.e., $M \equiv_{\infty,\omega} N \Leftrightarrow f(M) \equiv_{\infty,\omega} f(N)$).

As back-and-forth equivalence is the same as isomorphism for countable structures, λ-Borel reducibility when $\lambda = \omega$ is identical to Borel reducibility. As before, for any infinite λ, there is a maximal class under λ-Borel reducibility, and we say a theory is λ*-Borel complete* if it is in this maximal class. All of the 'classical' Borel complete theories, e.g., graphs, linear orders, groups, and fields, are λ-Borel complete for all λ. However, the theories T_h are not.

LEMMA 4.1. *If T is mutually algebraic in a countable language, then there are at most $\beth_2$ pairwise $\equiv_{\infty,\omega}$-inequivalent models (of any size).*

PROOF. We show that every model M has an (∞,ω)-elementary substructure of size $2^{\aleph_0}$, which suffices. So, fix M and choose an arbitrary countable $M_0 \preceq M$. By Proposition 4.4 of [4], $M \setminus M_0$ can be decomposed into countable components, and any permutation of isomorphic components induces an automorphism of M fixing M_0 pointwise. As there are at most $2^{\aleph_0}$ non-isomorphic components over M_0, choose a substructure $N \subseteq M$ containing M_0 and, for each isomorphism type of a component, N contains either all of copies in M (if there are only finitely many) or else precisely $\aleph_0$ copies if M contains infinitely many copies. It is easily checked that $N \preceq_{\infty,\omega} M$. ⊣

COROLLARY 4.2. *No mutually algebraic theory T in a countable language is λ-Borel complete for $\lambda \geq \beth_2$. In particular, T_h is Borel complete, but not λ-Borel complete for large λ.*

PROOF. Fix $\lambda \geq \beth_2$. It is readily checked that there is a family of 2^λ graphs that are pairwise not back and forth equivalent. As there are fewer than 2^λ $\equiv_{\infty,\omega}$-classes of models of T, there cannot be a λ-Borel reduction of graphs into $\mathrm{Mod}_\lambda(T)$. ⊣

In [7], another example of a Borel complete theory that is not λ-Borel complete for all λ is given (it is dubbed TK there) but the T_h examples are cleaner. In order to understand this behavior, in [7] we call a theory T *grounded* if every potential canonical Scott sentence σ of a model of T (i.e., in some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, σ is a canonical Scott sentence of some model), then σ is a canonical Scott

https://doi.org/10.1017/jsl.2021.78 Published online by Cambridge University Press

sentence of a model in $\mathbb{V}$. Proposition 5.1 of [7] proves that every theory of refining equivalence relations is grounded. By contrast, we have

PROPOSITION 4.3. *If T is Borel complete with a cardinal bound on the number of $\equiv_{\infty,\omega}$-classes of models, then T is not grounded. In particular, T_h is not grounded.*

PROOF. Let κ denote the number of $\equiv_{\infty,\omega}$-classes of models of T. If T were grounded, then κ would also bound the number of potential canonical Scott sentences. As the class of graphs has a proper class of potential canonical Scott sentences, it would follow from Theorem 3.10 of [7] that T could not be Borel complete. $\dashv$

Acknowledgment. Both authors were partially supported by NSF grant no. DMS-1855789.

REFERENCES

[1] H. FRIEDMAN and L. STANLEY, *A Borel reducibility theory for classes of countable structures*, this JOURNAL, vol. 54 (1989), no. 3, pp. 894–914.

[2] S. GAO, ***Invariant Descriptive Set Theory***, Chapman & Hall/CRC Pure and Applied Mathematics, vol. 293, CRC Press, Boca Raton, 2008.

[3] A. KECHRIS, ***Classical Descriptive Set Theory***, Graduate Texts in Mathematics, vol. 156, Springer, New York, 1995.

[4] M. C. LASKOWSKI, *Mutually algebraic structures and expansions by predicates*, this JOURNAL, vol. 78 (2013), no. 1, pp. 185–194.

[5] M. C. LASKOWSKI and S. SHELAH, *Borel complexity of some $\aleph_0$ -stable theories*. ***Fundamenta Mathematicae***, vol. 229 (2015), no. 1, pp. 1–46.

[6] M. C. LASKOWSKI and D. S. ULRICH, *Characterizing the existence of a Borel complete expansion*, submitted, arXiv:2109.06140.

[7] D. ULRICH, R. RAST, and M. C. LASKOWSKI, *Borel complexity and potential canonical Scott sentences*. ***Fundamenta Mathematicae***, vol. 239 (2017), no. 2, pp. 101–147.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MD, USA
E-mail: laskow@umd.edu
E-mail: ds_ulrich@hotmail.com