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ANALYSIS ON ROOT SYSTEMS

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Introduction. A great part of mathematical analysis relies directly on the methods of separation of variables and on the successive reduction of several variables problems to one-dimensional equations and to the theory of classical special functions; for example, the theory of elliptic or parabolic equations with regular coefficients (even with non constant coefficients) can be done because we know explicitly the fundamental solutions of the Laplace operator or of the heat equation; these fundamental solutions are functions of one variable; pseudodifferential or parametrices methods are thus basically small perturbations of an explicitly known problem in one variable.

On the other hand, there are many problems which are not of this type: they are related to the questions of operators with singular coefficients and to the global behaviour of the solutions; in that case, the local model cannot be reduced to a one variable problem but is fundamentally a several variables problem which cannot be treated in a detailed way by one variable methods or perturbation analysis of a one variable problem. Although a precise definition of what should be "regular singularities" for a partial differential operator is not yet understood, we can imagine that the singularities of the radial part (in the Cartan decomposition) of the Laplace-Beltrami operator of a symmetric space in rank greater than 1, should be typical examples of regular singularities. Moreover these operators present certain symmetries due to the action of a finite group, the Weyl group, and to the underlying root system. In fact, once we have fixed the root system on a euclidean space, and once we have fixed certain numbers called the multiplicities of the root, we can determine, in a unique way, a second order elliptic operators having its singularities on certain hyperplanes of $\mathbf{R}^n$ bounding the so called Weyl chamber (which is a cone in $\mathbf{R}^n$). This operator comes from the Laplace operator of a symmetric space for very precise values of the multiplicities of the roots (for example

$$\frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}$$

comes from a euclidean space only if α is an integer). Our purpose is to study such operators and in particular the fundamental solution of their

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heat equation. We can do this only in very few cases. Moreover, even in the symmetric space cases, we think that representation theory or Harish Chandra theory does not give explicit results and we have treated the problem directly. We shall concentrate only on the A_p and BC_p root systems (leaving aside D_p and the exceptional systems). Here is the content of the paper.

Section 1 gives general definitions on root systems, Laplace operators, volume element and their relation to symmetric spaces.

Section 2 studies the root system A_p (with the usual technical difficulty associated with the choice of coordinate). We obtain for $SU(p + 1)$ the invariant operators, their eigenvalues and eigenfunctions. From this, Section 3 deduces the heat kernel in term of Θ functions. Section 4 begins the study of the root system BC_p, their eigenfunctions, eigenvalues and heat kernel for certain compact spaces. The non compact case is treated in Section 8.

Section 5 gives several formulas for the analysis on a symmetric space in horospherical coordinates and Section 6 deduces an explicit expression for the quantum propagation of the open Toda lattice (a problem which was posed to us by the late Professor Mark Kac). Section 7 gives explicit recursion formulas for the heat kernels of ordinary hypergeometric equations. Section 9 treats the case of the rank 2 spaces with root system B_2 (or C_2) and symmetric spaces of rank 1. Section 10 applies the preceding analysis of Section 5 and Sections 7 and 9 to fundamental solution on solvable groups and to the quantum mechanics in the exponential potential or in the Morse potential.

1. Root systems and radial parts of Laplace operators.

1. *Root systems on a euclidean space*. a) Let E be a euclidean space of dimension p and $\mathfrak{R}$ a root system on E. If x, y are points in E, denote (x, y) the euclidean scalar product and if α is a root in $\mathfrak{R}$, we identify α with a vector in E so that we can define (x, α).

b) Call $\mathfrak{R}_+$ the set of positive roots for a certain order and call Λ the so called Weyl chamber

$$(1.1) \qquad \Lambda = \{x \in E/(x, \alpha) > 0 \quad \text{for } \alpha \in \mathfrak{R}_+\}.$$

c) To each $\alpha \in \mathfrak{R}_+$ we associate a number $\rho_\alpha > 0$ called multiplicity of the root and we define on E the *volume element*

$$(1.2) \qquad v_\rho(x) = \prod_{\alpha \in \mathfrak{R}_+} (\sin h(\alpha, x))^{\rho_\alpha}.$$

d) Let also $e_1, \ldots, e_n$ be an orthonormal basis of E and $x_i = (e_i, x)$ the corresponding orthonormal coordinate. Call

(1.3) $$\Delta_2 f = (v_\rho(x))^{-1} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(v_\rho(x)\frac{\partial}{\partial x_j} f\right).$$

Definition. This second order differential operator is called the Laplace-Beltrami operator for $(E, \mathfrak{R}, \rho)$.

2. *Relation to the symmetric space of negative curvature.* Let $X = G/K$ be a symmetric space of negative curvature, with G a semi simple Lie group with finite center, K a maximal compact subgroup. Call $\mathfrak{G}$ and $\mathfrak{K}$ the corresponding Lie algebras, $\mathfrak{P}$ the orthogonal complement of $\mathfrak{K}$ in $\mathfrak{G}$ for the Killing scalar product and $\mathfrak{A}_{\mathfrak{P}}$ a maximal abelian subalgebra contained in $\mathfrak{P}$: $\mathfrak{A}_{\mathfrak{P}}$ is a euclidean space of a certain dimension p (for the Killing scalar product) and we shall denote it by E. Now $\mathfrak{A}_{\mathfrak{P}} \equiv E$ can be obtained as follows: consider a complexified Lie algebra $\mathfrak{G}_{\mathbf{C}}$ of $\mathfrak{G}$ and $\mathfrak{H}_{\mathbf{C}}$ a maximal complex abelian subalgebra in $\mathfrak{G}_{\mathbf{C}}$. On $\mathfrak{H}_{\mathbf{C}}$, there is a natural system of complex roots defined as follows: for any $h \in \mathfrak{H}_{\mathbf{C}}$, we have a linear map

$$X \in \mathfrak{G}_{\mathbf{C}} \to [h, X] \in \mathfrak{G}_{\mathbf{C}}.$$

We can diagonalize these maps simultaneously, because $\mathfrak{H}_{\mathbf{C}}$ is abelian, and the eigenvalues are denoted $\alpha(h)$: they are linear forms on $\mathfrak{H}_{\mathbf{C}}$; as eigenvalues they have complex multiplicities equal to 1, and we denote by $\mathfrak{G}_{\mathbf{C}}^{(\alpha)}$ their eigenspace (of dimension 1). We denote by $\mathfrak{R}_{\mathbf{C}}$ this root system. Now $E \equiv \mathfrak{A}_{\mathfrak{P}}$ is just $\mathfrak{H}_{\mathbf{C}} \cap \mathfrak{P}$ (so it is a real abelian subalgebra of $\mathfrak{P}$). We can then restrict the linear form α to E, to obtain the so called restricted root system $\mathfrak{R}$. Each root α has then a multiplicity ρ_α, which can be greater than 1, and in this manner, one obtain a triplet $(E, \mathfrak{R}, \rho)$.

3. *Radial coordinate on X.* Now the symmetric space $X = G/K$ has a natural system of so called *radial coordinates*: if o is a given origin in X, then

(1.4) $$X = K \cdot A \cdot o$$

where K (resp. A) are the subgroups of G with Lie algebra $\mathfrak{K}$ (resp. $\mathfrak{A}_{\mathfrak{P}}$) and then any point $m \in X$ can be uniquely written as

$$m = k \cdot (\exp x) \cdot o$$

where x belongs to the Weyl chamber Λ of $\mathfrak{A}_{\mathfrak{P}}$ for a given order $\mathfrak{R}$. x is the generalization of the radial coordinate in usual euclidean space but: it is here in general a p-dimensional vector belonging to a kind of cone Λ.

Now, the riemannian volume element of the space X can be decomposed into a part depending only on K and a radial part depending only on x and this radial part is

(1.5) $$v_\rho(x)dx_1 \dots dx_n \quad (v_\rho \text{ given by } (1.1)).$$

We call radial function a function f on X which depends only of x in the decomposition (1.4): this means that f is a function on X such that

$$f(k \cdot m) = f(m) \quad \text{for any } m \in X, k \in K.$$

Then using the radial coordinate system, f can be viewed as a function $\widetilde{f}$ on Λ (or as a symmetric function on $\mathfrak{A}$ for the action of the Weyl group W generated by the reflections through the walls of Λ or by the adjoint action of K on $\mathfrak{A}_{\mathfrak{P}}$) and the integral of f on X is

$$(1.6) \qquad \int_X f(m)dv(m) = \text{Vol } K \int_\Lambda \widetilde{f}(x_1 \ldots x_n)v_\rho(x)dx_1 \ldots dx_n.$$

Moreover the action of the usual Laplace Beltrami operator $\Delta_2^{(X)}$ of X on such function f is exactly the action of the operator Δ_2 defined by (1.3) on $\widetilde{f}$:

$$(1.7) \qquad \widetilde{\Delta_2^{(x)}f}(x) = \Delta_2\widetilde{f}(x).$$

In particular Δ_2 is self adjoint with respect to v_ρ which is obvious by (1.6) and the formula (1.3).

Remark. In the first paragraph we have considered general positive ρ_α; in fact, for the symmetric spaces, the allowed ρ_α which come naturally from the structure of symmetric spaces are extremely special (see below for the examples that we treat).

4. *Higher order Laplace operators.* On X, one can define other Laplace operators as differential operators on X which are invariant by the isometries G. One can prove that there are p algebraically independent such operators (including the Laplace-Beltrami operator $\Delta_2^{(X)}$); because these operators are invariant by isometries of G, they transform radial functions into radial functions, and one can define for them a radial part, which is a differential operator on Λ; we denote by $\Delta_3, \ldots, \Delta_{p+1}$ these radial parts and $\Delta_3^{(X)}, \ldots, \Delta_{p+1}^{(X)}$ the corresponding operators on X.

One can prove that the higher order terms of these radial parts are polynomials in $\left(\dfrac{\partial}{\partial x_j}\right)_{j=1\ldots p}$ which are invariant by the action of the Weyl group W and conversely, any polynomial in $\left(\dfrac{\partial}{\partial x_j}\right)_{j=1\ldots p}$ which is invariant by the Weyl group W is the higher order part of the radial part of a Laplace operator in X. Moreover this polynomial defines the Laplace operator in a unique way (up to additive constants). Finally these operators commute with one another; very few things are known about these operators and it is one aim of this work to give rather explicit formulas.

5. *Weyl alcove and compact symmetric space.* a) Let $(E, \mathfrak{R}, \rho)$ be as before. Then, there is a maximal root $\alpha_{\max}$ for the order. Call C the set

(1.8) $\qquad C = \{x \in E/x \in \Lambda \quad \text{and} \quad 0 < (x, \alpha_{\max}) < \pi\}.$

C is a simplex of dimension p and is called the Weyl alcove. We can then define

$$(1.9) \qquad \hat{\nu}_\rho(x) = \prod_{\alpha \in \mathfrak{R}_+} (\sin(\alpha, x))^{\rho_\alpha}$$

and

$$(1.10) \qquad \hat{\Delta}_2 f = (\hat{\nu}_\rho(x))^{-1} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\hat{\nu}_\rho(x)\frac{\partial}{\partial x_j} f\right)$$

which is also called the Laplace-Beltrami operator for the triplet $(E, \mathfrak{R}, \rho)$.

b) If $X = G/K$ is a compact symmetric space; one can do the theory developed in 2, 3, 4. We associated to X a triplet $(E, \mathfrak{R}, \rho)$ as before and we can define the radial function and radial coordinates x, except now that x is restricted to the Weyl alcove C and that $\hat{\nu}_\rho(x)$ given by (1.9) is the volume element of X in these coordinates; in the same manner $\hat{\Delta}_2$ given by (1.10) is the radial part of the Laplace-Beltrami operator $\Delta_2^{(X)}$ of X.

6. *The Weyl group and the spheres in X.* a) Let E be a euclidean space, $\mathfrak{R}$ a root system; the Weyl group W is the group generated by reflections through the hyperplanes $(R, x) = 0$ for $R \in \mathfrak{R}$. It is also generated by the reflections through the walls of the Weyl chamber. The Weyl group permutes the set of all Weyl chambers; this means that for any $w, w' \in W$, $w\Lambda \cap w'\Lambda$ is empty if

$$w \neq w' \quad \text{and} \quad \bigcup_{w \in W} \overline{W\Lambda} = E.$$

In the case of a symmetric space, W is also the group $\mathrm{Ad}K|_{\mathfrak{A}_{\mathfrak{P}}}$. Once a Weyl chamber Λ is given, this induces an order on E by saying that $x \geqq 0$ if and only if $x \in \bar{\Lambda}$ and $x \geqq y$ if and only if $x - y \in \bar{\Lambda}$. We also say that $x > 0$ if and only if $x \in \Lambda$.

b) Let us now take a non compact symmetric space $X = G/K$. Let $m \in X$; then we can consider the orbit Km of the compact group (which is the analogue of a sphere in the euclidean case). Then the set of all Km is in bijection with the closed Weyl chamber $\bar{\Lambda}$. In the case where $X = G/K$ is a compact symmetric space, we have the same situation, but for the Weyl alcove, namely the set of all Km is in bijection with the closed Weyl alcove $\bar{C}$.

We shall call "*spheres*" the orbits of the compact group K. The set of spheres is indexed either by $\bar{\Lambda}$ or by $\bar{C}$.

c) The affine Weyl group $W^{(0)}$ is the group of affine isometries in E generated by the reflections through the walls of the Weyl alcove C; it contains the Weyl group as a subgroup, but also translations. It permutes transitively all the Weyl alcoves C, i.e., if $w, w' \in W^{(0)}$, $wC \cap w'C$ is empty when $w \neq w'$ and the wC induce a paving of E.

References. For general information and notation about symmetric spaces and root systems, see [**15**], [**17**], [**2**] (in case of complex space) and [**1**].

2. Invariant operators and their eigenfunctions for the root system A_p in the compact case $SU(p + 1)$.

1. *The root system A_p and its affine Weyl group.*

a) *The root system A_p.* Let E be the euclidean vector space of dimension p, (X, Y) its scalar product and $\mathfrak{R}$ the root system of type A_p. A basis of this system is given by the vectors $\{R_1, \ldots, R_p\}$ in E such that

$$(2.1)\qquad \begin{aligned}&(R_i, R_i) = 1\ (R_i, R_{i+1}) = -\frac{1}{2}\\ &(R_i, R_j) = 0 \quad \text{if } j \neq i - 1, i, i + 1.\end{aligned}$$

The Weyl alcove is defined here by

$$(2.2)\qquad C = \left\{X \in E/(R_i, X) > 0, 1 \leqq i \leqq p, \sum_{i=1}^{p} (R_i, X) < \pi\right\}.$$

The orthogonal symmetries with respect to the hyperplanes $\{X \in E/ (R_i, X) = 0\}$ generate the Weyl group W_p of this root system which is isomorphic to the symmetric group S_{p+1} of order $p + 1$.

b) *The affine Weyl group.* Call T_p the group generated by translations by vectors $2\pi R_i$ in E (considered as an affine space if one wants to be very formal) and let $W_p^{(0)}$ be the semi direct product of W_p and T_p; it is the affine Weyl group of A_p. By the action of $W_p^{(0)}$ into C, we obtain a paving of E by simplexes which are isometric to C

c) *Description of the paving.* Let Ω be the neighborhood of 0 in E given by

$$(2.3)\qquad \Omega = \bigcup_{\sigma \in W_p} \sigma(C).$$

We also denote $\check{R}_i \equiv 2\pi R_i$ so that

$$T_p = \sum_{i=1}^{p} \mathbf{Z}\check{R}_i.$$

Call τ_G the translation by vector G. Then the paving is

$$(2.4)\qquad E = \bigcup_{G \in T_p} \tau_G(\Omega).$$

Moreover if $G \neq G'$, then $\tau_G(\Omega) \cap \tau_{G'}(\Omega)$ is empty.

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We shall take as a basis of E the fundamental roots $\{R_1, \dots, R_p\}$; then any vector $X \in E$ has coordinates x_i on this basis namely

$$(2.5) \qquad X = \sum_{i=1}^{p} x_i R_i.$$

Let Q be the hypercube

$$(2.6) \qquad Q = \{X \in E/|x_i| < \pi \quad \text{for any } i = 1 \dots p\}.$$

Because Ω is a bounded neighborhood of the origin, there exists an integer K_0 such that

$$Q \subset \bigcup_{n_i=-K_0}^{+K_0} \tau_{\sum_{i=1}^p n_i \check{R}_i}(\Omega)$$

and we have thus a partition of Q

$$(2.7) \qquad Q = \bigcup_{n_i=-K_0}^{+K_0} \Omega'_{(n_1,\dots,n_p)}$$

with

$$\Omega'_{(n_1\dots n_p)} = Q \cap \tau_{\sum_{i=1}^p n_i \check{R}_i}(\Omega)$$

(it can happen that some Ω' are empty).

Now we can also make the action of T_p on Q to get an obvious paving of E and we also have

$$\Omega \subset \bigcup_{n_i=-K_0}^{+K_0} \tau_{\sum_{i=1}^p n_i \check{R}_i}(Q)$$

and a partition of Ω

$$(2.8) \qquad \Omega = \bigcup_{n_i=-K_0}^{+K_0} \Omega_{n_1\dots n_p}$$

where

$$\Omega_{n_1\dots n_p} = \Omega \cap \tau_{\sum_{i=1}^p n_i \check{R}_i}(Q).$$

But it is clear that

$$\Omega_{n_1\dots n_p} = \tau_{(\sum_{i=1}^p n_i \check{R}_i)}[\tau_{(-\sum_{i=1}^p n_i \check{R}_i)}(\Omega) \cap Q]$$

so that we obtain

$$(2.9) \qquad \Omega_{n_1\dots n_p} = \tau_{\sum_{i=1}^p n_i \check{R}_i}(\Omega'_{(-n_1,\dots,-n_p)}).$$

2. *A change of coordinates in E.*

a) *E as an hyperplane in* $\mathbf{R}^{p+1}$. We consider $\mathbf{R}^{p+1}$ with its euclidean structure and $q_1 \dots q_{p+1}$ the orthonormal coordinates in $\mathbf{R}^{p+1}$. Let E be the hyperplane of $\mathbf{R}^{p+1}$ with equation

$$\sum_{j=1}^{p+1} q_j = 0.$$

In this coordinate system, the roots are given by

$$(2.10) \quad (R, X) = \frac{1}{\sqrt{2}}(q_i - q_j)$$

for $X \in E$ with coordinate $(q_1 \dots q_{p+1})$, and the basis for roots is given by

$$(2.11) \quad (R_i, X) = \frac{1}{\sqrt{2}}(q_i - q_{i+1}) \quad i = 1, \dots, p.$$

We use here a normalization by $1/\sqrt{2}$ so that $\|R_i\| = 1$. Let us denote by $\tau_{i,i+1}$ the transposition of the q_i and q_{i+1}:

$$\tau_{i,i+1}(q_1, \dots, q_i, q_{i+1}, \dots, q_{p+1})$$
$$= (q_1, \dots, q_{i+1}, q_i, \dots, q_{p+1}).$$

It is clear that $\tau_{i,i+1}$ is exactly the element of the Weyl group W_p which is the reflexion through the wall $(R_i, X) = 0$

b) *Changing the coordinates from E to $\mathbf{R}^{p+1}$ from ξ_i to q_j.* Let $(e_1, \dots, e_p)$ an orthonormal basis of E and $(\xi_1, \dots, \xi_p)$ the coordinates of E in this basis. The roots in this basis can be written

$$(2.12) \quad \begin{cases} R_1 = e_1 & (R_1, X) = \xi_1 \\ R_2 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{\sqrt{2}}e_3 & (R_2, X) = -\frac{1}{2}\xi_1 - \frac{1}{2}\xi_2 + \frac{1}{\sqrt{2}}\xi_3 \\ R_3 = e_2 & (R_3, X) = \xi_2 \\ R_4 = -\frac{1}{2}e_2 - \frac{1}{2\sqrt{2}}e_3 + \frac{\sqrt{5}}{2\sqrt{2}}e_4 & (R_4, X) = -\frac{1}{2}\xi_2 - \frac{1}{2\sqrt{2}}\xi_3 + \frac{\sqrt{5}}{2\sqrt{2}}\xi_4 \\ R_5 = -\frac{\sqrt{2}}{\sqrt{5}}e_4 + \frac{\sqrt{3}}{\sqrt{5}}e_5 & \\ \text{and for } 5 \leqq k \leqq p & \\ R_k = -\sqrt{\frac{k-1}{2k}}e_{k-1} + \sqrt{\frac{k+1}{2k}}e_k & (R_k, X) = -\sqrt{\frac{k-1}{2k}}\xi_{k-1} + \sqrt{\frac{k+1}{2k}}\xi_{k+1} \end{cases}$$

Because $q_i - q_{i+1} = \sqrt{2}(R_i, X)$ we have the matrix changing the ξ_i's coordinates into the q_i's coordinates given by the matrix on the next page.

It is easy to check that

$$\sum_{i=1}^{p+1} q_i \equiv 0.$$

If $\epsilon_1 \dots \epsilon_{p+1}$ is the canonical basis of $\mathbf{R}^{p+1}$ (with respect to which the coordinates are $q_1 \dots q_{p+1}$), we have

$$
\begin{bmatrix} q_1\\ q_2\\ q_3\\ q_4\\ q_5\\ \vdots\\ q_k\\ q_{k+1}\\ \vdots\\ \\ q_p\\ q_{p+1}\end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & & & e_k & e_{k+1} & & e_p \end{array}\\
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2\sqrt{5}} & \cdots & \cdots & \frac{1}{\sqrt{k(k+1)}} & \frac{1}{\sqrt{(k+1)(k+2)}} & & \frac{1}{\sqrt{p(p+1)}}\\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{2\sqrt{5}} & \cdots & \cdots & \frac{1}{\sqrt{k(k+1)}} & \vdots & & \frac{1}{\sqrt{p(p+1)}}\\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{5}} & \cdots & \cdots & \frac{1}{\sqrt{k(k+1)}} & \vdots & & \vdots\\
0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{5}} & \cdots & \cdots & \frac{1}{\sqrt{k(k+1)}} & \cdots & \cdots & \vdots\\
0 & 0 & 0 & -\frac{2}{\sqrt{5}} & \cdots & \cdots & \frac{1}{\sqrt{k(k+1)}} & \cdots & \cdots & \vdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & \cdots\ 0 & -\sqrt{\frac{k-1}{k}} & \frac{1}{\sqrt{k(k+1)}} & \sqrt{\frac{1}{(k+1)(k+2)}} & \cdots & \sqrt{\frac{1}{p(p+1)}}\\
0 & 0 & 0 & 0 & & 0 & -\sqrt{\frac{k}{k+1}} & \sqrt{\frac{1}{(k+1)(k+2)}} & \cdots & \sqrt{\frac{1}{p(p+1)}}\\
0 & 0 & 0 & 0 & & \vdots & 0 & -\sqrt{\frac{k+1}{k+2}} & \cdots & \vdots\\
 & & & & & \vdots & \vdots & 0 & & \vdots\\
0 & 0 & 0 & 0 & & \vdots & \vdots & \vdots & & \sqrt{\frac{1}{p(p+1)}}\\
0 & 0 & 0 & 0 & & 0 & \cdots & \cdots & \cdots & -\sqrt{\frac{p}{p+1}}
\end{bmatrix}
\end{array}
\begin{bmatrix} \xi_1\\ \xi_2\\ \vdots\\ \vdots\\ \xi_p\end{bmatrix}
$$

$$
(2.13)\qquad
\begin{aligned}
e_1 &= \frac{1}{\sqrt{2}}\epsilon_1 - \frac{1}{\sqrt{2}}\epsilon_2\\
e_2 &= \frac{1}{\sqrt{2}}\epsilon_3 - \frac{1}{\sqrt{2}}\epsilon_4\\
e_3 &= \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 - \frac{1}{2}\epsilon_3 - \frac{1}{2}\epsilon_4
\end{aligned}
$$

and for $4 \leqq k \leqq p$

$$
(2.13)\qquad e_k = \frac{1}{\sqrt{k(k+1)}}\sum_{l=1}^{k}\epsilon_l - \sqrt{\frac{k}{k+1}}\epsilon_{k+1}.
$$

We shall denote the preceding matrix relation by

$$
(2.14)\qquad (q) = M(\xi)
$$

where M is defined above.

c) *Changing the coordinate from* ξ *to* x; recall also that in E, there is a third coordinate system

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$$X = \sum_{i=1}^{p} x_i R_i.$$

Define

(2.15) $\quad (\xi) = A(x).$

Then A is computed in (2.12). We shall need A^{-1}:

$$(2.16) \qquad A^{-1} = \left[\begin{array}{ccccccccc}
1 & 0 & (\sqrt{2})^{-1} & \sqrt{2}(\sqrt{4.5})^{-1} & \cdots & & \sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & \sqrt{2}(\sqrt{p(p+1)})^{-1} \\
0 & 0 & 2(\sqrt{2})^{-1} & 2\sqrt{2}(\sqrt{4.5})^{-1} & \cdots & & 2\sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & 2\sqrt{2}(\sqrt{p(p+1)})^{-1} \\
0 & 1 & (\sqrt{2})^{-1} & 3\sqrt{2}(\sqrt{4.5})^{-1} & \cdots & & 3\sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & 3\sqrt{2}(\sqrt{p(p+1)})^{-1} \\
0 & 0 & 0 & 4\sqrt{2}(\sqrt{4.5})^{-1} & \cdots & & 4\sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & & 5\sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & k\sqrt{2}(\sqrt{k(k+1)})^{-1)} & \cdots & k\sqrt{2}(\sqrt{p(p+1)})^{-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p\sqrt{2}(\sqrt{p(p+1)})^{-1}
\end{array}\right]$$

d) *Computation of differentials*. Denote

$${}^t(\partial q) = \left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_{p+1}}\right)$$

$${}^t(\partial \xi) = \left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_p}\right)$$

$${}^t(\partial x) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p}\right).$$

Then we have

$$\frac{\partial}{\partial q_i} = \sum_j \frac{\partial}{\partial \xi_j} \frac{\partial \xi_j}{\partial q_i}.$$

But

$$X = \sum_{i=1}^{p+1} q_i \epsilon_i = \sum_{j=1}^{p} \xi_j e_j,$$

so that

$$\xi_j = \sum_{i=1}^{p+1} q_i (\epsilon_i, e_j)$$

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and

$$\frac{\partial \xi_j}{\partial q_i} = (\epsilon_i, e_j)$$

and (ϵ_i, e_j) is given by the element of the j^{th} column and i^{th} line of M, so that

(2.17) $(\partial q) = M(\partial \xi)$.

Moreover from (2.15) we have

(2.18) $(\partial \xi) = {}^t A^{-1}(\partial x)$

and combining (2.17) and (2.18) we obtain

$$(\partial q) = M\,{}^tA^{-1}(\partial x).$$

A routine computation gives us the matrix $M\,{}^tA^{-1}$ and the expression of vector fields by Lemma 1.

LEMMA 1. *For* $1 \leqq r \leqq p + 1$, *we have the following formula*

$$(2.19) \quad \frac{\partial}{\partial q_r} = \frac{\sqrt{2}}{p+1}\left(\sum_{k=r}^{p} (p + 1 - k)\frac{\partial}{\partial x_k} - \sum_{l=1}^{r-1} l\frac{\partial}{\partial x_l}\right).$$

3. *The L^2 space of the Weyl alcove.*

a) *Symmetry and antisymmetry.* Let $f:E \to \mathbf{C}$ be a complex function; we define the action of the affine Weyl group $W_p^{(0)}$ on f by

(2.20) $(\sigma f)(X) = f(\sigma^{-1}(X))$ for $X \in E$.

We say that f is invariant if

$$\sigma f = f \quad \text{for any } \sigma \in W_p^{(0)}$$

and that it is antiinvariant if

$$\sigma f = (-1)^\sigma f \quad \text{for any } \sigma \in W_p^{(0)}$$

if $(-1)^\sigma$ is the determinant of σ.

We begin to work in $(x_1, \dots, x_p)$ coordinates.

LEMMA 2. *$f(x_1, \dots, x_p)$ is invariant by the affine group if and only if f is periodic of period 2π in each x_i and if*

$$\sigma_i f = f \quad \textit{for } i = 1, \dots, p$$

where σ_i is the reflexion through the wall $(R_i, X) = 0$. f is antiinvariant by $W_p^{(0)}$ if and only if f is periodic of period 2π in the x_i and

$$\sigma_i f = -f \quad \textit{for } i = 1, \dots, p.$$

The proof is obvious because the translation by $2\pi R_i$ and the reflexion σ_j

generate $W_p^{(0)}$ and precisely we consider the coordinate system of the x_i in the basis of R_i. Moreover it is clear that

$$(-1)^{\sigma_i} = -1.$$

b) *Antisymmetrisation operator A*. If f is a periodic function, of period 2π in the x_i, we define

$$(2.21) \qquad (Af)(x_1, \dots, x_p) = \sum_{\sigma \in W_p} (-1)^{\sigma} f(\sigma(x_1 \dots x_p)).$$

It is clear that Af is an antiinvariant function by $W_p^{(0)}$. If f is already antiinvariant $Af = (p + 1)!f$ and so $A^2 = (p + 1)!A$.

c) *The $f_{(\mathbf{n})}$ function*. Define $(\mathbf{n}) = (n_1, \dots, n_p)$ where n_j are integers and

$$(2.22) \qquad f_{(\mathbf{n})}(x_1, \dots, x_p) = \exp\left(i \sum_{j=1}^{n} n_j x_j\right).$$

LEMMA 3. *If one of the n_k is 0, then $Af_{(\mathbf{n})} = 0$.*

Proof. If $n_k = 0$, let $W_p(k)$ be the subgroup generated by σ_k so that it is $\{I, \sigma_k\}$. Then

$$W_p = \bigcup_{g \in G} W_p(k) \cdot g$$

for a suitable subset $G \subset W_p$. Then

$$(Af_{(n)})(x_1 \dots x_p) = \sum_{\sigma \in W} (-1)^{\sigma} e^{i(\sigma \mathbf{n}, X)}$$

$$= \sum_{g \in G} (-1)^{q} (e^{i(gX, Id\mathbf{n})} - e^{i(gX, \sigma_k \mathbf{n})}).$$

But $\sigma_k \mathbf{n} = \mathbf{n} = Id\mathbf{n}$.

LEMMA 4. *The functions $Af_{(\mathbf{n})}|_C$ (where $n_k > 0$ for any k) give a complete orthogonal system on $L^2(C, dx_1 \dots dx_p)$ and*

$$(2.23) \qquad \|Af_{(\mathbf{n})}\|_{L^2(C,dx)} = (2\pi)^{p/2}.$$

Proof. We take a function $f_{(\mathbf{m})}$ and recall the definitions of $\Omega_{(\mathbf{n})}$ and $\Omega'_{(\mathbf{n})}$ given in (2.7) and (2.8). Because of (2.9) and because $f_{(\mathbf{m})}\overline{f}_{(\mathbf{m}')}$ is invariant by the translation $\sum n_i \check{R}_i$, we have

$$\int_{\Omega_{(\mathbf{n})}} \overline{f}_{(\mathbf{m}')} f_{(\mathbf{m})}(x) dx = \int_{\Omega'_{(\mathbf{n})}} \overline{f}_{(\mathbf{m}')} f_{(\mathbf{m})}(x) dx$$

and so by (2.7), (2.8) and (2.9)

$$(2.24) \qquad \int_{\Omega} \overline{f}_{(\mathbf{m}')} f_{(\mathbf{m})} dx = \int_{Q} \overline{f}_{(\mathbf{m}')} f_{(\mathbf{m})} dx.$$

But now Q being the hypercube (2.6) in the x_i coordinates, the second member of this last relation is 0 if $(\mathbf{m}') - (\mathbf{m}) \neq (\mathbf{0})$ and so the $f_{(\mathbf{m})}$ are orthogonal on Ω.

Let $\mathfrak{H}$ be the Hilbert space of the 2π-periodic functions in the x_i which are square integrable on Ω with the L^2 norm on Ω and $A\mathfrak{H}$ the subspace of antiinvariant functions (which is meaningful because of the definition (2.3) of Ω

$$\Omega = \bigcup_{\sigma \in W} \sigma(C)).$$

If g is a function defined on C, it has a unique antiinvariant extension in E.

It is clear that if g is antiinvariant, we have

$$\int_{\sigma(C)} g(x)\bar{g}(x)dx = \int_C g(x)\bar{g}(x)dx$$

for any $\sigma \in W_p$ and so the mapping

$$(2.25) \qquad g \in L^2(C, dx) \rightarrow \frac{Ag}{\sqrt{(p+1)!}} \in A\mathfrak{H}$$

gives an isometry between $L^2(C, dx)$ and $A\mathfrak{H}$. It is also clear that if f, g are in $\mathfrak{H}$

$$(2.26) \qquad (Af|g)_{L^2(\Omega)} = (Af|Ag)_{L^2(C)} = (f|Ag)_{L^2(\Omega)}$$

because by definition of Ω

$$(Af|g)_{L^2(\Omega)} = \int_\Omega Af(x)\overline{g(x)}dx = \sum_{\sigma \in W} \int_{\sigma(C)} Af(x)\overline{g(x)}dx$$

and because Af is antiinvariant

$$= \sum_{\sigma \in W} (-1)^\sigma \int_C Af(x')\overline{g(\sigma x')}dx' = (Af|Ag)_{L^2(C)}.$$

Let now $\mathfrak{H}_{(\mathbf{n})}$ be the subspace of $\mathfrak{H}$ generated by the function $\sigma f_{(\mathbf{n})}$ where $\sigma \in W_p$; it is clear that

$$\sigma f_{(\mathbf{n})} = f_{t_{\sigma(\mathbf{n})}}.$$

It is also clear by (2.24) that

$$\mathfrak{H}_{(\mathbf{n})} \cap \mathfrak{H}_{(\mathbf{n}')} = \{0\} \quad \text{if } n_i, n_i' \geqq 0 \text{ and } (\mathbf{n}) \neq (\mathbf{n}').$$

We shall write $(\mathbf{n}) \geqq 0$ if all $n_i \geqq 0$. It is also clear that for any $(\mathbf{n}_i) \in \mathbf{Z}^p$, there exists a unique $(\mathbf{n}') \geqq 0$ such that

$$f_{(\mathbf{n})} \in \mathfrak{H}_{(\mathbf{n}')}$$

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because $(\mathbf{n}) = \sigma(\mathbf{n}')$ for some $\sigma \in W_p$. So we can deduce that we have an orthogonal decomposition

$$\mathfrak{H} = \sum_{(\mathbf{n}) \geqq 0} \mathfrak{H}_{(\mathbf{n})}$$

(2.27) $$A\mathfrak{H} = \sum_{(\mathbf{n}) \geqq 0} A\mathfrak{H}_{(\mathbf{n})}$$

and $A\mathfrak{H}_{(\mathbf{n})}$ is generated by the only function $Af_{(\mathbf{n})}$ which is 0 if one of the n_k is 0 and is not 0 if all n_k are not 0. Moreover using (2.26) and because $A^2 = (p + 1)!A$, we have

$$(Af_{(\mathbf{n})}|Af_{(\mathbf{n}')})_{L^2(\Omega)} = (p + 1)!(Af_{(\mathbf{n})}|Af_{(\mathbf{n}')})_{L^2(C)}$$

so by (2.26)

$$= (p + 1)!(f_{(\mathbf{n})}|A^2 f_{(\mathbf{n}')})_{L^2(\Omega)}$$

and because $A^2 = (p + 1)!A$

$$= ((p + 1)!)^2 (f_{(\mathbf{n})}|Af_{(\mathbf{n}')})_{L^2(\Omega)}$$

$$= ((p + 1)!)^2 \sum_{\sigma \in W_p} (-1)^{\sigma} (f_{(\mathbf{n})}|f_{{}^t\sigma(\mathbf{n}')})_{L^2(\Omega)}.$$

But if $(\mathbf{n}) \neq (\mathbf{n}')$, for all $\sigma \in W_p$, we have ${}^t\sigma(\mathbf{n}') \neq (\mathbf{n})$ ($(\mathbf{n}) > 0$, $(\mathbf{n}') > 0$, so that σ change the "order") and so the

$$(f_{(\mathbf{n})}|f_{{}^t\sigma(\mathbf{n}')})_{L^2(\Omega)} = 0$$

because of (2.24).

This means that the $Af_{(\mathbf{n})}$ for $(\mathbf{n}) > 0$ are a complete orthogonal system in A. Moreover by (2.26)

$$\|Af_{(\mathbf{n})}\|^2_{L^2(C)} = (Af_{(\mathbf{n})}|f_{(\mathbf{n})})_{L^2(\Omega)}$$

$$= \sum_{\sigma \in W_p} (-1)^{\sigma} (f_{{}^t\sigma(\mathbf{n})}|f_{(\mathbf{n})})_{L^2(\Omega)}.$$

But ${}^t\sigma(\mathbf{n})$ is in a different Weyl chamber of that of $(\mathbf{n})$ (i.e., at least one of the $({}^t\sigma(\mathbf{n}))_k$ is < 0), so that ${}^t\sigma(\mathbf{n}) = (\mathbf{n})$ if and only if $\sigma = \mathrm{Id}$ and so the integral is $(2\pi)^p \delta_{\sigma,\mathrm{Id}}$ (using (2.24)), and

$$\|Af_{(\mathbf{n})}\|^2_{L^2(C)} = (2\pi)^p.$$

As a corollary, we can prove.

LEMMA 5. *Let*

$$f(x) = \sum_{(\mathbf{n})} a_{(\mathbf{n})} f_{(\mathbf{n})}(x)$$

be a trigonometric polynomial ($a_{(\mathbf{n})} \in \mathbf{C}$). *If f is antiinvariant we have*

$$f(x) = \sum_{(\mathbf{n})>0} a_{(\mathbf{n})}(Af_{(\mathbf{n})})(x).$$

Proof. Because f is antiinvariant, we have

$$Af = (p + 1)!f$$

and also

$$\sigma f = (-1)^{\sigma} f$$

so that

$$a_{t_{\sigma(\mathbf{n})}} = (-1)^{\sigma} a_{(\mathbf{n})}.$$

But also by Lemma 4, f can be decomposed on the basis of the $Af_{(\mathbf{n})}$ with $(\mathbf{n}) > 0$ which gives

$$f = \sum_{(\mathbf{n})>0} b_{(\mathbf{n})} Af_{(\mathbf{n})}.$$

But

$$\begin{aligned} Af &= \sum_{(\mathbf{n})} a_{(\mathbf{n})} Af_{(\mathbf{n})} \\ &= \sum_{(\mathbf{n})>0} \left(\sum_{\sigma \in W} (-1)^{\sigma} a_{t_{\sigma(\mathbf{n})}} \right) Af_{(\mathbf{n})} \\ &= (p + 1)! \sum_{(\mathbf{n})>0} a_{(\mathbf{n})} Af_{(\mathbf{n})} \end{aligned}$$

and this is equal also to $(p + 1)!f$, so $b_{(\mathbf{n})} = a_{(\mathbf{n})}$ for $(\mathbf{n}) > 0$.

4. *Differential operators with constant coefficients on E.*

a) *Action of the affine Weyl group.* If P is a differential operator of order k on E, and if σ is an element of the affine Weyl group $W_p^{(0)}$, then σ acts on P by an action denoted $d\sigma$

$$(2.28) \qquad d\sigma(P)(f) = \sigma(P(\sigma^{-1}f))$$

where σ acts on f by (2.20). P is said to be *invariant* if

$$d\sigma(P) = P \quad \text{for any } \sigma \in W_p^{(0)}$$

and *antiinvariant* if

$$d\sigma(P) = (-1)^{\sigma} P \quad \text{for any } \sigma \in W_p^{(0)}.$$

To investigate the invariant or antiinvariant differential operators, it is better to use the coordinates $q_1 \dots q_{p+1}$ described in n° 2, although the coordinates $x_1, \dots, x_p$ were better suited to investigate functions (because these coordinates are adapted to the translational part of the affine Weyl group).

b) *Invariant operators and their action on antiinvariant function.*

LEMMA 6. *Let $P(Y_1 \dots Y_{p+1})$ be a symmetric polynomial in the indeterminates $Y_1 \dots Y_{p+1}$. Then*

$$P\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right)$$

is invariant by the Weyl group W_p.

Proof. The Weyl group W_p is generated by the reflections σ_i in the walls $(R_i, X) = 0$. But we have seen in n° 2 that this reflection is the transposition exchanging q_i and q_{i+1}; this means that W_p is the symmetric group S_{p+1} acting on the q_i's and the lemma is proved.

LEMMA 7. *Let $P(Y_1, \dots, Y_{p+1})$ be a symmetric polynomial in the Y_i. Then, for* $(\mathbf{n}) > 0$

$$P\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right)Af_{(\mathbf{n})} = P(\rho_1, \dots, \rho_{p+1})Af_{(\mathbf{n})} \tag{2.29}$$

where

$$\rho_k((\mathbf{n})) = \frac{i\sqrt{2}}{p+1}\left(\sum_{l=k}^{p} (p+1-l)n_l - \sum_{l=1}^{k-1} ln_l\right). \tag{2.30}$$

Proof. P being invariant by W_p and $Af_{(\mathbf{n})}$ being antiinvariant, it is clear that

$$P\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right)Af_{(\mathbf{n})}$$

is a trigonometric polynomial which is antiinvariant by W_p. Moreover by Lemma 1, (2.19), it is obvious that

$$\frac{\partial}{\partial q_r} f_{(\mathbf{n})} = \rho_r((\mathbf{n}))f_{(\mathbf{n})}$$

so that

$$P\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right)f_{(\mathbf{n})} = P(\rho_1((\mathbf{n})) \dots \rho_{p+1}((\mathbf{n})))f_{(\mathbf{n})}.$$

Now Lemma 5 implies the result.

c) *Example: the usual Laplacian.* Let

$$P(Y_1 \dots Y_{p+1}) = \sum_{k=1}^{p+1} Y_k^2.$$

Because the basis of the e_j is orthonormal on E, we have

$$(2.31)\qquad \sum_{k=1}^{p+1} \frac{\partial^2}{\partial q_k^2} = \sum_{j=1}^{p} \frac{\partial^2}{\partial \xi_j^2} \equiv \Delta.$$

In fact we have seen in n° 2 that

$$\frac{\partial}{\partial q_k} = \sum_j \frac{\partial}{\partial \xi_j}(\epsilon_k, e_j)$$

so that

$$\sum_k \frac{\partial^2}{\partial q_k^2} = \sum_{k,j,l} \frac{\partial^2}{\partial \xi_j \partial \xi_l}(\epsilon_k, e_j)(\epsilon_k, e_l) = \sum \frac{\partial^2}{\partial \xi_j^2}.$$

We obtain that in coordinates $(x_1, \dots, x_p)$ the Laplacian is

$$(2.32)\qquad \Delta = \frac{2}{p+1}\left(\sum_{r=1}^{p} r(p+1-r)\frac{\partial^2}{\partial x_r^2} + 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)\frac{\partial^2}{\partial x_r \partial x_l}\right)$$

and that $Af_{(\mathbf{n})}$ is an eigenfunction of Δ of eigenvalue

$$(2.33)\qquad \lambda^{(0)}_{(\mathbf{n})} = -\frac{2}{p+1}\left(\sum_{r=1}^{p} r(p+1-r)n_r^2 + 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)n_r n_l\right).$$

5. *Differential operators in radial coordinates for $SU(p + 1)$.*
a) *The function $\mathfrak{F}(X)$.* Let

$$\mathfrak{F}(X) = Af_{(1,1,\dots,1)}.$$

($(1, \dots, 1)$ corresponds to the smallest element $(\mathbf{n})$ in the Weyl chamber which is not on one of the wall). It is easy to see that

$$(2.34)\qquad \mathfrak{F}(X) = \bigcap_{\substack{R \in A_p \\ R > 0}} (e^{i(R,X)} - e^{-i(R,X)})$$

because if we denote by $\psi(X)$ the second member of this equality, then

$$\psi(\sigma X) = \bigcap_{\substack{R \in A_p \\ R > 0}} (e^{i(\sigma R,X)} - e^{-i(\sigma R,X)})$$

but σR is a positive root if and only if $(-1)^\sigma = 1$ so that

$$\psi(\sigma X) = (-1)^\sigma \psi(X);$$

then ψ is antiinvariant. Moreover let $2\rho(X)$ be the sum of the positive roots:

(2.35) $$2\rho(X) = \bigcap_{\substack{R \in A_p \\ R>0}} R(X) = x_1 + \ldots + x_p.$$

Then

$$\psi(X) = f_{(1,\ldots,1)}(X) \bigcap_{R>0} (1 - e^{-2i(R,X)})$$

and if we use Lemma 5, we see that

$$\psi = Af_{(1,\ldots,1)}.$$

b) *The differential operators Q_p and their eigenfunctions.* We denote

(2.36) $$p_{(\mathbf{n})}(X) = (2i)^{p(p+1)/2} \frac{Af_{(\mathbf{n}+\mathbf{1})}(X)}{Af_{(\mathbf{1})}(X)}$$

where $(\mathbf{1}) = (1, 1, \ldots, 1)$ and

$$(\mathbf{n} + \mathbf{1}) = (n_1 + 1, \ldots, n_p + 1) \quad \text{and} \quad (\mathbf{n}) > 0.$$

These are trigonometric polynomials in $x_1, \ldots, x_p$ which are invariant by W_p^0. Moreover they are orthogonal in

$$L^2\left(C, \frac{\Im^2(x)dx}{(2i)^{p(p+1)}}\right)$$

because

$$\int_C p_{(\mathbf{n})}\bar{p}_{(\mathbf{n}')}\frac{\Im^2(x)dx}{(2i)^{p(p+1)}} = \int_C Af_{(\mathbf{n}+\mathbf{1})}\overline{Af_{(\mathbf{n}'+\mathbf{1})}}dx = \delta_{(\mathbf{n}),(\mathbf{n}')}(2\pi)^p$$

(Lemma 4 and (2.23)).

Then we define the following differential operators. If P is a symmetric polynomial in $Y_1, \ldots, Y_{p+1}$

(2.37) $$Q_p\left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_{p+1}}\right) = \frac{1}{\Im} P\left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_{p+1}}\right)\Im - P(p_1(\mathbf{1}), \ldots, p_{p+1}(\mathbf{1})).$$

Then we obtain the following theorem.

THEOREM 1. *The operators Q_p are invariant by the affine Weyl group $W_p^{(0)}$, and they have the following properties.*

(i) Q_p *(constant)* $= 0$

(ii) $Q_p p_{(\mathbf{n})} = {}^{(P)}\lambda_{(\mathbf{n})} p_{(\mathbf{n})}$ *where*

$$(2.38)\quad {}^{(P)}\lambda_{(\mathbf{n})} = P(\rho_1(\mathbf{n} + \mathbf{1}), \dots, \rho_{p+1}(\mathbf{n} + \mathbf{1})) - P(\rho_1(\mathbf{1}), \dots, \rho_{p+1}(\mathbf{1})).$$

(iii) *They commute with each other.*

Proof. It is clear by the fact that $\mathfrak{F} = Af_{(\mathbf{1})}$ and by Lemma 7. Moreover because the $P_{(\mathbf{n})}$ is a complete orthogonal system in the space $L^2(C, \mathfrak{F}^2(x)dx)$, and because they are joint eigenfunctions of all the Q_p, these operators commute with each other.

c) Particular case: the Laplace operator. We take for $P = \Delta$ as in (2.31). Then $p_{(\mathbf{n})}$ is an eigenfunction of Q_Δ with eigenvalue

$$(2.39)\quad {}^{(\Delta)}\lambda_{(\mathbf{n})} = \lambda^{(0)}_{(\mathbf{n}+\mathbf{1})} - \lambda^{(0)}_{(\mathbf{1})} \quad (\lambda^{(0)}_{(\mathbf{n}+\mathbf{1})} \text{ given by (2.33)}).$$

Taking into account the explicit form (2.33) of $\lambda^{(0)}_{(\mathbf{n})}$ we obtain

$$\begin{aligned}
{}^{(\Delta)}\lambda_{(\mathbf{n})} = -\frac{2}{p+1}\Bigg[&\sum_{r=1}^{p} r(p+1-r)n_r^2 \\
&+ 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)n_r n_l + 2\sum_{r=1}^{p} r(p+1-r)n_r \\
&+ 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)(n_r + n_l)\Bigg] \\
= -\frac{2}{p+1}\Bigg(&\sum_{r=1}^{p} r(p+1-r)n_r^2 \\
&+ 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)n_r n_l\Bigg) - 2\sum_{r=1}^{p} r(p+1-r)n_r.
\end{aligned}$$

A direct computation gives also

$$(2.40)\quad \lambda^{(0)}_{(\mathbf{1})} = -\frac{2}{p+1}\left(\sum_{r=1}^{p} r(p+1-r) + 2\sum_{1 \leqq r < l \leqq p} r(p+1-l)\right)$$

$$\lambda^{(0)}_{(\mathbf{1})} = -\frac{p(p+1)(p+2)}{6}.$$

6. *The group $SU(p+1)$ as a symmetric space.* We can consider $SU(p+1) = M$ as the symmetric space $M = G/K$ where

$$G = SU(p+1) \times SU(p+1)$$

and K is the diagonal identified to $SU(p+1)$; G acts on $M = SU(p+1)$ by

$$(g_1, g_2) \cdot m = g_1^{-1} m g_2.$$

Any $m \in M$ has $p + 1$ eigenvalues $e^{i\varphi_k}$ $1 \leqq k \leqq p + 1$ with

$$\sum_{k=1}^{p+1} \varphi_k = 0.$$

At the level of Lie algebras we have the Cartan decomposition

$$\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}.$$

The maximal abelian subalgebra in $\mathfrak{P}$ is the matrices which are diagonal of trace 0. Let $q_i/\sqrt{2}$ be the diagonal elements. The roots are

$$R(q) = \frac{q_i - q_j}{\sqrt{2}}$$

and they have multiplicity 2; we are exactly in the situation of n° 1 and 2. Moreover

$$\frac{\mathfrak{I}^2(x)}{(2i)^{p(p+1)}} dx_1 \dots dx_n$$

is exactly $d\hat{v}(x)$ defined in (1.9) with the $\rho_\alpha = 2$. By [**2**], one knows that the radial part (in the Cartan decomposition) of the invariant operators are exactly the Q_p described in n° 5 and Theorem 1 describes exactly the radial eigenfunctions and their eigenvalues.

We could have also obtained the same result abstractly because we know that the radial eigenfunctions are central functions on $SU(p + 1)$; but the characters are a basis of central function and are given by Weyl [**32**]; they are just the $p_{(\mathbf{n})}$; but the invariant operators on M are the operators which are biinvariant on $SU(p + 1)$; so the $p_{(\mathbf{n})}$ are eigenfunctions of these operators. Here we have obtained, by our explicit procedures, the eigenvalues which will be useful in the next section.

References for Section 2. The study of eigenvalues and eigenfunctions on $SU(3)$ has been done by Koornwinder [**18**, **19**]. For general multiplicities of the roots, it was done completely in [**6**]. See also [**32**] for another presentation without using root systems.

3. The heat kernels on $SU(p + 1)$ and $SL(p + 1, \mathbf{C})/SU(p + 1)$.

1. *The heat kernel on symmetric spaces.* Let $X = G/K$ be a symmetric space (compact or not compact). We can consider the heat equation

$$(3.1) \qquad \begin{cases} \dfrac{\partial f}{f} = \Delta_2^{(X)} f \\ f|_{t=0} = f_0 \end{cases}$$

where f_0 is a given function (say C^0 with compact support) and $\Delta_2^{(X)}$ is the Laplace-Beltrami operator. This solution can be written as

$$f(t, m) = (P_t f_0)(m) = \int_X p_t(m, m') f_0(m') dv(m')$$

where $p_t(m, m')dv(m')$ is the heat kernel, $dv(m')$ being the volume element of X. It is clear that $p_t(m, m') = p_t(m', m)$ because $\Delta_2^{(X)}$ is self-adjoint with respect to $dv(m')$ and

$$(3.1)' \quad \begin{cases} \dfrac{\partial p_t(m, m')}{t} = \Delta_{2,m}^{(X)} p_t(m, m') \\ \\ p_t(m, m')dv(m') \to \delta_m(m') \quad \text{if } t \to 0 \end{cases}$$

where δ_m is the Dirac mass at point m.

Moreover let g be an element of G such that $m = g \cdot 0$ (0 being a chosen origin in X); the heat equation is invariant by action of G, so that

$$(3.2) \quad \int p_t(g \cdot 0, m') f_0(m') dv(m') = \int p_t(0, m'') f_0(g_0 \cdot m'') dv(m'').$$

But $p_t(0, m'')$ is invariant also by the action of $k \in K$, so that finally $p_t(0, m'')$ depends only on the radial part of m'' and we can write

$$(3.3) \quad p_t(m, m') = q_t(0, x)$$

$x \in \mathfrak{A}_p$ being the radial coordinates of $g^{-1} \cdot m'$ where g is such that $g \cdot 0 = m$.

In the case of a non compact symmetric spaces, (3.2) becomes

$$(3.4) \quad f(t, g \cdot 0) = \int_\wedge q_t(0, x) \left(\int_K f_0(g \cdot ke^x 0) dk \right) v_p(x) dx$$

where $v_p(x)$ is defined in (1.2) (and is the volume element in radial coordinate), and in the case of a compact symmetric space

$$(3.4)' \quad f(t, g \cdot 0) = \int_C q_t(0, x) \left(\int_K f_0(gke^x \cdot 0) dk \right) \hat{v}_p(x) dx$$

where $\hat{v}_p(x)$ is defined in (1.9).

In both cases dk is the invariant measure on K of mass 1. Now $q_t(0, x)$ is a function on $\mathfrak{A}_p$ which is naturally $p_t(0, ke^x \cdot 0)$; so it is invariant with respect to the action of the Weyl group in the non-compact case. In the compact case it will be invariant by the action of the affine Weyl group.

Moreover if $t \to 0^+$, we have by (3.1)′ that

$$q_t(0, x) v_p(x) \, dx \to \delta_0(x).$$

Finally, $p_t(m, m')$ is symmetric in (m, m') and satisfies the heat equation with respect to both variables, so that $q_t(0, x)$ satisfies the heat equation for the radial operators Δ_2 or $\hat{\Delta}_2$ (defined by (1.3) or (1.10)). Our problem is to find $q_t(0, x)$ in the non compact case with the properties

$$\text{(3.5)}\qquad \begin{aligned} &1)\ q_t(0, x) \text{ is invariant on } \mathfrak{A}_{\mathfrak{P}} \text{ by the Weyl group } W \\ &2)\ \frac{\partial q_t}{\partial t} = \Delta_2^{(X)} q_t \quad \text{on } \Lambda \\ &3)\ q_t(0, x) v_p(x) dx \to \delta_0(x) \quad \text{if } t \to 0^+ \text{ on } \Lambda \end{aligned}$$

and in the non compact case

$$\text{(3.5)}'\qquad \begin{aligned} &1)\ q_t(0, x) \text{ is invariant on } \mathfrak{A}_{\mathfrak{P}} \text{ by the affine Weyl group } W^{(0)} \\ &2)\ \frac{\partial q_t}{\partial t} = \hat{\Delta}_2^{(X)} q_t \\ &3)\ q_t(0, x) \hat{v}_p(x) dx \to \delta_0(x) \quad \text{if } t \to 0^+ \text{ on } C. \end{aligned}$$

In the non compact case, $q_t(0, x)v_p(x)dx$ is a kernel acting on functions on Λ (or on functions on $\mathfrak{A}_{\mathfrak{P}}$ invariant by W), and in the compact case, it acts on functions on C (or an functions on $\mathfrak{A}_{\mathfrak{P}}$ invariant by $W^{(0)}$).

2. *The heat kernel on the Weyl chamber or alcove.* We can generalize the problems (3.5) or (3.5)′ by asking for a kernel $q_t(x, x')v_p(x')dx'$ or $\hat{q}_t(x, x')\hat{v}_\rho(x')dx'$ such that

$$f(t, x) = \int_\Lambda q_t(x, x') f_0(x') v_p(x') dx' \quad \text{or}$$

$$f(t, x) = \int_C \hat{q}_t(x, x') f_0(x') \hat{v}_\rho(x') dx'$$

solves

$$\frac{\partial f}{\partial t} = \Delta_2 f \quad \text{or} \quad \frac{\partial f}{\partial t} = \hat{\Delta}_2 f$$

and f is invariant by action of W or $W^{(0)}$ respectively and

$$f(t, x) \to f_0(x) \quad \text{if } t \to 0.$$

Because Δ_2 and $\hat{\Delta}_2$ are self adjoint with respect to v_p and $\hat{v}_\rho$, we see that q_t or $\hat{q}_t$ are symmetric and we can ask the problem of finding

$$\text{(3.6)}\qquad \left\{ \begin{aligned} &1)\ q_t(x, x') \text{ symmetric in } (x, x'), \text{ invariant in both variables by} \\ &\quad \text{the Weyl group } W \\ &2)\ \frac{\partial q_t}{\partial t} = \Delta_2 q_t \\ &3)\ q_t(x, x') v_p(x') dx' \to \delta_x(x') \text{ if } t \to 0^+ \text{ on } \Lambda. \end{aligned} \right.$$

(Or the analogous problem (3.6)′ in the compact case.)

Remark. The interpretation of this problem in the symmetric space is that we start at $t = 0$ with the uniform mass on the sphere $K \cdot e^x \cdot 0$ and we let this uniform measure diffuse by heat diffusion until time t; then $q_t(x, x')v_p(x')$ is the fraction of this unit mass uniformly distributed on the sphere $Ke^{x'}0$ at time t.

Now, in general, we shall obtain a formula for $q_t(x, x')$ for x, x' in (or in C), but we want to obtain $q_t(0, x')$ (recall that 0 is not in Λ or C but on the boundary). We prove the following lemma.

LEMMA 1. *Suppose that $q_t(x, x')$ is obtained on $\Lambda \times \Lambda$ (or $C \times C$) and satisfies* (3.6) *and that $q_t(x, x')$ has limit when x and x or x' tends to the walls of Λ (or of C) and we denote this limit by $q_t(x, x')$ also. Then $q_t(0, x)$ satisfies* (3.5) *or* (3.5)′.

Proof of Lemma 1. We shall do it only in the non compact case. First of all the condition 1) is trivially satisfied at the limit and 2) also if x' stays in Λ. To prove 3), we take a function f continuous with compact support, invariant by W, on $\mathfrak{A}_{\mathfrak{P}}$ and we want to prove that

$$\lim_{t \to 0} \int_\Lambda f(x')q_t(0, x')v_p(x')dx' = f(0) \quad \text{or}$$

$$\lim_{t \to 0} \lim_{x \to 0} \int_\Lambda f(x')q_t(x, x')v_p(x')dx' = f(0).$$

Fix ϵ_0 such that for $\epsilon < \epsilon_0$,

$$\|f(x) - f(x')\| < h \quad \text{for any } x, x' \in B(0, \epsilon).$$

Now fix such an ϵ and fix x in the ball $B\left(0, \frac{\epsilon}{2}\right)$: then

$$\int_\Lambda f(x')q_t(x, x')v_p(x')dx'$$

$$= f(x) + \int_{B(0,\epsilon)\cap\Lambda} (f(x') - f(x))q_t(x, x')v_p(x')dx'$$

$$+ \int_{(\complement B(0,\epsilon))\cap\Lambda} (f(x') - f(x))q_t(x, x')v_p(x')dx'$$

because

$$\int_\Lambda q_t(x, x')v_p(x')dx' = 1.$$

Let us now fix ϵ; if x stays fixed in $B\left(0, \frac{\epsilon}{2}\right)$, then

$$\int_{(\complement B(0,\epsilon))\cap\Lambda} (f(x') - f(x))q_t(x, x')v_p(x')dx'$$

tends to 0 if $t \to 0^+$ because

$$q_t(x, x') \to \delta_x(x')$$

and because the distance from x to x' is bounded from below by $\epsilon/2$; and so for h given, there exists t_0 such that if $t < t_0$, this integral is less than h. Moreover by the definition of ϵ, for any t, we have

$$\left|\int_{B(0,\epsilon)} (f(x') - f(x))q_t(x, x')v_p(x')dx'\right|$$

$$< h \int q_t(x, x')v_p(x')dx' = h$$

because q_t is positive of total mass 1 which proves the lemma.

3. *The formula for the heat kernel for* $SU(p + 1)$. We shall now treat the problem (3.6)′ in the case of $SU(p + 1)$. We can forget about $SU(p + 1)$ and treat the problem on the euclidean space E (of dimension p) with the operator $\Delta_2 \equiv Q_\Delta$ given by (2.37) (where we take

$$P = \sum_{i=1}^{p+1} \frac{\partial^2}{\partial q_i^2}).$$

The volume element is just

$$\frac{J^2(x)dx}{(2i)^{p(p+1)}}$$

(see part 6 of Section 2) and Δ_2 is self-adjoint with respect to $J^2(x)dx$. Moreover we look for a function $q_t(x, x')$ which is symmetric in (x, x') and is invariant in both variables by the action of the affine Weyl group $W^{(0)}$, which means that it is periodic of period 2π in the x_i's coordinate and invariant by W. Then $q_t(x, x')$ defines a symmetric kernel on

$$L^2\left(C, \frac{J^2(x)dx}{(2i)^{p(p+1)}}\right)$$

and has a natural decomposition on the orthonormal basis of the

$$\frac{1}{(2\pi)^{p/2}} p_{(\mathbf{n})}$$

(given by (2.36)) which are eigenfunctions of Δ_2 for the eigenvalues ${}^{(\Delta)}\lambda_{(\mathbf{n})}$ given by (2.39): so we obtain

$$q_t(x, x') = \frac{1}{(2\pi)^p} \sum_{(\mathbf{n})\geqq 0} e^{{}^{(\Delta)}\lambda_{(\mathbf{n})}t} p_{(\mathbf{n})}(x)p_{(\mathbf{n})}(x')^*$$

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$$(3.7)\qquad q_t(x, x') = \frac{e^{-\lambda^{(0)}_{(\mathbf{1})}t}}{(2\pi)^p}\frac{2^{p(p+1)}}{Af_{(\mathbf{1})}(x)(Af_{(\mathbf{1})}(x'))^*} \times \sum_{(\mathbf{n})>0} e^{\lambda^{(0)}_{(\mathbf{n})}t} Af_{(\mathbf{n})}(x)(Af_{(\mathbf{n})}(x'))^*$$

(where in the last summation we have changed $(\mathbf{n} + \mathbf{1})$ with $(\mathbf{n}) \geqq 0$ in $(\mathbf{n})$ with $(\mathbf{n}) > 0$ and

$$^{(\Delta)}\lambda_{(\mathbf{n})} = \lambda^{(0)}_{(\mathbf{n}+\mathbf{1})} - \lambda^{(0)}_{(\mathbf{1})}.$$

In (3.7) we can also extend the summation in $(\mathbf{n})$ to $(\mathbf{n}) \geqq 0$ because we know that $Af_{(\mathbf{n})} = 0$ if one of the n_k is 0. But we also know that any $(\mathbf{n}) \in \mathbf{Z}^p$ is the image by some $\sigma \in W$ of an $(\mathbf{n}) \in \bar{\Lambda}$ and if $\sigma \neq \mathrm{Id}$ (or $(\mathbf{n}) \notin \bar{\Lambda}$) one of the n_k is negative. Suppose that $(\mathbf{n})$ is in $\mathbf{Z}^p$ and $n_k \neq 0$ for all k; there exists a $\sigma \in W$ unique such that $\sigma(\mathbf{n})$ is in Λ; but then

$$(3.8)\qquad Af_{(\mathbf{n})} = (-1)^{\sigma} Af_{\sigma(\mathbf{n})}.$$

Also $\lambda^{(0)}_{(\mathbf{n})}$ is exactly $\lambda^{(0)}_{\sigma(\mathbf{n})}$. On the other hand if one of the n_k is 0, then $Af_{(\mathbf{n})} = 0$. So we can extend in (3.7) the summation to all $\mathbf{Z}^p$ if we divide by the number of elements of W_p:$(p + 1)!$ and using (3.8) and denoting by A_x, $A_{x'}$ the action of A on x or x' variables:

$$(3.9)\qquad \begin{aligned} q_t(x, x') &= \frac{e^{-\lambda^{(0)}_{(\mathbf{1})}t}}{(2\pi)^p(p + 1)!}\frac{2^{p(p+1)}}{Af_{(\mathbf{1})}(x)(Af_{(\mathbf{1})}(x'))^*} \times \sum_{(\mathbf{n})\in\mathbf{Z}^p} e^{\lambda^{(0)}_{(\mathbf{n})}t} Af_{(\mathbf{n})}(x)(Af_{(\mathbf{n})}(x'))^* \\ q_t(x, x') &= \frac{e^{-\lambda^{(0)}_{(\mathbf{1})}t} \times 2^{p(p+1)} \times A_x A_{x'}}{(2\pi)^p(p + 1)! Af_{(\mathbf{1})}(x)(Af_{(\mathbf{1})}(x'))^*} \times \sum_{(\mathbf{n})\in\mathbf{Z}^p} e^{\lambda^{(0)}_{(\mathbf{n})}t} f_{(\mathbf{n})}(x)(f_{(\mathbf{n})}(x'))^* \end{aligned}$$

Now the series on the left side of (3.9) is a generalized theta function. By (2.33) we have

$$(3.10)\qquad \lambda^{(0)}_{(\mathbf{n})} = -{}^t(\mathbf{n})M(\mathbf{n})$$

where M is the symmetric matrix of Hilbert type:

$$(3.11)\qquad M_{kl} = 2\left[\mathrm{Inf}(k, l) - \frac{kl}{p + 1}\right] \quad 1 < k, l \leqq p.$$

Now if S is a complex symmetric $(p \times p)$ matrix and X is a vector in $\mathbf{C}^p$, we define

$$(3.12)\qquad \Theta(S, X) = \sum_{(\mathbf{n}) \in \mathbf{Z}^p} \exp(i\pi\,{}^t(\mathbf{n})S(\mathbf{n}) + 2i\pi({}^t(\mathbf{n}), (X))).$$

Here we have

$$(3.13)\qquad q_t(x, x') = \frac{2^{p(p+1)}e^{-\lambda^{(0)}_{(1)}t}}{(2\pi)^p(p+1)!}$$

$$\times\ (Af(\mathbf{1})(x)Af_{(1)}(x')^*)^{-1}A_xA_{x'}\Theta\left(-\frac{t}{i\pi}M, \frac{x - x'}{2\pi}\right)$$

M being defined in (3.11).

We have now to go to the limit $x \to 0$. Let us fix $x' \neq 0$ in C; we have to compute

$$\lim_{x \to 0} \frac{A\varphi(x)}{A_{f_{(1)}}(x)}$$

for some given function φ; it is clear that the numerator and the denominator of this quotient tend to 0 when $x \to 0$. Now $Af_{(1)} \equiv \mathfrak{Z}$ has been computed in (2.34) and it is equivalent to:

$$Af_{(1)}(x) \sim \prod_{\substack{R>0\\ R\in A_p}} 2i(R, x) \quad \text{when } x \to 0$$

and

$$(3.14)\qquad \lim_{x \to 0} \frac{A\varphi(x)}{Af_{(1)}(x)} = \left(\prod_{\substack{R>0\\ R\in A_p}} \frac{\partial}{2i\partial(R, x)}\right)(A\varphi)(x)\,|_{x=0}.$$

Call

$$(3.15)\qquad L = \prod_{\substack{R>0\\ R\in A_p}} \left(\frac{\partial}{2i\partial(R, x)}\right);$$

this is clearly an antiinvariant operator for W_p (this is the one of least order); then

$$(LA\varphi)(x) = \sum_{\sigma\in W_p} (-1)^{\sigma}(L(\sigma\varphi))(x) = \sum_{\sigma\in W_p} (L\varphi)(\sigma x)$$

and for $x = 0$ this is $(p + 1)!(L\varphi)(0)$, so that when $x \to 0$, we obtain from (3.13), (3.14)

$$q_t(0, x) = \frac{2^{p(p+1)}e^{-\lambda^{(0)}_{(1)}t}}{(2\pi)^p}$$

$$\times\ (Af_{(1)}(x))^{-1}A_x\left[L_{x'}\Theta\left(-\frac{t}{i\pi}M, \frac{x - x'}{2}\right)\right]\Big|_{x'=0}.$$

We can now simplify slightly this expression; call

$$\varphi(x - x') = \Theta\left(-\frac{t}{i\pi}M, \frac{x - x'}{2\pi}\right);$$

we have to compute

$$\begin{aligned} A_x[L_{x'}\varphi(x - x')]\,|_{x'=0} &= \sum (-1)^{\sigma}[L_{x'}\varphi(\sigma^{-1}x - x')]\,|_{x'=0} \\ &= \sum (-1)^{\sigma}[L_{x'}\varphi(\sigma^{-1}(x - \sigma x'))]\,|_{x'=0}. \end{aligned}$$

But by antiinvariance of L

$$\begin{aligned} L_{x'}\varphi(\sigma^{-1}(x - \sigma x')) &= (-1)^{\sigma}L_{x''}\varphi(\sigma^{-1}(x - x''))\,|_{x''=\sigma x'} \\ &= (-1)^{\sigma}L_x(\varphi(\sigma^{-1}(x - x'')))\,|_{x''=\sigma x'} \end{aligned}$$

so that

$$\begin{aligned} A_x[L_{x'}\varphi(x - x')]\,|_{x'=0} &= \sum_{\sigma} L_x(\varphi(\sigma^{-1}x)) \\ &= \sum(-1)^{\sigma}(L_x\varphi)(\sigma^{-1}x) \end{aligned}$$

and finally

$$(3.16)\qquad q_t(0, x) = \frac{2^{p(p+1)}e^{-\lambda_{(1)}^{(0)}t}}{(2\pi)^p}(Af_{(1)}(x))^{-1}A_xL_x\Theta\left(-\frac{t}{i\pi}M, \frac{x}{2\pi}\right).$$

THEOREM 1. *The heat kernel of $SU(p + 1)$ in radial coordinates is given by formula* (3.16) *with L defined in* (3.15) Θ *in* (3.12) *and M in* (3.11). *The heat kernel of the Weyl alcove C* (*or the heat kernel on $SU(p + 1)$ between* 2 *regular spheres of $SU(p + 1)$ is given by formula* (3.13) *for $x, x' \in C$.*

4. *The heat kernel on the non compact symmetric space $SL(p + 1, \mathbf{C})/SU(p + 1)$.* $SL(p + 1, \mathbf{C})/SU(p + 1)$ is the non compact dual of $SU(p + 1)$. This means that it has the same roots with same multiplicity 2 and it can be formally obtained from $SU(p + 1)$ by taking purely imaginary variables. The volume element is

$$(3.17)\qquad v_{(2)}(x)dx = \left(\prod_{\substack{R\in A_p\\ R>0}} \sinh(R, x)\right)^2 dx$$

and we define as usual

$$(3.18)\qquad J(x) = \prod_{\substack{R\in A_p\\ R>0}} 2\sinh(R, x)$$

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so that the volume element is

$$v_2(x) = \frac{J^2(x)}{2^{p(p+1)}}.$$

It is known from [2] (and in fact it is easy to check directly using (1.3)), that the radial part of the Laplace-Beltrami operator on $(E, A_p, (2 \dots 2))$ is then

$$(3.19) \qquad \Delta_2 = \frac{1}{J(x)}\Delta\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right)J(x) + \lambda_{(1)}^{(0)}$$

with $\lambda_{(\mathbf{1})}^{(0)}$ given by (2.40) (the + is due to the fact that J involves now real exponentials) and where

$$\Delta\left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{p+1}}\right) = \sum_{i=1}^{p+1} \frac{\partial^2}{\partial q_i^2}.$$

We want to find the heat kernel of problem (3.6) and then (3.5). As usual, it is inconvenient to work in the q_i coordinates (because they have to obey the relation

$$\sum_{i=1}^{p+1} q_i = 0).$$

We have then to work in x_i coordinates, which are not orthogonal. But we know from (2.32) that

$$(3.20) \qquad \Delta e^{i(\xi,x)} = -\frac{2}{p+1}\left(\sum_{r=1}^{p} r(p+1-r)\xi_r^2 \right.$$
$$\left. + 2 \sum_{1 \leqq r < l \leqq p} r(p+1-l)\xi_r\xi_l\right).$$

Call $p_t^{(e)}(x, x')$ the heat kernel (with respect to the measure dx') of the operator Δ given by (2.32): we have

$$\Delta = -({}^t\partial_x)M(\partial_x)$$

and thus

$$(3.21) \qquad p_t^{(e)}(x, x') = \frac{1}{(4\pi t)^{p/2}(\det M)^{+1/2}}$$
$$\times \exp -\frac{1}{4t}({}^t(x - x')M^{-1}(x - x'))$$

where M is the matrix (3.11), and so

$$\frac{\partial}{\partial t} p_t^{(e)}(x, x') = \Delta_x p_t^{(e)}(x, x')$$

$$p_t^{(e)}(x, x')dx' \to \delta(x - x').$$

Now if we define

$$r_t(x, x') = \frac{e^{+\lambda_{(1)}^{(0)}t} p_t^{(e)}(x, x') 2^{p(p+1)}}{J(x)J(x')}$$

it is easy to see that

$$f_t(x) = \int_\Lambda f_0(x') r_t(x, x') \frac{J^2(x')}{2^{p(p+1)}} dx'$$

is a solution of

$$\frac{\partial f_t}{\partial t} = \Delta_2 f_t \quad (\Delta_2 \text{ given by (3.19)})$$

$$f_t \to f_0 \qquad \text{if } t \to 0^+$$

provided that supp $f_0 \subset \Lambda$ (and so does not touch the walls of Λ). We now define

(3.22) $$q_t(x, x') = \frac{e^{\lambda_{(1)}^{(0)}t}}{(p + 1)!} \frac{2^{p(p+1)}}{J(x)J(x')} A_x A_{x'} p_t^{(e)}(x, x').$$

It is clear that

$$q_t(x, x')dv(x') \to \delta(x - x')$$

in $\Lambda \times \Lambda$ because in the summation

$$\frac{1}{(p + 1)!} \sum_\sigma \sum_\tau (-1)^\sigma (-1)^\tau p_t^{(e)}(\sigma x, \tau x')$$

only the terms $\sigma = \tau$ give a $\delta(x - x')$ for functions defined only on Λ, so on functions with support in $\overline{\Lambda}$, this double summation tends to $\delta(x - x')$ because we have divided by $(p + 1)!$.

Moreover q_t satisfies the heat equation in Λ

(3.23) $$\frac{\partial}{\partial t} q_t = \Delta_2 q_t$$

because Δ_2 commute with the action of W, so that

$$e^{\lambda_{(1)}^{(0)}t} \frac{p_t^{(e)}(\sigma x, x')}{J(x)J(x')}$$

satisfies the heat equation in x, (3.23), for any x' and any $\sigma \in W$.

Finally it is clear that q_t is smooth on $\mathfrak{A}_{\mathfrak{P}}$ is symmetric in x, x' and is invariant by action of W.

We now have to compute the limit when $x \to 0$. Then by (3.18)

$$J(x) \sim \prod_{\substack{R \in A_p \\ R > 0}} 2(R, x).$$

We can do the same computation as in part 3; we call now

$$(3.24) \qquad L' = \prod_{\substack{R > 0 \\ R \in A_p}} \frac{\partial}{2\partial(R, x)}$$

and we finally find

$$(3.25) \qquad q_t(0, x) = 2^{p(p+1)} \frac{e^{\lambda_{(1)}^{(0)} t}}{J(x)} A_x L'_x p_t^{(e)}(0, x)$$

and

THEOREM 2. *The heat kernel* $SL(p + 1, \mathbf{C})/SU(p + 1)$ *is given by formula* (3.25). *On the Weyl chamber* Λ, *the heat kernel is given by formula* (3.22) *for* $x, x' \in \Lambda$ (*this represents the heat diffusion between two spheres* $SU(p + 1) \cdot e^x \cdot 0$ *and* $SU(p + 1) \cdot e^{x'} \cdot 0$ *in the symmetric space*).

References for Section 3. The heat kernel for $SU(2)$ was given by Schulman [**28**] and for the general $SU(p)$ by Dowker [**9**]. One method is slightly different because we proceed through the eigenfunctions. Moreover we stress the importance of the matrix M which will be very useful in a later publication.

The Laplace operators for non compact $SL(P + 1, \mathbf{C})/SU(p + 1)$ are given by Berezin [**2**] and their expressions are used by Dynkin to obtain the Green's kernel [**10**].

4. Invariant operators, their eigenfunctions and heat kernels for the root system BC_p and certain related spaces.

1. *The root system* BC_p *and its affine Weyl group.*

a) *The root system* BC_p. We start with a euclidean space of dimension p; q will denote a point in E, $(q_1, \ldots, q_p)$ its coordinates with respect to an orthonormal base $e_1, \ldots, e_p$ of E. The root system BC_p contains by definition the following roots

(i) $R_i^{(1)}(q) = q_i$ and their opposites $-q_i$

(ii) $R_i^{(2)}(q) = 2q_i$ and their opposites $-2q_i$

(iii) $R_{ij}^{(1)}(q) = q_i - q_j$ and $R_{ij}^{(2)}(q) = q_i + q_j$ with their opposites.

We choose as positive roots the q_i, $2q_i$, $q_i - q_j$ and $q_i + q_j$ for $i < j$. The Weyl chamber is

(4.1) $\Lambda_q = \{q \in E_p/0 < q_p < q_{p-1} < \ldots < q_1\}.$

We must also associate to each root α a multiplicity ρ_α; we shall choose the following multiplicities:

(i) the $R_i^{(1)}(q) = q_i$ have equal multiplicities ρ_1 (and also their opposite roots)

(ii) the $R_i^{(2)}(q) = 2q_i$ have equal multiplicities ρ_2

(iii) the $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$ have equal multiplicities ρ_3.

Here the ρ_i are positive or 0 numbers.

b) *Related symmetric spaces.* For example, let us consider the case $p = 2$; we have two non compact symmetric spaces with the root system BC_2 namely

$$\frac{S_p(2, \mathbf{R})}{U(2)} \rho_1 = 0, \rho_2 = 1, \rho_3 = 1$$

$$\frac{SU(2, 2)}{S(U(2)xU(2))} \rho_1 = 0, \rho_2 = 1, \rho_3 = 2.$$

c) *The Weyl group.* As usual, the Weyl group is generated by reflexion through the walls of Λ. It is the semi direct product of the group S_p of permutations of the q_i's and p representatives of the group $\mathbf{Z}/2\mathbf{Z}$ operating on q_i by $q_i \to \pm q_i$.

d) *The affine Weyl group.* We define the Weyl alcove by

(4.2) $C_q = \{q \in E/0 < q_p < q_{p-1} < \ldots < q_1 < \pi\}$

and we define also the group T_p generated by the translations of vectors $2\pi e_i$ (so $q_i \to q_i + 2\pi$ for $i = 1 \ldots p$). The affine Weyl group $W^{(0)}$ is then the semi direct product of the Weyl group and the group T_p. The images by the elements of $W^{(0)}$ of the Weyl alcove C_q generate a paving of E.

2. *The space of functions on C_q and volume element.*

a) *Invariant functions by the affine Weyl group.* Let $f(q)$ be a function on E. Then f is invariant by the affine Weyl group $W^{(0)}$ if and only if

(i) f is periodic of period 2π in each of the coordinates q_i

(ii) f is symmetric with respect to all permutations of the q_i

(iii) f is even with respect of all coordinates q_i. Such a function is known, when its values on C_q are known.

b) *The volume element and the associated Laplace operator.* On C_q, we shall define the following volume element

$$(4.3) \quad d\hat{V}^{(\rho_1,\rho_2,\rho_3)} = \prod_{i=1}^{p} \left(\sin \frac{q_i}{2}\right)^{\rho_1} (\sin q_i)^{\rho_2}$$

$$\times \prod_{1 \leqq i < j \leqq p} \left(\sin\left(\frac{q_i - q_j}{2}\right) \sin\left(\frac{q_i + q_j}{2}\right)\right)^{\rho_3} \prod_{i=1}^{p} dq_i.$$

We call $\hat{W}^{(\rho_1,\rho_2,\rho_3)}$ the function appearing in front of $\prod_{i=1}^p dq_i$. We can define an associated Laplace operator

(4.4) $$\hat{\Delta}^{(\rho_1,\rho_2,\rho_3)} = \frac{1}{\hat{W}^{(\rho_1,\rho_2,\rho_3)}} \sum_{i=1}^{p} \frac{\partial}{\partial q_i}\left(\hat{W}^{(\rho_1,\rho_2,\rho_3)} \frac{\partial}{\partial q_i}\right)$$

It is clear that $\hat{\Delta}^{(\rho_1,\rho_2,\rho_3)}$ gives a self adjoint operator on $L^2(C, d\hat{V}^{(\rho_1,\rho_2,\rho_3)})$.

Remark. We remark that in (4.3) and (4.4) we have divided the roots appearing in the sine functions by 2. The formulas are not exactly the same as the general formulas of Section 1 (1.9) and (1.10). This is only a question of tradition.

3. *The change of coordinates* $x_i = \cos q_i$ (*algebraic variables*).

a) *Definition of the "algebraic" variables*. In the case of BC_p root system, it is very convenient to use the so called algebraic variables $x_i = \cos q_i$. Then the Weyl alcove becomes

(4.5) $$C_x = \{x = (x_1, \dots, x_p) \in \mathbf{R}^p \\ -1 < x_1 < x_2 < \dots < x_p < 1\}.$$

Moreover if f is a function on E_q, it becomes a function (still denoted f) on $[-1, +1]^p$ and $f(x_1, \dots, x_p)$ is invariant with respect to the affine Weyl group if and only if it is symmetric with respect to all permutations of the x_i (the periodicity and the evenness of f is taken into account by the cosine function). It is also clear that the passage from x-variables to q-variables induces a bijection from functions defined on C_x to functions defined on C_q.

b) *The volume element in algebraic variables*. We call

(4.6) $$\rho_1 = 2\alpha - 2\beta, \quad \rho_2 = 2\beta + 1, \quad \rho_3 = 2\gamma + 1.$$

Then

$$W^{(\rho_1,\rho_2,\rho_3)} \equiv W^{(\alpha,\beta,\gamma)} = W_1 W_2$$

with

$$W_1 = \prod_{i=1}^{p}\left[\left(\sin\frac{q_i}{2}\right)^{2\alpha-2\beta}(\sin q_i)^{2\beta+1}\right]$$

$$W_2 = \prod_{1\leqq i<j\leqq p}\left(\sin\left(\frac{q_i - q_j}{2}\right)\sin\left(\frac{q_i + q_j}{2}\right)\right)^{2\gamma+1}.$$

This can be rewritten as

$$W_1 = 2^{p\beta}\left(\prod_{i=1}^{p}\sin^2\frac{q_i}{2}\right)^{\alpha}\left(\prod_{i=1}^{p}\cos^2\frac{q_i}{2}\right)^{\beta}\prod_{i=1}^{p}\sin q_i$$

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$$= 2^{\mathfrak{p}(\beta-\alpha)}\left(\prod_{i=1}^{p}(1-x_i)\right)^{\alpha}\left(\prod_{i=1}^{p}(1+x_i)\right)^{\beta}\prod_{i=1}^{p}\sin q_i$$

and

$$W_2 = 2^{-[\mathfrak{p}(\mathfrak{p}-1)/2](2\gamma+1)}\left(\prod_{1\leqq j<i\leqq p}(x_i-x_j)\right)^{2\gamma+1}.$$

Moreover

$$dq_1 \ldots dq_p = (-1)^p\left(\prod_{i=1}^{p}\sin q_i\right)dx_1 \ldots dx_p$$

and we obtain

LEMMA 1. *In coordinates x_i, the volume element is*

$$(4.7)\qquad dV^{(\alpha,\beta,\gamma)} = C\prod_{i=1}^{p}(1-x_i)^{\alpha}(1+x_i)^{\beta} \times \prod_{1\leqq j<i\leqq p}(x_i-x_j)^{2\gamma+1}dx_1 \ldots dx_p$$

where

$$C = 2^{\mathfrak{p}(\beta-\alpha-[(\mathfrak{p}-1)/2](2\gamma+1))}.$$

We denote

$$(4.8)\qquad m^{(\alpha\beta\gamma)} = \prod_{i=1}^{p}(1-x_i)^{\alpha}(1+x_i)^{\beta}\prod_{1\leqq j<i\leqq p}(x_i-x_j)^{2\gamma+1}$$

c) *The Laplace-Beltrami operator in algebraic coordinates.*

LEMMA 2. *We have in algebraic coordinates*

$$(4.9)\qquad \Delta^{(\alpha,\beta,\gamma)} = \frac{1}{m^{(\alpha,\beta,\gamma)}}\sum_{i=1}^{p}\frac{\partial}{\partial x_i}\left((1-x_i^2)m^{(\alpha,\beta,\gamma)}\frac{\partial}{\partial x_i}\right)$$

or by expanding

$$(4.10)\qquad \Delta^{(\alpha,\beta,\gamma)} = \sum_{i=1}^{p}\Bigg\{(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + \Bigg[\beta-\alpha-(\alpha+\beta+2)x_i + (2\gamma+1)(1-x_i^2)\sum_{\substack{j\neq i\\ j=1}}^{p}\frac{1}{x_i-x_j}\Bigg]\frac{\partial}{\partial x_i}\Bigg\}.$$

Proof. This follows by direct computation.

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Remark. We see that the variables x_i are separated if and only if $2\gamma + 1 = 0$, but this does not correspond to a symmetric space of type BC_p (it can correspond for certain values of α and β to a product of compact symmetric spaces).

4. *The case where* $2\gamma + 1 = 0$.

a) *Jacobi polynomials in one variable.* Let α, $\beta > -1$. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthonormal for the weight

$$\mu^{(\alpha,\beta)} = (1 - x)^\alpha(1 + x)^\beta \quad \text{on } [-1, +1];$$

they satisfy the differential equations

$$(4.11) \quad \left((1 - x^2)\frac{d^2}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x]\frac{d}{dx}\right)P_n^{(\alpha,\beta)} = -n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}$$

and also the ladder equation:

$$(4.12) \quad \frac{d}{dx}P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x).$$

The preceding differential equation is

$$(4.13) \quad \frac{1}{\mu^{(\alpha,\beta)}}\frac{d}{dx}\left(\mu^{(\alpha+1,\beta+1)}\frac{d}{dx}P_n^{(\alpha,\beta)}\right) = -n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}.$$

We suppose also that

$$\|P_n^{(\alpha,\beta)}\|^2_{L^2([-1,+1],\mu^{(\alpha,\beta)}dx)} = 1.$$

b) *The case* $\gamma = -1/2$. In case $\gamma = -1/2$, the operator $\Delta^{(\alpha,\beta,-1/2)}$ appears to be a sum of identical independent operators in one variable of type (4.11). It is clear that

$$(4.14) \quad \Delta^{(\alpha,\beta,-(1/2))}\prod_{l=1}^p P_n^{(\alpha,\beta)}(x_l) = -\left(\sum_{l=1}^p n_l(n_l + \alpha + \beta + 1)\right)\prod_{l=1}^p P_{n_l}^{(\alpha,\beta)}(x_l)$$

and the products

$$\prod_{l=1}^p P_n^{(\alpha,\beta)}(x_l)$$

are orthogonals for the weight $m^{(\alpha,\beta,-(1/2))}$ on the cube $[-1, +1]^p$. If we want to obtain an invariant function, we just have to symmetrise with respect to the symmetric group S_p. We define an order on the p-uplets of integers by

$$n_1 \leqq n_2 \leqq \ldots \leqq n_p.$$

If $n_1 < n_2 < \ldots < n_p$, we define

$$(4.15) \quad p_{(\mathbf{n})}^{(\alpha,\beta,-(1/2))}(x) = \sum_{\sigma \in S_p} \prod_{k=1}^{p} P_{n_{\sigma(k)}}^{(\alpha,\beta)}(x_k)$$

where as in Section 2 $(\mathbf{n}) = (n_1, \ldots, n_p)$.

In the case when some integers n_j are equals, we need only to sum over a subgroup of S_p. We obtain easily

LEMMA 3. *The polynomials* $p_{(\mathbf{n})}^{(\alpha,\beta,-(1/2))}(x)$ *for* $(\mathbf{n}) = (n_1, \ldots, n_p)$ $(n_1 \leqq n_2 \leqq \ldots \leqq n_p)$ *are an orthogonal basis on* C_n *for the volume* $dV^{(\alpha,\beta,-(1/2))}$ *of eigenfunctions of* $\Delta^{(\alpha,\beta,-(1/2))}$ *with the eigenvalues*

$$(4.16) \quad \lambda_{(\mathbf{n})}^{(\alpha,\beta,-(1/2))} = -\sum_{i=1}^{p} n_i(n_i + \alpha + \beta + 1).$$

5. *The case* $2\gamma + 1 = 2\left(\text{or } \gamma = \dfrac{1}{2}\right)$.

a) *Notations*. Let us define

$$(4.17) \quad \varphi(x) = \prod_{1 \leqq i < j \leqq p} (x_i - x_j)$$

$$\theta_i = \prod_{\substack{1 \leqq k < l \leqq p \\ k \text{ and } l \neq i}} (x_k - x_p)$$

(so θ_i does not contain x_i)

$$\varphi_i = \frac{\varphi}{\theta_i}$$

$$\mu_i^{(\alpha,\beta)} = (1 - x_i)^{\alpha}(1 + x_i)^{\beta}.$$

It is clear that, from (4.10) or (4.9) we obtain

$$(4.18) \quad \Delta^{(\alpha,\beta,\gamma)} = \sum_{i=1}^{p} \frac{1}{\mu_i^{\alpha,\beta}\varphi_i^2}\left(\frac{\partial}{\partial x_i}\mu_i^{(\alpha+1,\beta+1)}\varphi_i^2\frac{\partial}{\partial x_i}\right).$$

b) *The polynomials* $p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}$. For $(\mathbf{n}) = (n_1, \ldots, n_p)$ $(n_1 \leqq n_2 \leqq \ldots \leqq n_p)$ we define the antisymmetric polynomials (with respect to S_p)

$$(4.19) \quad p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x) = \sum_{\sigma \in S_p} (-1)^{\sigma} \frac{\prod_{j=1}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x_{\sigma(j)})}{\varphi(x)}.$$

It is easy to see that the $\varphi \cdot p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x)$ are orthogonal on C_x for the volume

$$\prod_{i=1}^{p}(1 - x_i)^{\alpha}(1 + x_i)^{\beta}dx$$

so that we obtain:

LEMMA 4. *The* $p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x)$ *are invariant polynomials orthogonal on* C_x *for the volume* $dV^{(\alpha,\beta,1/2)}(x)$.

Proof. We have

$$dV^{(\alpha,\beta,1/2)}(x) = \prod_{i=1}^{p}(1 - x_i)^{\alpha}(1 + x_i)^{\beta}\varphi(x)^2dx$$

from (4.8) in the case $\gamma = 1/2$, using the definition (4.17) of φ.

Remark. See also [**16**].

We now want to prove the following result.

THEOREM 1. *The invariant polynomials*

$$p_{(\mathbf{n})}^{(\alpha,\beta,1/2)} \quad ((\mathbf{n}) = (n_1, \dots, n_p) \quad n_1 \leqq n_2 \leqq \dots \leqq n_p)$$

are an orthonormal basis on C_x *for the volume* $dV^{(\alpha,\beta,1/2)}$ *of eigenfunctions of* $\Delta^{(\alpha,\beta,1/2)}$ *for the eigenvalue*

$$\lambda_{(\mathbf{n})}^{(\alpha,\beta,1/2)} = -\sum_{i=1}^{p} n_i(n_i + \alpha + \beta + 2i - 1). \tag{4.20}$$

6. *Proof of Theorem* 1. We abbreviate $\Delta^{(\alpha,\beta,1/2)}$ by Δ; we have to compute $\Delta p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}$. Define

$$Q_i^{(\alpha,\beta)} = \prod_{\substack{j=1 \\ j\neq i}}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x_j) \tag{4.21}$$

$$A = \Delta\left[\frac{\prod_{j=1}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x_j)}{\varphi}\right]. \tag{4.22}$$

Then we obtain from the expression (4.18) for Δ

$$A = \sum_{i=1}^{p} \frac{Q_i^{(\alpha,\beta)}(x)}{\mu_i^{(\alpha,\beta)}\theta_i\varphi_i^2}\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)} \times \varphi_i^2\frac{\partial}{\partial x_i}\frac{P_{n_i+i-1}^{(\alpha,\beta)}(x_i)}{\varphi_i}\right).$$

Let us compute each term in the sum $\sum_{i=1}^{p}$ giving A:

$$\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\varphi_i^2\frac{\partial}{\partial x_i}\frac{P_{n_i+i-1}^{(\alpha,\beta)}(x_i)}{\varphi_i}\right)$$

$$= \mu_i^{(\alpha+1,\beta+1)}\left[\frac{\partial}{\partial x_i}\varphi_i + \varphi_i^2\frac{\partial}{\partial x_i}\frac{1}{\varphi_i}\right]$$

$$\times \frac{\partial}{\partial x_i}P_{n_i+i-1}^{(\alpha,\beta)}(x_i) + \varphi_i\left[\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\left(\frac{\partial}{\partial x_i}P_{n_i+i-1}^{(\alpha,\beta)}(x_i)\right)\right)\right]$$

$$+ \frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\varphi_i^2\frac{\partial}{\partial x_i}\frac{1}{\varphi_i}\right)P_{n_i+i-1}^{(\alpha,\beta)}(x_i).$$

We have $\varphi_i^2\partial_i\varphi_i^{-1} = -\partial_i\varphi_i$

$$A = \frac{1}{\varphi}\sum_{i=1}^{p} Q_i^{(\alpha,\beta)}(x)\left[\frac{1}{\mu_i^{(\alpha,\beta)}}\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\frac{\partial}{\partial x_i}P_{n_i+i-1}^{(\alpha,\beta)}\right)\right.$$

$$\left. + \frac{P_{n_i+i-1}^{(\alpha,\beta)}(x_i)}{\varphi_i\mu_i^{(\alpha,\beta)}}\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\varphi_i^2\frac{\partial}{\partial x_i}\varphi_i^{-1}\right)\right].$$

But, by equation (4.13)

$$\frac{1}{\mu_i^{(\alpha,\beta)}}\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\frac{\partial}{\partial x_i}P_{n_i+i-1}^{(\alpha,\beta)}\right)$$

$$= -(n_i + i - 1)(n_i + \alpha + \beta + i)$$

$$\times P_{n_i+i-1}^{(\alpha,\beta)}\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\varphi_i^2\frac{\partial}{\partial x_i}\varphi_i^{-1}\right)$$

$$= -\frac{\partial}{\partial x_i}\left(\mu_i^{(\alpha+1,\beta+1)}\frac{\partial}{\partial x_i}\varphi_i\right)$$

$$= \mu_i^{(\alpha,\beta)}\left(\alpha - \beta + x_i(\alpha + \beta + 2)\frac{\partial\varphi_i}{\partial x_i} + (x_i^2 - 1)\frac{\partial^2}{\partial x_i^2}\varphi_i\right)$$

and so finally

$$A = \frac{\prod_{j=1}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x_j)}{\varphi}\left[-\sum_{i=1}^{p}(n_i + i - 1)(n_i + \alpha + \beta + i)\right.$$

$$\left. + \sum_{i=1}^{p}\frac{1}{\varphi_i}\left(\alpha - \beta + x_i(\alpha + \beta + 2)\frac{\partial\varphi_i}{\partial x_i} + (x_i^2 - 1)\frac{\partial^2\varphi_i}{\partial x_i^2}\right)\right].$$

But we have by definition of φ_i

$$\frac{\partial}{\partial x_i}\varphi_i = (-1)^{i-1}\sum_{\substack{j=1\\ j\neq i}}^{p}\prod_{\substack{k=1\\ k\neq i,j}}^{p}(x_i - x_k)$$

$$= \varphi_i\left[\sum_{k=i+1}^{p} \frac{1}{x_i - x_k} - \sum_{k=1}^{i-1} \frac{1}{x_k - x_i}\right]$$

and so

$$\sum_{i=1}^{p} \frac{1}{\varphi_i}\frac{\partial \varphi_i}{\partial x_i} = 0$$

$$\sum_{i=1}^{p} x_i \frac{\partial \varphi_i}{\partial x_i}$$

$$= \sum_{i=1}^{p} x_i\left(-\sum_{k=1}^{i-1} \frac{1}{x_k - x_i} + \sum_{k=i+1}^{p} \frac{1}{x_i - x_k}\right) = \frac{p(p-1)}{2}$$

$$\frac{1}{\varphi_i}\frac{\partial^2 \varphi_i}{\partial x_i^2} = \sum_{\substack{k=1 \\ k\neq i}}^{p} \sum_{\substack{l=1 \\ \rho\neq i}}^{p} \frac{1}{(x_i - x_e)(x_i - x_k)}$$

$$\sum_{i=1}^{p} \frac{1}{\varphi_i}\frac{\partial^2 \varphi_i}{\partial x_i^2} = \sum_{\substack{i=1 \\ i\neq k\neq l}}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \frac{1}{(x_i - x_k)(x_i - x_l)} = 0$$

because the terms (i, k, l) and (k, i, l) are of opposite sign.

Finally

$$\sum_{i=1}^{p} x_i^2 \frac{(\partial^2 \varphi_i)/(\partial x_i^2)}{\varphi_i} = \sum_{\substack{i=1 \\ i\neq k\neq l}}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} x_i^2((x_i - x_k)(x_i - x_l))^{-1}$$

$$= \frac{p(p-1)(p-2)}{3}.$$

In fact, we have

$$\sum_{i=1}^{p} x_i^2 \frac{(\partial^2 \varphi_i)/(\partial x_i^2)}{\varphi_i} = \frac{1}{\varphi} \sum_{\substack{i=1 \\ i\neq k\neq l}}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} x_i^2 \varphi((x_i - x_k)(x_i - x_l))^{-1}$$

$$\equiv \frac{Q}{\varphi}$$

and it is clear that Q is antisymmetric of degree $p(p - 1)/2$; so it is of minimal degree (to be also antisymmetric) and it is then a multiple of φ. To evaluate the coefficient we compute the coefficient of

$$x_1^{p-1} x_2^{p-2} \dots x_{p-1}$$

to obtain $p(p - 1)(p - 2)/3$.

We put together all the formulas obtained

$$\Delta\left(\frac{\prod_{j=1}^{p} P^{(\alpha,\beta)}_{n_j+j-1}(x_j)}{\varphi}\right)$$

$$= \frac{\prod_{j=1}^{p} P^{(\alpha,\beta)}_{n_j+j-1}(x_j)}{\varphi}\left[-\sum_{i=1}^{p}(n_i + i - 1)(n_i + \alpha + \beta + i)\right.$$

$$\left. + (\alpha + \beta + 2)\frac{p(p - 1)}{2} + \frac{p(p - 1)(p - 2)}{3}\right]$$

the eigenvalue obtained is

$$-\sum_{i=1}^{p} n_i(n_i + \alpha + \beta + 2i - 1).$$

Now it is clear that if $\sigma \in S_p$, we have

$$\sum_{i=1}^{p} n_{\sigma(i)}(n_{\sigma(i)} + \alpha + \beta + 2\sigma(i) - 1)$$

$$= \sum_{i=1}^{p} n_i(n_i + \alpha + \beta + 2i - 1)$$

so that

$$\Delta p^{(\alpha,\beta,1/2)}_{(\mathbf{n})}(x) = \lambda^{(\alpha,\beta,1/2)}_{(\mathbf{n})} p^{(\alpha,\beta,1/2)}_{(\mathbf{n})}(x)$$

with

$$\lambda^{(\alpha,\beta,1/2)}_{(\mathbf{n})} = -\sum_{i=1}^{p} n_i(n_i + \alpha + \beta + 2i - 1)$$

which proves the first part of Theorem 1.

We want to compute the norm of

$$p^{(\alpha,\beta,1/2)}_{(\mathbf{n})}(x) \quad \text{in} \quad L^2(C_x, dV^{(\alpha,\beta,1/2)}).$$

Let us denote by A the operation of antisymmetrisation with respect to the group S_p acting by permutations on $X_1 \dots X_p$; then

$$\varphi(x)p^{(\alpha,\beta,1/2)}_{(\mathbf{n})}(x) = A\left(\prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j)\right)$$

and because of the expression (4.8) for $dV^{(\alpha,\beta,1/2)}$ and (4.17) of φ

$$(4.23)\quad \| p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x)\|^2_{L^2(C_x, dV^{(\alpha,\beta,1/2)})} = \left\| A\left(\prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j)\right)\right\|^2_{L^2(C_x, \Pi\mu_i^{(\alpha,\beta)}dx_i)}.$$

As in Section 2, we have

$$(Af|g)_{L^2([-1,+1]^p, \Pi_i\mu_i^{(\alpha,\beta)}dx_i)} = (Af|Ag)_{L^2(C_x, \Pi_i\mu_i^{(\alpha,\beta)}dx_i)} = (f|Ag)_{L^2([-1,+1]^p, \Pi_i\mu_i^{(\alpha,\beta)}dx_i)}$$

because

$$\int_{[-1,+1]^p} (Af)(x)\overline{g(x)} \prod_i \mu_i^{(\alpha,\beta)}(x_i)dx_i = \sum_{\sigma\in S_p} \int_{\sigma(C_x)} (Af)(x)\overline{g(x)} \prod_i \mu^{(\alpha,\beta)}(x_i)dx_i = \sum_{\sigma\in S_p} (-1)^\sigma \int_{C_x} f(x)\overline{g(\sigma x)} \prod_i \mu^{(\alpha,\beta)}(x_i)dx_i = (Af/Ag)_{L^2(C_x, \Pi\mu_i^{(\alpha,\beta)}dx_i)}.$$

Then

$$(4.24)\quad \left\| A\left(\prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j)\right)\right\|^2_{L^2(C_x, \Pi\mu_i^{(\alpha,\beta)}dx^i)} = \left(A\left(\prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j)\right)\right. \times \left|\left.\prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j)\right)_{L^2([-1,+1]^p, \Pi\mu_i^{(\alpha,\beta)}dx^i)}.$$

If $\sigma \neq \mathrm{Id}$, a term like

$$(4.25)\quad \int_{[-1,+1]^p} \prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_{\sigma(j)}) \times \prod_{j=1}^{p} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j) \prod_{j=1}^{p} \mu^{(\alpha,\beta)}(x_j)dx_j = 0$$

because for every j, one has to compute

$$(4.26)\quad \int_{-1}^{+1} P^{(\alpha,\beta)}_{(n_j+j-1)}(x_j) P^{(\alpha,\beta)}_{n_{\sigma(j)}+\sigma(j)-1}(x_j)\mu^{(\alpha,\beta)}(x_j)dx_j$$

and this is not 0 if and only if for every j

$$(*)\quad n_j = n_{\sigma(j)} + \sigma(j) - j.$$

But if $\sigma \neq \mathrm{Id}$, there exists at least one j such that $\sigma(j) > j$; then $n_j \leqq n_{\sigma(j)}$ and we arrive at contradiction with (*). This implies that if $\sigma \neq \mathrm{Id}$ for at least one j the integral (4.26) is 0 by orthogonality of the $P_n^{(\alpha,\beta)}(x_i)$ with respect to $\mu^{(\alpha,\beta)}(x_i)$ and so for $\sigma \neq \mathrm{Id}$, (4.25) is satisfied.

Using (4.24), (4.25) and (4.23), we deduce

$$\| p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x) \|^2_{L^2(C_x, dV^{(\alpha,\beta,1/2)})}$$

$$= \prod_{j=1}^{p} \| P_{n_j+j-1}^{(\alpha,\beta)} \|^2_{L^2([-1,+1],\mu^{(\alpha,\beta)}dx)} = 1$$

if the $P_{n_j+j-1}^{(\alpha,\beta)}$ are normalized with L^2 norm equal to 1.

7. *The heat kernel for* $\Delta^{(\alpha,\beta,1/2)}$ *on* E.

a) *The Cauchy problem on* E. We want now to find the fundamental solution of the Cauchy problem on the symmetric function $f(x)$:

$$(4.27) \quad \begin{cases} \dfrac{\partial f}{\partial t} = \Delta^{(\alpha,\beta,1/2)} f \\ f|_{t=0} = f_0. \end{cases}$$

We write this solution as

$$(4.28) \quad f(t, x) = \int_{C_{x'}} f_0(x') q_t(x, x') dV^{(\alpha,\beta,1/2)}(x')$$

where $q_t(x, x')$ is the fundamental solution of

$$\frac{\partial}{\partial t} = \Delta^{(\alpha,\beta,1/2)}$$

with respect to the volume element $dV^{(\alpha,\beta,1/2)}$.

b) But the $p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}$ are an orthonormal basis for

$$L^2(C_x, dV^{(\alpha,\beta,1/2)}(x));$$

we can write

$$(4.29) \quad q_t(x, x') = \sum_{\substack{(\mathbf{n}) \\ 0 \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_p}} e^{\lambda_{(\mathbf{n})}^{(\alpha,\beta,1/2)} t} p_{(\mathbf{n})}^{(\alpha,\beta,1/2)} p_{(\mathbf{n})}^{(\alpha,\beta,1/2)}(x')$$

$$= \frac{1}{\varphi(x)\varphi(x')} \sum_{\substack{(\mathbf{n}) \\ 0 \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_p}} e^{\lambda_{(\mathbf{n})}^{(\alpha,\beta,1/2)} t}$$

$$\times A_x \left(\prod_{j=1}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x_j) \right) A_{x'} \left(\prod_{j=1}^{p} P_{n_j+j-1}^{(\alpha,\beta)}(x'_j) \right)$$

where, as before, A_x and $A_{x'}$ denote antisymmetrisation in x and x' respectively. But by (4.20)

$$-\lambda_{(\mathbf{n})}^{(\alpha,\beta,1/2)} = \sum_{j=1}^{p} n_j(n_j + \alpha + \beta + 2j - 1)$$

$$= \sum_{j=1}^{p} [(n_j + j - 1)(n_j + j - 1 + \alpha + \beta + 1) - j^2 + \alpha + \beta - j(\alpha + \beta - 1)]$$

so that

$$(4.30) \qquad \lambda_{(\mathbf{n})}^{(\alpha,\beta,1/2)} = \sum_{j=1}^{p} \lambda_{n_j+j-1}^{(\alpha,\beta)} + K_p$$

where

$$K_p = -(\alpha + \beta) + (\alpha + \beta - 1)\frac{p(p+1)}{2} + \sum_{j=1}^{p} j^2$$

is a constant depending only on α, β, p and where

$$\lambda_k^{(\alpha,\beta)} = -k(k + \alpha + \beta + 1)$$

is the eigenvalue of $P_k^{(\alpha,\beta)}(x)$ for the operator (4.1).

Let us define $k_j = n_j + j - 1$; then we can rewrite (4.29) as

$$(4.31) \qquad q_t(x, x') = \frac{e^{K_p t}}{\varphi(x)\varphi(x')} \sum_{\substack{(\mathbf{k}) \\ 0 \leqq k_1 < k_2 < \ldots < k_p}} e^{\sum_{j=1}^{p} \lambda_{k_j}^{(\alpha,\beta)} t}$$

$$\times A_x\left(\prod_{j=1}^{p} P_{k_j}^{(\alpha,\beta)}(x_j)\right) A_{x'}\left(\prod_{j=1}^{p} P_{k_j}^{(\alpha,\beta)}(x'_j)\right).$$

But

$$A_x\left(\prod_{j=1}^{p} P_{l_j}^{(\alpha,\beta)}(x_j)\right) = 0$$

if two of the l'_j are equal, so that we can extend the summation to the case $0 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_p$.

Now let us take any p-uplet $(\mathbf{k}) = (k_1, \ldots, k_p) \in \mathbf{N}^p$. Then there exists a unique $\sigma \in S_p$ such that

$$\sigma(\mathbf{k}) = (k_{\sigma(1)} \ldots k_{\sigma(p)})$$

satisfies $0 \leqq k_{\sigma(1)} \leqq \ldots \leqq k_{\sigma(p)}$; then

$$A_x\left(\prod_{j=1}^{p} P_{k_j}^{(\alpha,\beta)}(x_j)\right) = (-1)^{\sigma} A_x\left(\prod_{j=1}^{p} P_{k_{\sigma(j)}}^{(\alpha,\beta)}(x_j)\right)$$

and in the summation (4.31) we can extend the summation to all $(\mathbf{k}) \in \mathbf{N}^p$ provided that we divide by $p!$, so that

$$q_t(x, x') = \frac{e^{K_p t}}{p!\varphi(x)\varphi(x')} A_x A_{x'} \sum_{(\mathbf{k}) \in \mathbf{N}^p} \prod_{j=1}^{p} \Big(e^{\lambda_{k_j}^{(\alpha,\beta)} t} \times P_{k_j}^{(\alpha,\beta)}(x_j) P_{k_j}^{(\alpha,\beta)}(x'_j) \Big)$$

$$(4.32) \qquad = \frac{e^{K_p t}}{p!\varphi(x)\varphi(x')} A_x A_{x'} \left\{ \prod_{j=1}^{p} \left(\sum_{k_j=0}^{+\infty} e^{\lambda_{k_j}^{(\alpha,\beta)} t} P_{k_j}^{(\alpha,\beta)}(x_j) P_{k_j}^{(\alpha,\beta)}(x'_j) \right) \right\}$$

$$q_t(x, x') = \frac{e^{K_p t}}{p!\varphi(x)\varphi(x')} A_x A_{x'} \left\{ \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x_j, x'_j) \right\}$$

where $p_t^{(\alpha,\beta)}(x_j, x'_j)$ denotes the heat kernel of the operator (4.11) (with respect to $\mu^{(\alpha,\beta)}(x_j)dx_j$).

Recalling Lemma 1 of 2 of Section 3 and the definition (4.17) of $\varphi(x)$, we see that when $x \to 1$ in C_x, (1 corresponds to all $q_i = 0$), we obtain

$$q_t(\mathbf{1}, x) = \frac{e^{K_p t}}{p!\varphi(x)} \lim_{x' \to \mathbf{1}} \left[\frac{A_{x'} A_x \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x_j, x'_j)}{\varphi(x')} \right]$$

$$= \frac{e^{K_p t}}{p!\varphi(x)} L_{x'} A_{x'} A_x \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x_j, x'_j) \,|_{x'_j = 1}$$

where

$$L_{x'} = \prod_{1 \leqq i < j \leqq p} \left(\frac{\partial}{\partial x'_i} - \frac{\partial}{\partial x'_j} \right)$$

is the simplest antisymmetric operator for the action of S_p and this is

$$(4.33) \qquad q_t(0, x) = \frac{e^{K_p t}}{\varphi(x)} \left[A_x L_{x'} \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x'_j, x_j) \right]_{x'_j = 1 \forall j}$$

and we thus obtain

THEOREM 2. *The heat kernel of problem* (4.27) *is given by formula* (4.32) *if* $x, x' \in C_x$ *and by* (4.33) *if* $x' = 1$, $x \in C_x$.

We see that the computation of the heat kernel for the operator $\Delta^{(\alpha,\beta,1/2)}$ on a euclidean space of dimension p is reduced to the computation of the heat kernel for the operator $L^{(\alpha,\beta)}$ in dimension 1 defined by (4.11).

5. Analysis on symmetric spaces in horospherical coordinates.

1. *The Iwasawa decompositions and horospherical coordinates.* a) We started in Section 1 with a semi simple Lie algebra $\mathfrak{G}$ and its complexified

form $\mathfrak{G}_{\mathbf{C}}$; then we defined for any complex root α in $\mathfrak{R}_{\mathbf{C}}$ a one-dimensional subspace $\mathfrak{G}_{\mathbf{C}}^{(\alpha)}$ of $\mathfrak{G}_{\mathbf{C}}$ and we defined a nilpotent Lie algebra

$$(5.1) \qquad \mathfrak{N}_{\mathbf{C}}^{(+)} = \sum_{\alpha \in \mathfrak{R}_{\mathbf{C}}^{+}} \mathfrak{G}_{\mathbf{C}}^{(\alpha)}$$

and its real part

$$\mathfrak{N}^{(+)} = \mathfrak{G} \cap \mathfrak{N}_{\mathbf{C}}^{(+)}.$$

Then $\mathfrak{G}$ is decomposed in

$$(5.2) \qquad \mathfrak{G} = \mathfrak{N}^{(+)} \oplus \mathfrak{A}_{\mathfrak{P}} \oplus \mathfrak{K} \quad \text{(Iwasawa decomposition).}$$

b) Then the symmetric space $X = G/K$ (of negative curvature) is $X = NA \cdot O$ where O is as usual the origin and N the nilpotent Lie group of the Lie algebra $\mathfrak{N}^{(+)}$. Any point $m \in X$ can be uniquely decomposed in (n, a) where $n \in N$ and $a \in A$ and $(n, \log a)$ are the horospherical coordinates of $m \in X$.

2. *The Laplace-Beltrami operator in horospherical coordinates*. a) The Laplace-Beltrami operator $1/2\Delta_2^{(X)}$ is the generator of a diffusion process on X, denoted $m_\omega(t)$ (ω being in some sample path) and so, we have only to compute the infinitesimal increment of this process in time dt to find $1/2\Delta_2^{(X)}$. As we want to write $\Delta_2^{(X)}$ in horospherical coordinates we shall denote

$$m_\omega(t) = (n_\omega(t), a_\omega(t)).$$

To be self contained we redo here the computation of **[22]** (in fact we do a slightly simpler computation because we do not need their full result).

b) We suppose that $m_\omega(0) = m_0$ so that

$$(n_\omega(0), a_\omega(0)) = (n_0, a_0)$$

(so that $m_0 = n_0 a_0 \cdot O$). The tangent space at O of X is exactly the space $\mathfrak{P}$ and it has an orthonormal basis $\epsilon_1, \ldots, \epsilon_n$ (where $n = \dim \mathfrak{P} = \dim X$) that we split in $\epsilon_1 = e_1, \ldots, \epsilon_p = e_p$ which is an orthonormal basis of $\mathfrak{A}$ and $\epsilon_{p+1}, \ldots, \epsilon_n$ which is an orthonormal basis of the orthogonal complement $\mathfrak{A}^{\perp}$ of $\mathfrak{A}$ inside $\mathfrak{P}$. Then it is well known, by **[22]**, that the stochastic process $m_\omega(t)$ is exactly

$$m_\omega(t) = g_\omega(t) \cdot O$$

where $g_\omega(t)$ is the so called horizontal process on G so that the logarithm of its increment is a white noise inside $\mathfrak{P}$, i.e.,

$$(5.3) \qquad g_\omega(t + dt)g_\omega(t)^{-1} = \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} + o(\sqrt{dt})\right)$$

where the G_q are gaussian random variables of mean 0 and variance 1. We can assume here that

(5.4) $\quad g_\omega(0) = n_0 a_0.$

c) We want to find the processes $(n_\omega(t), a_\omega(t))$ for $t = dt$ so that

(5.5) $\quad g_\omega(dt) \cdot O = n_\omega(dt) a_\omega(dt) \cdot O.$

We write

(5.6) $\quad n_\omega(dt) = n_0 \exp(\Delta n) \quad a_\omega(dt) = a_0 \exp(\Delta a).$

Call $\Delta' n = (\mathrm{Ad}\, a_0^{-1})\Delta n$, so that

$$\exp(\Delta n) a_0 = a_0 \exp(\Delta' n).$$

Then (5.5) becomes, after a simplification by $n_0 a_0$, (we take (5.6) and (5.4) into account)

$$\exp(\Delta' n)\exp(\Delta a) \cdot O = \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} + o(dt) \right). \tag{5.7}$$

But

$$\exp(\Delta' n)\exp(\Delta a) = \exp\left(\Delta' n + \Delta a + \frac{1}{2}[\Delta' n, \Delta a] \right) \tag{5.8}$$

(up to terms of order $o(dt)$, we shall from now forget $o(dt)$ once and for all). Now

$$\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} = \left(\sum_{q=1}^{p} e_q G_q + \sum_{q=p+1}^{n} \epsilon_q' G_q + \sum_{q=p+1}^{n} \epsilon_q'' G_q \right) \sqrt{dt} \tag{5.9}$$

where ϵ_q' (resp ϵ_q'') are for $q = r + 1, \dots, n$ the orthogonal projection of $\epsilon_q \in \mathfrak{A}^\perp$ into $\mathfrak{N}^{(+)}$ and $\mathfrak{K}$ respectively. $\mathfrak{A}^\perp$ is exactly $\mathfrak{N}^{(+)} \oplus \mathfrak{K}$ by Iwasawa's formula (5.2) and so we have

$$\begin{aligned} \text{(5.9)}' \quad & \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} \right) \\ &= \exp\left[\left(\sum_{q=1}^{p} e_q G_q + \sum_{q=p+1}^{n} \epsilon_q' G_q \right) \sqrt{dt} + \left(\sum_{q=p+1}^{n} \epsilon_q'' G_q \right) \sqrt{dt} \right] \end{aligned}$$

where we have separated the $\mathfrak{A} \oplus \mathfrak{N}^{(+)}$ part and the $\mathfrak{K}$ part. But for any $k \in \mathfrak{K}$ of the order of $O(\sqrt{dt})$

$$\begin{aligned} \text{(5.10)} \quad & \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} \right) O \\ &= \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} \right) \cdot \exp(k)\ O \end{aligned}$$

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$$= \exp\left(\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} + k + \frac{1}{2}\left[\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt}, k\right]\right) \cdot O.$$

Now we come back to (5.7) using (5.8) and (5.10) and a k such that the term inside the exponential in (5.10) has a projection into $\mathfrak{K}$ equal to 0. We deduce

$$(5.11) \quad \sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt} + k + \frac{1}{2}\left[\sum_{q=1}^{n} \epsilon_q G_q \sqrt{dt}, k\right]$$

$$= \Delta' n + \Delta a + \frac{1}{2}[\Delta' n, \Delta a].$$

We can then identify the term in $\mathfrak{K}$ of order $\sqrt{dt}$ in (5.11): it is, using (5.9),

$$\sum_{q=p+1}^{n} \epsilon_q'' G_q \sqrt{dt} + k$$

and we impose this to be 0 mod $O(dt)$; so

$$k = -\sum_{p+1}^{n} \epsilon_q'' G_q \sqrt{dt} + O(dt).$$

Then identifying the $\mathfrak{N}^{(+)}$ and $\mathfrak{A}$ projections of the equality (5.11) mod $O(dt)$ we have

$$(5.12) \quad \begin{cases} \sum_{p+1}^{n} \epsilon_q' G_q' \sqrt{dt} + n_1(dt) = \Delta' n \\ \sum_{1}^{p} e_q G_q \sqrt{dt} + a_1(dt) = \Delta a \\ -\sum_{p+1}^{n} \epsilon_q'' G_q \sqrt{dt} + k_1(dt) = k \end{cases}$$

where $n_1(dt)$, $a_1(dt)$, $k_1(dt)$ are of order dt in $\mathfrak{N}^{(+)}$, $\mathfrak{A}$ and $\mathfrak{K}$ respectively. We put (5.12) in (5.11) to obtain

$$0 = n_1(dt) + a_1(dt) - k_1(dt) + \frac{1}{2}\left[\sum_{p+1}^{n} \epsilon_q' G_q', \sum_{1}^{p} e_q G_q\right] dt$$

$$- \frac{1}{2}\left[\sum_{1}^{n} \epsilon_q G_q^{\sqrt{dt}}, -\sum_{q=p+1}^{n} \epsilon_q'' G_q \sqrt{dt}\right] (\text{mod } o(dt)).$$

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Now take the expectation: because the G_q are independent the fourth term is of 0 expectation and in the fifth term only

$$\sum_{p+1}^{n} \epsilon_q G_q$$

survives: so

$$(5.13) \qquad 0 = E(n_1(dt)) + E(a(dt)) - E(k_1(dt)) - \sum_{p+1}^{n} [\epsilon_q', \epsilon_q'']E(G_q^2)dt.$$

But ϵ_q' is in $\mathfrak{N}^{(+)}$ and ϵ_q'' is in $\mathfrak{K}$, so $[\epsilon_q', \epsilon_q'']$ is in $\mathfrak{A}$. Identifying the $\mathfrak{N}^{(+)}$, $\mathfrak{K}$ and $\mathfrak{A}$ parts of (5.13), we obtain

$$(5.14) \qquad \begin{cases} n_1(dt) = 0 \\ k_1(dt) = 0 \bmod O(dt) \\ a_1(dt) = \dfrac{1}{2} \displaystyle\sum_{q=p+1}^{n} [\epsilon_q', \epsilon_q'']dt \end{cases}$$

so that by (5.12) and (5.14) and the fact that $\Delta' n = (\mathrm{Ad}\, a_0^{-1})\Delta n$

$$(5.15) \qquad \begin{cases} \Delta a = \displaystyle\sum_{1}^{p} e_q G_q \sqrt{dt} + \dfrac{1}{2} \sum_{q=p+1}^{n} [\epsilon_q', \epsilon_q'']dt \\ \Delta n = (\mathrm{Ad}\, a_0)\left(\displaystyle\sum_{q=p+1}^{n} \epsilon_q' G_q \sqrt{dt} \right). \end{cases}$$

d) We now obtain from (5.15) the analytic expression of $\Delta_2^{(X)}$ in coordinates $(n, \log a)$ as follows:

$$(5.16) \qquad \Delta_2^{(X)} = \Delta_{\text{eucl}}^{(a)} + Z + \sum_{q=p+1}^{n} ((\mathrm{Ad}\, a)\tilde{\epsilon}_q')^2$$

with the following notations

$$\text{(i)} \qquad \Delta_{\text{eucl}}^{(a)} = \sum_{j=1}^{p} \frac{\partial^2}{\partial X_j^2}$$

is the usual euclidean Laplace operator on the euclidean space $\mathfrak{A} \equiv E$.

(ii) Z is the constant vector field on $\mathfrak{A} \equiv E$

$$(5.17) \qquad Z = \sum_{q=p+1}^{n} [\epsilon_q', \epsilon_q''].$$

(iii) At the point $(n, \log a) \in N \times A \simeq X$, we define the vector field $(\mathrm{Ad}\, a)\tilde{\epsilon}_q'$ for $q = p + 1, \ldots, n$ and

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$$\sum_{q=p+1}^{n} ((\mathrm{Ad}\ a)\tilde{\epsilon}'_q)^2$$

is the differential operator of second order which is the sum of the squares of these vector fields.

(iv) ϵ'_q and ϵ''_q are the projections of ϵ_q on $\mathfrak{N}^{(+)}$ and $\mathfrak{K}$, ϵ_q being an orthonormal basis of $\mathfrak{P}$ and in particular ϵ'_q generate in N a left invariant vector field denoted $\tilde{\epsilon}'_q$.

3. *Algebraic structure of the nilpotent algebra* $\mathfrak{N}^{(+)}$. We want to obtain a more explicit expression for the last operator appearing in (5.16). For a fixed $a \in A$, this last operator is a second order left-invariant operator in the nilpotent group N.

a) *Coming back to the complexified Lie algebras*. We have seen in (5.1) that

$$\mathfrak{N}_{\mathbf{C}}^{(+)} = \sum_{\alpha \in \mathfrak{R}_{\mathbf{C}}^{+}} \mathfrak{G}_{\mathbf{C}}^{(\alpha)}.$$

Moreover $\mathfrak{G}_{\mathbf{C}}^{(\alpha)}$ is a 1-dimensional complex vector space and we define

$$\mathfrak{G}_{\mathbf{C}}^{(\alpha)} = \mathbf{C}X_\alpha \cdot X_\alpha \in \mathfrak{G}_{\mathbf{C}}.$$

We define $\theta:\mathfrak{G}_{\mathbf{C}} \to \mathfrak{G}_{\mathbf{C}}$ the complex conjugation in $\mathfrak{G}_{\mathbf{C}}$ with respect to the real part $\mathfrak{G}$ of $\mathfrak{G}_{\mathbf{C}}$. It is known (see **[22]**, Lemma 4.3) that $(\mathfrak{A}^{\perp}{}_{\mathfrak{P}}\mathfrak{P})_{\mathbf{C}}$ (i.e., the complexified space of the orthogonal complement in $\mathfrak{P}$ of $\mathfrak{A}_{\mathfrak{P}}$) is generated by the elements $X_\alpha - \theta X_\alpha$. Then it is easy to see that the projection of $X_\alpha - \theta X_\alpha$ in $\mathfrak{N}_{\mathbf{C}}^{(+)}$ is $2X_\alpha$.

b) *Computation in the real nilpotent algebra* $\mathfrak{N}^{(+)}$. Now, two roots of the maximal abelian subalgebra $\mathfrak{H}_{\mathbf{C}}$ of $\mathfrak{G}_{\mathbf{C}}$ may have the same restriction to the $\mathfrak{A}_{\mathfrak{P}}$ part. Recall that here $\mathfrak{R}$ is the restricted root system of $\mathfrak{R}_{\mathbf{C}}$. We can then split $\mathfrak{N}^{(+)}$ in the eigenspaces of the roots

$$(5.18)\qquad \mathfrak{N}^{(+)} = \sum_{\alpha \in \mathfrak{R}^{+}} \mathfrak{N}^{(+)(\alpha)}$$

$\mathfrak{N}^{(+)(\alpha)}$ being the space of all X such that

$$(5.19)\qquad [h, X] = \alpha(h)X \quad \text{for any } h \in \mathfrak{A}.$$

Then $\mathfrak{N}^{(+)(\alpha)}$ is the vector space generated by the

$$\{X_\beta/\beta \in \mathfrak{R}_{\mathbf{C}} \text{ and } \beta|_{\mathfrak{A}} = \alpha\}.$$

We have

$$\dim_{\mathbf{R}} \mathfrak{N}^{(+),(\alpha)} = \rho_\alpha.$$

It is also known (see **[15]**) that

$$\begin{aligned}[\mathfrak{G}_{\mathbf{C}}^{(\alpha)}, \mathfrak{G}_{\mathbf{C}}^{(\beta)}] &\subset \mathfrak{G}_{\mathbf{C}}^{(\alpha+\beta)} && \text{if } \alpha + \beta \text{ is a root in } \mathfrak{R}_{\mathbf{C}} \\ &= 0 && \text{if } \alpha + \beta \text{ is not a root.}\end{aligned}$$

From this it follows that

$$(5.20) \qquad \begin{cases}[\mathfrak{N}^{(+)(\alpha)}, \mathfrak{N}^{(+)(\beta)}] \subset \mathfrak{N}^{(+)(\alpha+\beta)} & \text{if } \alpha + \beta \text{ is a root in } \mathfrak{R} \\ \qquad\qquad = 0 & \text{if not.}\end{cases}$$

c) *Decomposition of* $\mathfrak{N}^{(+)}$. We call a *fundamental root* a positive root $\alpha \in \mathfrak{R}$ which is not a non trivial linear combination of other roots. It is known that any positive root is a linear combination with positive or 0 integers coefficients of the fundamental roots and this decomposition is unique. We call $\mathfrak{F}^{+}$ the set of all fundamental roots.

We can decompose $\mathfrak{N}^{(+)}$ as follows:

$$(5.21) \qquad \mathfrak{N}^{(+)} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \mathfrak{V}_3 \oplus \dots$$

where the $\mathfrak{V}_j$ are vector subspaces defined as follows

$$(5.22) \qquad \begin{cases}\mathfrak{V}_1 = \bigoplus_{\alpha \in \mathfrak{F}^{+}} \mathfrak{N}^{(+),\alpha} \\ \mathfrak{V}_2 = \bigoplus_{\alpha \in (\mathfrak{F}^{+}+\mathfrak{F}^{+}) \cap \mathfrak{R}^{+}} \mathfrak{N}^{(+),\alpha} \\ \dots\dots\dots\dots\dots\dots\dots \\ \mathfrak{V}_s = \bigoplus_{\alpha \in \underbrace{(\mathfrak{F}^{+}+\dots+\mathfrak{F}^{+})}_{s \text{ times}} \cap \mathfrak{R}^{+}} \mathfrak{N}^{(+),\alpha}\end{cases}$$

where $(\mathfrak{F}^{+} + \dots + \mathfrak{F}^{+}) \cap \mathfrak{R}^{+}$ ($\mathfrak{F}^{+}$ taken s times) is the set of positive roots which can be written as the sum of s fundamental roots (not necessarily different). Then for $i, j > 0$ we have by (5.20)

$$(5.23) \qquad [\mathfrak{V}i, \mathfrak{V}_j] \subset \mathfrak{V}_{i+j}.$$

4. *Structure of the nilpotent part of* $\Delta_2^{(X)}$.

a) *Coordinates on* N. We shall define exponential coordinates on N as follows. For any $\alpha \in \mathfrak{R}^{+}$, the subspace $\mathfrak{N}^{(+),\alpha}$ is generated by $X_{\alpha,j}$, $j = 1, \dots, \rho_\alpha$ (these $X_{\alpha,j}$ are the X_β's in $\mathfrak{G}_{\mathbf{C}}$ such that $\beta|_{\mathfrak{A}} = \alpha$). Then any element in N can be written as

$$\exp\left(\sum_{\alpha \in \mathfrak{R}^{+}} \sum_{j=1}^{\rho_\alpha} \xi_{\alpha,j} X_{\alpha,j}\right)$$

where $(\xi_{\alpha,j})$ is the set of exponential coordinates.

Among these coordinates, we distinguish a special subset that we call *fundamental coordinates* which are the $\xi_{\alpha,j}$ for $\alpha \in \mathfrak{F}^{+}$, $j = 1, \dots, \rho_\alpha$ (they are the coordinates on $\mathfrak{V}_1$). The multiplication law in N can be expressed as

$$(\xi'_{\alpha,j}) \cdot (\xi_{\alpha,j}) = (\xi''_{\alpha,j})$$

where

$$(5.24) \quad \exp(\textstyle\sum \xi'_{\alpha,j}X_{\alpha,j}) \exp(\sum \xi_{\alpha,j}X_{\alpha,j}) = \exp(\sum \xi''_{\alpha,j}X_{\alpha,j}).$$

By Campbell-Hausdorff, the first member is

$$(5.25) \quad \exp\left[\sum_{\alpha,j} (\xi_{\alpha,j} + \xi'_{\alpha,j})X_{\alpha,j} + \frac{1}{2}\sum_{\alpha,j}\sum_{\beta,k} \xi'_{\alpha,j}\xi_{\beta,k}[X_{\alpha,j}, X'_{\beta,k}] + \dots\right]$$

where the dots in (5.25) denote terms containing at least two brackets of the $X_{\alpha,j}$. In particular, if α is a fundamental root, $X_{\alpha,j}$ cannot appear in the bracket or in the dot in this last expression (5.25). So we obtain by identifying to (5.24)

$$(5.26) \quad \xi''_{\alpha,j} = \xi'_{\alpha,j} + \xi_{\alpha,j} \quad \text{if } \alpha \in \mathfrak{F}^+.$$

Then if $\alpha \in (\mathfrak{F}^+ + \mathfrak{F}^+) \cap \mathfrak{R}^+$ and is of the form $\alpha = \beta + \gamma$ where $\beta, \gamma \in \mathfrak{F}^+$, we define

$$[X_{\beta,j}, X_{\gamma,k}] = \sum_l C_{j,k}^{\beta,\gamma,l} X_{\alpha,l}$$

(where we use (5.20)) and $C_{j,k}^{\beta,\gamma,l}$ are constants. We can identify (5.25) and (5.24) for $\alpha \in (\mathfrak{F}^+ + \mathfrak{F}^+) \cap \mathfrak{R}^+$ to obtain

$$(5.27) \quad \xi''_{\beta+\gamma,l} = \xi'_{\beta+\gamma,l} + \xi_{\beta+\gamma,l} + \frac{1}{2}\sum_{j,k} (\xi'_{\beta,j}\xi_{\gamma,k} - \xi'_{\gamma,k}\xi_{\beta,j})C_{jk}^{\beta,\gamma,l} \quad \text{for } \beta, \gamma \in \mathfrak{F}^+$$

(because the higher brackets denoted by dots in (5.25) do not give contribution to the computation of these terms).

b) *Left invariant vector fields on N.* Let $\widetilde{X}_{\alpha,j}$ be the left invariant vector field on N associated to $X_{\alpha,j}$; if f is a function on N and if $(\xi_{\alpha,j}) = \xi$ is a point in N, we have by definition

$$(\widetilde{X}_{\alpha,j}f)(\xi) = \frac{\partial}{\partial \xi'_{\alpha,j}} f(\xi' \cdot \xi)\big|_{\xi'=0}.$$

If α is a fundamental root, we deduce from (5.26), (5.27)

$$(5.28) \quad (\widetilde{X}_{\alpha,j}f)(\xi) = \frac{\partial f}{\partial \xi_{\alpha,j}}(\xi) + \frac{1}{2}\sum_{\gamma,k,l} (C_{jk}^{\alpha,\gamma,l} - C_{kj}^{\gamma,\alpha,l})\xi_{\gamma,k}\frac{\partial f}{\partial \xi_{\alpha+\gamma,l}} + \dots$$

where the dots in (5.28) involve derivatives $\partial/\partial\xi_{\beta,r}$ where β is a root which is a combination of three or more fundamental roots. (i.e., $X_{\beta,r} \in \mathfrak{V}_3 \oplus \mathfrak{V}_4 \oplus \dots$). We have also that if $\alpha = \beta + \gamma$ where β, γ are fundamental roots

(5.29) $$(\widetilde{X}_{\beta+\gamma,j}f)(\xi) = \frac{\partial f}{\partial \xi_{\beta+\gamma,l}}(\xi) + \dots$$

with the same meaning as before for the dots in (5.29) and in general $\widetilde{X}_{\beta,j}f$ involves only derivatives of f with respect to the $\xi_{\sigma,k}$ where $X_{\sigma,k} \in \mathfrak{V}_s$ and $s \geqq s_0$ and $X_{\beta,j} \in \mathfrak{V}_{s_0}$ (i.e., σ is a root which is longer than β).

c) *The vector fields $\widetilde{\xi}'_q$ for $q = p + 1, \dots, n$ and the nilpotent part of $\Delta_2^{(X)}$.* Recall that the $(X_\alpha - \theta X_\alpha)_{\alpha \in \mathfrak{R}_C^+}$ give a basis of $\mathfrak{A}_{\mathfrak{P}}^{\perp}\mathfrak{P}$ (orthogonal complement of $\mathfrak{A}_{\mathfrak{P}}$ in $\mathfrak{P}$). They are not orthonormal, (but see **[22]**, Lemma 4.3 of Chapter II); also we have that the orthogonal projection on $\mathfrak{N}^{(+)}$ of $X_\alpha - \theta X_\alpha$ is $2X_\alpha$; finally the ξ'_q appear as linear combinations of the $2X_{\alpha,j}$ and the $\widetilde{\xi}'_q$ as the same combinations of the $2\widetilde{X}_{\alpha,j}$. Now the action of Ad a on $X_{\alpha,j}$ is obvious; in fact

$$\text{Ad } a = \exp(\text{ad} \log a),$$

and so

(5.30) $$(\text{Ad } a)(\widetilde{X}_{\alpha,j}) = \exp[\,(\alpha, \log a)\,] \cdot \widetilde{X}_{\alpha,j}$$

because

$$(\text{ad} \log a)X_{\alpha,j} = [\log a, x_{\alpha,j}] = (\alpha, \log a)X_{\alpha,j}$$

by definition of the root α.

Let us write now

(5.31) $$\widetilde{\xi}'_q = \sum_{\alpha,j} d_q^{\alpha,j}\widetilde{X}_{\alpha,j}$$

where the $d_q^{\alpha j}$ are constants. Then by (5.30),

$$(\text{Ad } a)\widetilde{\xi}'_q = \sum_{\alpha,j} d_q^{\alpha j} \exp[\,(\alpha, \log a)\,]\widetilde{X}_{\alpha,j}$$

and so

(5.32) $$\sum_{q=p+1}^{n} (\,(\text{Ad } a)\widetilde{\xi}'_q)^2 = \sum_{q=p+1}^{n} \left(\sum_{\alpha,j} d_q^{\alpha j} \exp[\,(\alpha, \log a)\,]\widetilde{X}_{\alpha,j}\right)^2$$

and when we compute the squares in (5.32) we can consider the

$$d_q^{\alpha j} \exp[\,(\alpha, \log a)\,]$$

as constants with respect to the differential operators $X_{\beta,k}$.

5. *Restriction of the nilpotent part of $\Delta_2^{(X)}$ to a special class of functions.* Suppose now that we want to compute $\Delta_2^{(X)}f$ on the function f on $X = N \cdot A \cdot O$ in the horospherical coordinates. Then f becomes a function $f(n, h)$ where $n \in N$, $h \in \Lambda$ (so that the corresponding point of X is $m = ne^h \cdot O$).

We define the following class of functions on X. We say that a function f in X is a *fundamental function* if, in horospherical coordinates on X, f depends only on the $\mathfrak{A}$ part and on the fundamental coordinates

$$(\xi_{\alpha,j})_{\alpha\in\mathfrak{F}^+,j=1\dots\rho_\alpha}$$

of the exponential chart of N.

In particular, we see that on such fundamental function we have from (5.28)

$$(5.33)\qquad (\widetilde{X}_{\alpha,j}f)(\xi, h) = \frac{\partial f}{\partial\xi_{\alpha,j}}(\xi, h) \quad \text{if } \alpha \in \mathfrak{F}^+$$

and from (5.29) and its obvious generalization

$$(5.34)\qquad (\widetilde{X}_{\beta,j}f)(\xi, h) = 0 \quad \text{if } \alpha \in \mathfrak{R}^+ \cap \complement\mathfrak{F}^+.$$

In that case the nilpotent part in (5.32) becomes extremely simple namely

$$(5.35)\qquad \sum_{q=p+1}^{n} (\mathrm{Ad}\ a\widetilde{\xi}'_q)^2 f = \sum_{q=p+1}^{n}\left(\sum_{\alpha\in\mathfrak{F}^+}\sum_{j=1}^{\rho_\alpha} d_q^{\alpha j}e^{(\alpha,\log a)}\frac{\partial}{\partial\xi_{\alpha,j}}\right)^2 f.$$

In particular, we take a fundamental function

$$f = f(\,(\xi_{\alpha,j})_{\alpha\in\mathfrak{F}^+,j=1\dots\rho_\alpha}, h)$$

and we define its partial Fourier transform with respect to the fundamental variables $(\xi_{\alpha,j})_{\alpha\in\mathfrak{F}^+}$ as

$$(5.36)\qquad \hat{f}(\,(\hat{\xi}_{\alpha,j}), h) = \int e^{i\sum_{\alpha\in\mathfrak{F}^+}\sum_{j=1}^{\rho_\alpha}\hat{\xi}_{\alpha,j}\xi_{\alpha,j}} f(\,(\xi_{\alpha,j}), h)\prod_{\alpha\in\mathfrak{F}^+}\prod_{j=1}^{\rho_\alpha} d\xi_{\alpha,j}.$$

Then we obtain if $a = \exp h$ from (5.35)

$$(5.37)\qquad \left(\sum_{q=p+1}^{n} (\mathrm{Ad}\ a\widetilde{\xi}'_q)^2 f\right)^{\wedge}(\,(\hat{\xi}'_{\alpha,j}), h) = -\sum_{q=p+1}^{n}\left(\sum_{\alpha\in\mathfrak{F}^+}\sum_{j=1}^{\rho_\alpha} d_q^{\alpha j}\hat{\xi}_{\alpha,j}e^{(\alpha,\log a)}\right)^2 \hat{f}(\,(\hat{\xi}_{\alpha,j}), h).$$

Remark. It is clear that the class of fundamental functions is intrinsically defined; we could have defined such a function f by saying that

$$f(n_0 \cdot m) = f(m)$$

for any $m \in X$ and any $n_0 \in \exp(\mathfrak{V}_2 \oplus \mathfrak{V}_3 \oplus \dots) \subset N$;

in horospherical coordinates, this means that

$$f(n_0 n, h) = f(n, h) \quad \text{for any } n_0 \in \exp(\mathfrak{V}_2 \oplus \mathfrak{V}_3 \oplus \dots);$$

this implies that $f(n, h)$ depends only on the fundamental coordinates of n. It is also clear that the operator $\Delta_2^{(X)}$ transform a fundamental function into a fundamental functions and that the heat semi-group $e^{t\Delta_2^{(X)}}$ leaves also invariant the class of fundamental functions.

6. *The volume element in horospherical coordinates and the vector Z.* a) We shall need in the next chapter an explicit analytical expression of the volume element of the symmetric space X in horospherical coordinates. Let $m = (n, \log h)$ the horospherical coordinates of a point m in X; here n is in the nilpotent group N, and h is in the abelian group A and $\log h$ is an element of $\mathfrak{A}_{\mathfrak{P}}$. This means that if O is the fixed origin of X,

$$m = n \cdot h \cdot o = n \cdot \exp(\log h) \cdot o.$$

We start with a volume element $dv(0)$ at point o; then $dv(m)$ at point m is obtained by transport of $dv(0)$ by $n \cdot \exp(\log h)$. Now, let $n' = 1 + \nu + \dots$ and $h' = 1 + \epsilon + \dots$ be very small elements of N and A. Then $n \cdot h$ acts on $n'h'o$ (which is very near o) by

$$n \cdot hn'h'o = (nhn'h^{-1}) \cdot (hh') \cdot o. \tag{5.38}$$

But

$$\begin{aligned} hn'h^{-1} &= \exp(\operatorname{ad} \log h) \cdot n' \\ &= 1 + \exp(\operatorname{ad} \log h) \cdot \nu + \dots \end{aligned}$$

But

$$\nu = \sum_{\alpha \in \Re^+} \sum_{j=1}^{\rho_\alpha} \nu_{\alpha,j} \quad \text{where } \nu_{\alpha,j} \in \mathfrak{G}^{(\alpha)}$$

and then by definition

$$\exp(\operatorname{ad} \log h)\nu = \sum_{\alpha \in \Re^+} \sum_{j=1}^{\rho_\alpha} e^{(\alpha, \log h)} \nu_{\alpha,j}.$$

This means that the jacobian of the mapping

$$n' \to hn'h^{-1}$$

at $n' = 1$ is exactly

$$\prod_{\alpha \in \mathfrak{R}^+} \prod_{j=1}^{\rho_\alpha} \exp(\alpha, \log h)$$

and this is

$$\text{(5.39)} \qquad \exp\left(\sum_{\alpha \in \mathfrak{R}^+} \rho_\alpha \alpha, \log h\right).$$

It is then clear by (5.1) and (5.2) that the jacobian of the translation by $n \cdot h$ at point o is given by (5.2); this implies immediately:

LEMMA 1. *The volume element of X at point $m = (n, \log h)$ in horospherical coordinates is*

$$\text{(5.40)} \qquad dv(m) = \exp\left(- \sum_{\alpha \in \mathfrak{R}^+} \rho_\alpha \alpha, \log h\right) d(\log h) dn$$

where $d(\log h)$ is the usual Lebesgue measure on $\mathfrak{A}_{\mathfrak{P}}$ and dn is the invariant measure on N.

b) We can also compute the measure dn in the exponential chart given previously. For any $\alpha \in \mathbf{R}^+$, any $j = 1 \dots \rho_\alpha$ the multiplicative law of N in exponential coordinates $(\xi_{\alpha,j})$ is given by

$$\xi''_{\alpha,j} = \xi_{\alpha,j} + \xi'_{\alpha,j} + \dots$$

where the dots indicate expressions in the $\xi_{\beta,l}$ and $\xi'_{\beta,l}$ involving only roots which are strictly smaller than α (i.e., roots β such that $\alpha = \beta +$ other positive roots). This implies that the translation (left or right) by a fixed element of N has a jacobian matrix (in these exponential coordinates) which is upper triangular with entries 1 on the diagonal, and then dn is the Lebesgue measure

$$\text{(5.41)} \qquad dn = \prod_{\alpha \in \mathfrak{R}^+} \prod_{j=1}^{\rho_\alpha} d\xi_{\alpha,j}$$

in the $\xi_{\alpha,j}$.

c) *We can now come back to the Laplace-Beltrami operator* $\Delta_2^{(X)}$. We know that $\Delta_2^{(X)}$ is self-adjoint with respect to the volume element. In particular, the abelian part $\Delta + Z$ must be self-adjoint with respect to the abelian part of the volume element, namely

$$\exp\left(- \sum_{\alpha \in \mathfrak{R}^+} \sum_{j=1}^{\rho_\alpha} \rho_\alpha \alpha, \log h\right) d(\log h).$$

This implies that Z must be necessarily

$$\text{(5.43)} \qquad Z = - \sum_{\alpha \in \mathfrak{R}^+} \rho_\alpha \alpha$$

so if we compare to the expression of Z given in (5.17),

LEMMA 2. *$-Z$ is the sum of the positive roots counted with their multiplicities. In particular we see that we obtain the following identity: if $(\epsilon_k)_{k=p+1,\dots,n}$ is an orthonormal basis of the orthogonal complementary $\mathfrak{A}_{\mathfrak{P}}^{\perp}$ of $\mathfrak{A}_{\mathfrak{P}}$ in $\mathfrak{P}$, then*

$$\sum_{\alpha \in \mathfrak{R}^+} \rho_\alpha \alpha = -\sum_{k=p+1}^{n} [\epsilon_k', \epsilon_k'']$$

where $\epsilon_k = \epsilon_k' + \epsilon_k''$ and $\epsilon_k' \in \mathfrak{N}^{(+)}$, $\epsilon_k'' \in \mathfrak{K}$.

References for Section 5. Information about Iwasawa decomposition is given in [**15**]. Karpelevic gives expression for the abelian part of the Laplace operator [**17**]. M. P. Malliavin and P. Malliavin give the complete expression of the Laplace operator in horospherical coordinates [**22**].

6. Quantization of the open Toda lattice.

1. *The structure of the Lie algebra $\mathscr{SL}(p+1, \mathbf{C})$.* a) $\mathscr{SL}(p+1, \mathbf{C})$ is the Lie algebra of complex $(p+1) \times (p+1)$ matrices with trace 0. If M, M' are two such matrices, the Killing scalar product is

(6.2) $\quad (M|M') = \operatorname{Tr} M'M.$

A Cartan subalgebra of $\mathscr{SL}(p+1, \mathbf{C})$ is the abelian algebra $\mathfrak{X}$ of diagonal matrices of trace 0.

A maximal compact subalgebra is the algebra $\mathfrak{K} = \mathscr{SU}(p+1)$ of antihermitian matrices of trace 0 and we can write

$$\mathscr{SL}(p+1, \mathbf{C}) = \mathfrak{K} \oplus \mathfrak{P}$$

where $\mathfrak{P}$ is the vector space of hermitian matrices of trace 0. Then the algebra $\mathfrak{A}_{\mathfrak{P}}$ is the set of diagonal matrices of trace 0 with real elements on the diagonal.

The nilpotent algebra $\mathfrak{N}^{(+)}$ is the algebra of upper triangular matrices with 0 on the diagonal

a) If h is an element in $\mathfrak{X}$, h has diagonal elements denoted $q_i/\sqrt{2}$ with

$$\sum_{i=1}^{p+1} q_i = 0.$$

The roots are the linear forms on $\mathfrak{X}$

(6.2) $$R_{ij}(h) = \frac{1}{\sqrt{2}}(q_i - q_j) \quad (i \neq j)$$

such that if M_{ij} is the matrix with 1 on the i^{th} line and j^{th} column and 0 elsewhere, we have

$$[h, M_{ij}] = \frac{1}{\sqrt{2}}(q_i - q_j)M_{ij}.$$

The positive roots are the one with $i < j$ and they correspond to the nilpotent algebra $\mathfrak{N}^+$; we have the decomposition

$$\mathfrak{N}^{(+)} = \bigoplus_{i<j} \mathfrak{G}_{(R_{ij})}$$

where $\mathfrak{G}_{(R_{ij})}$ is the vector space $\mathbf{C}M_{ij}$ considered as a two dimensional real space. The roots are of multiplicity 2.

c) $\mathfrak{P}$ is considered as the tangent space at point 0 on

$$X = SL(p + 1, \mathbf{C})/SU(p + 1);$$

its metric at point 0 is given by the Killing form so that for $M \in \mathfrak{P}$, we have with $M = (m_{ij})$

$$\|M\|^2 = \operatorname{Tr} M^2 = \sum_{i,j} m_{ij}m_{ji} = \sum|m_{ij}|^2$$

because M is hermitian. This has to be considered as a real scalar product. An orthonormal basis of $\mathfrak{P}$ is then given by the following elements: the matrices P_{kl}, Q_{kl}

$$(6.3) \qquad P_{kl} = \begin{array}{c c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} & (\sqrt{2})^{-1} \\ (\sqrt{2})^{-1} & \end{pmatrix} \end{array} \text{ (zeros elsewhere)}$$

$$Q_{kl} = \begin{array}{c c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} & i(\sqrt{2})^{-1} \\ -i(\sqrt{2})^{-1} & \end{pmatrix} \end{array} \text{ (zeros elsewhere) for } k < l$$

and an orthonormal basis (e_k) of $\mathfrak{A}_{\mathfrak{P}}$, $k = 1 \dots p$. Because of the computations given in the preceding chapter, we need to compute the projection of the P_{kl} and Q_{kl} on $\mathfrak{N}^+$, which we call P'_{kl} and Q'_{kl} in accordance to the notation of Section 5. We have trivially

$$(6.4) \qquad P'_{kl} = \sqrt{2}M_{kl} \quad Q'_{kl} = \sqrt{2}iM_{kl}$$

because $P_{kl} - P'_{kl}$ and $Q_{kl} - Q'_{kl}$ are antihermitian matrices and are in $\mathfrak{K}$.

2. *The Laplace operator acting on fundamental functions on $SL(p + 1)$; $\mathbf{C}/Su(p + 1)$*. In our case the fundamental roots are the roots $R_{j,j+1}$ for $j = 1, \dots, p$; for each i, the fundamental coordinates of an element of nilpotent group N are $2p$ real coordinates ξ_{j,η_j} which define the general element of the corresponding root space $\mathfrak{G}_{(R_{j,j+1})}$ (as a 2-dimensional real vector space) by

$(\xi_j + i\eta_j)M_{j,j+1}$.

Now we take the orthonormal basis P_{kl} and Q_{kl} $(k < l)$ of $\mathfrak{P}$. The only P'_{kl} and Q'_{kl} corresponding to the fundamental variables will be $P'_{j,j+1}$ and $Q'_{j,j+1}$ because of (6.4) and they correspond to the vector fields

$$\sqrt{2}\frac{\partial}{\partial \xi_j} + \dots \quad \text{and} \quad \sqrt{2}\frac{\partial}{\partial \xi_j} + \dots$$

respectively, where, as usual the dots are derivatives with respect to non fundamental variables. Finally the Laplace operator in horospherical coordinates acting on fundamental functions f will be

$$(6.5) \qquad \Delta_2^{(X)}f = \Delta f + Zf + 2\sum_{j=1}^{p} e^{2(q_j - q_{j+1})/\sqrt{2}}\left(\frac{\partial^2}{\partial \xi_j^2} + \frac{\partial^2}{\partial \eta_j^2}\right)f$$

and by Fourier transforming the fundamental variables we obtain

$$(6.5)' \qquad \hat{\Delta}_2^{(X)}\hat{f} = \Delta \hat{f} + Z\hat{f} - 2\sum_{j=1}^{p} e^{2(R_j,X)}(\hat{\xi}_j^2 + \hat{\eta}_j^2)f.$$

Here, as we have seen in Section 5, $-Z$ is the constant vector field on $\mathfrak{A}$ which is the sum of the positive roots. We can write

$$(\Delta + Z)\hat{f} = e^{-(1/2)(Z,q)}\Delta(e^{(1/2)(Z,q)}\hat{f}) - \frac{1}{4}\|Z\|^2\hat{f}$$

and the heat equation

$$(6.6) \qquad \begin{cases} \left.\dfrac{\partial \hat{f}}{\partial t}\right|_{t=0} = \hat{\Delta}_2^{(X)}\hat{f} \\ \hat{f}|_{t=0} = \hat{f}_0 \end{cases}$$

becomes

$$(6.7) \qquad \begin{cases} \dfrac{\partial \hat{g}}{\partial t} = \Delta \hat{g} - 2\displaystyle\sum_{j=1}^{p} \exp(\sqrt{2}(q_j - q_{j+1}))(\hat{\xi}_j^2 + \hat{\eta}_j^2)\hat{g} \\ \hat{g}|_{t=0} = \hat{g}_0 \end{cases}$$

where $\hat{f}$ denotes, as in Section 5, the Fourier transform of f with respect to the fundamental variables and where

$$(6.8) \qquad \begin{cases} \hat{g} = e^{(1/2)(Z,q)+(1/4)\|Z\|^2 t}\hat{f} \\ \hat{g}_0 = e^{(1/2)(Z,q)}\hat{f}_0. \end{cases}$$

3. *The fundamental solution of the problems* (6.7) *and* (6.6).

a) *The heat kernel on X*. We call, as in Section 3, $p(m^{(1)}, t|m^{(0)})$ the heat kernel on

$$X \equiv SL(p + 1, \mathbf{C})/SU(p + 1)$$

with respect to the volume element $dv(m^{(0)})$ of X. This means that the function

$$(6.9) \qquad f(t, m^{(1)}) = \int_X p(m^{(1)}, t|m^{(0)})f_0(m^{(0)})dv(m^{(0)})$$

satisfies the heat equation

$$(6.10) \qquad \begin{cases} \dfrac{\partial}{\partial t}f(t, m^{(1)}) = \Delta_2^{(X)}f(t, m^{(1)}) \\ f(0, m^{(0)}) = f_0(m^{(0)}). \end{cases}$$

We also know by Section 5 that this heat kernel respects the class of fundamental functions; this means that if we take a function

$$f_0(q^{(0)}, (\xi^{(0)}), (\eta^{(0)}))$$

on X depending only of the abelian part of the Iwasawa decomposition and of the fundamental variables $(\xi_j^{(0)})$ and $(\eta_j^{(0)})$ of the nilpotent group, then its transform by (6.9) is a function

$$f(t, q^{(1)}, (\xi^{(1)}), (\eta^{(1)}))$$

depending only of the coordinates $(q^{(1)}, (\xi^{(1)}), (\eta^{(1)}))$ of $m^{(1)}$.

We also know that the volume element is in horospherical coordinates

$$dv(m^{(0)}) = e^{(Z, q^{(0)})}dq^{(0)}dn$$

where dn is the invariant measure on N, and in our case it is easy to see that dn is the Lebesgue measure with respect to the exponential chart. For fundamental functions, we can rewrite (6.9) as

$$(6.11) \qquad \begin{aligned} &f(t, q^{(1)}, (\xi^{(1)}), (\eta^{(1)})) \\ &= \int_{\mathfrak{A} \times R^{2p}} K(q^{(1)}, (\xi^{(1)}), (\eta^{(1)}), t|q^{(0)}, \xi^{(0)}, \eta^{(0)})f_0(q^{(0)}, \xi^{(0)}, \eta^{(0)}) \\ &\times e^{(Z, q^{(0)})}dq^{(0)}d\xi^{(0)}d\eta^{(0)} \end{aligned}$$

with

$$d\xi^{(0)} = \prod_{j=1}^{p} d\xi_j^{(0)} \quad d\eta^{(0)} = \prod_{j=1}^{p} d\eta_j^{(0)}$$

and

$$(6.12) \qquad K(q^{(1)}, \xi^{(1)}, \eta^{(1)}, t|q^{(0)}, \xi^{(0)}, \eta^{(0)}) = \int p(m^{(1)}, t|m^{(0)})d\lambda(m^{(0)})$$

where $d\lambda(m^{(0)})$ is the Lebesgue measure with respect to all the non-fundamentals exponential coordinates on N.

The function (6.11) satisfies

(6.13) $$\begin{cases} \dfrac{\partial f}{\partial t} = (\Delta + Z)f + 2\displaystyle\sum_{j=1}^{p} \exp(\sqrt{2}(q_j - q_{j+1}))\left(\dfrac{\partial^2}{\partial \xi_j^2} + \dfrac{\partial^2}{\partial \eta_j^2}\right)f \\ f|_{t=0} = f_0(q^{(0)}, \xi^{(0)}, \eta^{(0)}). \end{cases}$$

If we do the Fourier transform in the fundamental coordinates and define

$$\hat{f}(t, q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}) = \int f(t, q^{(1)}, \xi^{(1)}, \eta^{(1)}) e^{i(\sum_{j=1}^{p} \xi_j^{(1)}\hat{\xi}_j^{(1)} + \eta_j^{(1)}\hat{\eta}_j^{(1)})} d\xi^{(1)} d\eta^{(1)}$$

we see that $\hat{f}$ satisfies (6.6) and is

(6.14) $$\hat{f}(t, q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}) = \int_{\mathfrak{A}\times \mathbf{R}^{2p}} \hat{\hat{K}}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t|q^{(0)}, \xi^{(0)}, \eta^{(0)}) \times \hat{f}_0(q^{(0)}, \hat{\xi}^{(0)}, \hat{\eta}^{(0)}) e^{(Z,q^{(0)})} dq^{(0)} d\xi^{(0)} d\eta^{(0)}$$

where

(6.15) $$\hat{\hat{K}}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t|q^{(0)}, \hat{\xi}^{(0)}, \hat{\eta}^{(0)}) = \frac{1}{(2\pi)^{2p}} \int K(q^{(1)}, \xi^{(1)}, \eta^{(1)}, t|q^{(0)}, \xi^{(0)}, \eta^{(0)}) \times \exp i \sum_{j=1}^{p} (\xi_j^{(1)}\hat{\xi}_j^{(1)} + \eta_j^{(1)}\hat{\eta}_j^{(1)} - \xi_j^{(0)}\hat{\xi}_j^{(0)} - \eta_j^{(0)}\hat{\eta}_j^{(0)}) \times d\xi^{(1)} d\eta^{(1)} \times d\xi^{(0)} d\eta^{(0)}.$$

b) Let us now suppose that

(6.16) $$\hat{f}_0(q^{(0)}, \hat{\xi}^{(0)}, \hat{\eta}^{(0)}) = \varphi_0(q^{(0)})$$

depends only on $q^{(0)}$. Then the corresponding f_0 is

(6.17) $$f_0(q^{(0)}, \xi^{(0)}, \eta^{(0)}) = \varphi_0(q^{(0)}) \delta(\xi^{(0)}) \delta(\eta^{(0)})$$

and from (6.11) (or (6.14)) we have

(6.18) $$\hat{f}(t, q^{(1)}, (\hat{\xi}^{(1)}), (\hat{\eta}^{(1)})) = \int_{\mathfrak{A}} \hat{K}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t|q^{(0)}) \varphi_0(q^{(0)}) e^{(Z,q^{(0)})} dq^{(0)}$$

where

(6.19) $$\hat{K}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t|q^{(0)})$$

$$= \int K(q^{(1)}, (\xi^{(1)}), (\eta^{(1)}), t | q^{(0)}, 0, 0)$$
$$\times e^{i(\Sigma_{j=1}^{p}) \xi_j^{(1)} \hat{\xi}_j^{(1)} + \eta_j^{(1)} \hat{\eta}_j^{(1)})} d\xi^{(1)} d\eta^{(1)}.$$

c) Finally the solution of problem (6.7) is given by

THEOREM 1. *The solution of problem* (6.7) *is the integral formula*

$$\hat{g}(t, q^{(1)}) = \int_{\mathfrak{A}} G(t, q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)} | q^{(0)}) \hat{g}_0(q^{(0)}) dq^{(0)} \tag{6.20}$$

with

$$\begin{aligned} &G(t, q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)} | q^{(0)}) \\ &= \hat{K}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t | q^{(0)}) e^{(1/2)(Z, q^{(1)} + q^{(0)}) + (1/4)\|Z\|^2 t} \end{aligned} \tag{6.21}$$

and $\hat{K}$ *is given by the Fourier transform* (6.19) *of* K *defined by the integral* (6.12).

Proof. We use (6.8); this formula converts problem (6.7) into problem (6.6) for which we have the solution given by formula (6.18) (in which we take $\varphi_0(q^{(0)}) = \hat{f}_0(q^{(0)})$). Then

$$\begin{aligned} \hat{g}(t, q^{(1)}) &= e^{(1/2)(Z, q^{(1)}) + (1/4)\|Z\|^2 t} \hat{f}(t, q^{(1)}, (\hat{\xi}^{(1)}), (\hat{\eta}^{(1)})) \\ &= e^{(1/2)(Z, q^{(1)})} e^{(1/4)\|Z\|^2 t} \int_{\mathfrak{A}} \hat{K}(q^{(1)}, \hat{\xi}^{(1)}, \hat{\eta}^{(1)}, t | q^{(0)}) \\ &\quad \times e^{-(1/2)(Z, q^{(0)})} \hat{g}_0(q^{(0)}) e^{(Z, q^{(0)})} dq^{(0)}. \end{aligned}$$

4. *The hamiltonian of the open Toda lattice and its propagator.* The open Toda lattice is a system of $p + 1$ point particles on a line with coordinates $q_1, \dots, q_{p+1}$ and momentum $p_1, \dots, p_{p+1}$ interacting via the pair potential

$$V(q) = 2 \sum_{j=1}^{p} \hat{\xi}_j^2 \exp \sqrt{2}(q_j - q_{j+1}) \tag{6.22}$$

where the $\hat{\xi}_j^2$ are given coupling constants. The interaction is exponential through nearest neighbours. The hamiltonian of the system is

$$H(p, q) = \sum_{j=1}^{p+1} p_j^2 + V(q)$$

and as a classical system, it is a completely integrable system which has been integrated in explicit form by Olshanetsky and Perelomov in [**25**]. The hamiltonian $H(p, q)$ can be quantized and gives the operator

$$-H = \sum_{j=1}^{p+1} \frac{\partial^2}{\partial q_j^2} - V(q). \tag{6.23}$$

Because the interaction is by pairs, the total momentum is conserved; it is

$$i \sum_{j=1}^{p+1} \frac{\partial}{\partial q_j}$$

and we can suppose that it is 0 by changing the reference frame. The wave function f is defined on $\mathbf{R}^{p+1}$ and satisfies

$$f(q_1 + a, \dots, q_{p+1} + a) = f(q_1, \dots, q_{p+1})$$

because

$$\sum_{j=1}^{p+1} \frac{\partial}{\partial q_j} f = 0.$$

We can reduce to the case where f is defined on the hyperplane

$$E = \left\{ q \in \mathbf{R}^{p+1} / \sum_{j=1}^{p+1} q_j = 0 \right\} \tag{6.24}$$

which is in the usual hyperplane where the root system A_p lives (see Section 3).

For fixed coupling constants $\hat{\xi}_j^2$ we want to solve the problem (6.25)

$$\begin{cases} \dfrac{\partial \boldsymbol{\varphi}}{\partial t} = -H\boldsymbol{\varphi} \\ \boldsymbol{\varphi}|_{t=0} = \boldsymbol{\varphi}_0 \end{cases} \tag{6.25}$$

where $\boldsymbol{\varphi}_0$ is a function on $q^{(0)}$. We can solve (6.25) by a kernel

$$\boldsymbol{\varphi}(t, q^{(1)}) = \int_{\mathfrak{A}} P(q^{(1)}, (\hat{\xi}_j), t|q^{(0)})\boldsymbol{\varphi}_0(q^{(0)})dq^{(0)}.$$

If we write explicitly $-H$ in (6.25) using (6.22) and (6.23) we see that the problem (6.25) is identical to problem (6.7) with all $\hat{\eta}_j = 0$:

$$\begin{cases} \dfrac{\partial \boldsymbol{\varphi}}{\partial t} = \Delta\boldsymbol{\varphi} - 2 \displaystyle\sum_{j=1}^{p} \hat{\xi}_j^2 \exp(\sqrt{2}(q_j - q_{j+1}))\,\boldsymbol{\varphi} \\ \boldsymbol{\varphi}|_{t=0} = \boldsymbol{\varphi}_0 \end{cases} \tag{6.7}$$

and, by Theorem 1, this is solved by the kernel

$$P(q^{(1)}, (\hat{\xi}_j), t|q^{(0)}) = G(t, q^{(1)}, (\hat{\xi}_j), 0|q^{(0)}) \tag{6.26}$$

and we obtain

THEOREM 2. *For fixed coupling constants* $(\hat{\xi}_j)$, *the Cauchy problem for the Schrödinger* (*or heat*) *equation*

$$(6.25)\quad \begin{cases} \dfrac{1}{i}\dfrac{\partial \boldsymbol{\varphi}}{\partial t} = -H\boldsymbol{\varphi} \\ \boldsymbol{\varphi}|_{t=0} = \boldsymbol{\varphi}_0 \end{cases}$$

is given by the formula

$$(6.27)\quad \boldsymbol{\varphi}(t, q^{(1)}) = \int_{\mathfrak{A}} \boldsymbol{\varphi}_0(q^{(0)}) P(q^{(1)}, (\hat{\xi}_j), it|q^{(0)}) dq^{(0)}$$

where

$$(6.28)\quad P(q^{(1)}, (\hat{\xi}_j), it|q^{(0)}) = e^{(1/2)(Z, q^{(1)} + q^{(0)}) + (i/4)\|Z\|^2 t} \times \int K(q^{(1)}, (\xi^{(1)}), (\eta^{(1)}), it|q^{(0)}, 0, 0) \times e^{i(\sum_{j=1}^{p} \hat{\xi}_j^{(1)} \xi_j^{(1)})} d\xi^{(1)} d\eta^{(1)}$$

and $K(q^{(1)}, \xi^{(1)}, \eta^{(1)}, t|q^{(0)}, 0, 0)$ *is given by*

$$(6.29)\quad K(q^{(1)}, \xi^{(1)}, \eta^{(1)}, t|q^{(0)}, 0, 0) = \int_{\mathbf{R}^{p^2-p}} p(m^{(1)}, t|q^{(0)}, 0, 0, \lambda^{(0)}) d\lambda^{(0)}$$

where $m^{(1)}$ *is the point of* X *with horospherical coordinates*

$$m^{(1)} = (q^{(1)}, \xi^{(1)}, \eta^{(1)}, \lambda^{(1)} = 0)$$

λ *being the nonfundamental coordinates on* N (*so they belong to* $\mathbf{R}^{p^2-p}$) *and* $p(m^{(1)}, t|m^{(0)})$ *is the heat kernel on* X *which depends only on the radial coordinates of* $m^{(1)}$ *with respect to* $m^{(0)}$ *in the radial decomposition of* X (*so, it depends only on p variables*). $p(m^{(1)}, t|m^{(0)})$ *has an explicit expression given in Section* 4 *in term of radial coordinates.*

5. *Transforming the radial coordinates in horospherical coordinates.* a) If we want to obtain slightly more constructive expression for the kernels, we must first take the heat kernel $p(m^{(1)}, t|m^{(0)})$ in radial coordinates, and then do the two integrals involved in (6.28) and (6.29). For this, we need to change radial coordinates into horospherical coordinates. In our case where

$$X = SL(p + 1, \mathbf{C})/SU(p + 1),$$

X can be identified with the space of hermitian positive definite matrices of order $(p + 1) \times (p + 1)$ with determinant 1. The action of a $g \in SL(p + 1, \mathbf{C})$ on $x \in X$ is just

$$g \cdot x = g \times g^*$$

where g^* is the adjoint matrix of g. Moreover any matrix x positive definite hermitian of determinant 1 can be represented as the product

$$x = gg^*$$

for a $g \in SL(p + 1, \mathbf{C})$ which is defined uniquely up to the right multiplication by an element $k \in SU(p + 1)$. The radial coordinates of x are very easy; they are just the eigenvalues of the matrix x.

b) To find the horospherical coordinates, we write any x as the product

$$x = z(x)h(x)z^*(x)$$

where $z(x)$ is an upper triangular matrix with 1 on the diagonal (in particular it is in N), and $h(x)$ is a diagonal matrix

$$h = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_{p+1} \end{pmatrix}.$$

The expression of the h_k are given in term of x by

$$h_1 = \frac{1}{\Delta_p}, \; h_2 = \frac{\Delta_p}{\Delta_{p-1}}, \ldots, h_p = \frac{\Delta_2}{\Delta_1}, \; h_{p+1} = \Delta_1$$

where the $\Delta_j(x)$ are the lower principal minors of the matrix x of order j: this means that

$$\Delta_j = \begin{vmatrix} x_{j+1,j+1} & \cdots\cdots & x_{j+1,p+1} \\ \vdots & \ddots & \vdots \\ \vdots & & x_{p,p+1} \\ x_{p+1,j+1} & \cdots \; x_{p+1,p} & x_{p+1,p+1} \end{vmatrix}$$

(in particular Δ_0 is 1, because x is of determinant 1). Then $\sqrt{h(x)}$ is the abelian part of x in the horospherical coordinates.

References to Section 6. Olshanetsky and Perelomov **[25]** give the expression for the classical motion of the non periodic Toda lattice using the horospherical decomposition of $SL(p + 1, \mathbf{C})/SU(p + 1)$. In **[26]** they study other quantum systems with potential $\sin h^{-2}(q_i - q_j)$, $\sin^{-2}(q_i - q_j)\ldots$ but they do not give the time dependent propagator. We cannot obtain these systems using the symmetric space we study here.

7. The heat kernels on the non compact symmetric spaces of rank 1.

1. *The symmetric spaces of rank* 1 *and their root systems.*

a) *General notations.* Let $X = G/K$ be a symmetric space of rank 1. In the Cartan decomposition $G = KEK$, E is a commutative group of dimension 1 with a natural euclidean structure coming from the metric of X. Let E be a euclidean space of dimension 1, q will denote a point in E and also its coordinate with respect to a half length vector. The root system contains at most two positive roots

$$(7.1) \quad \begin{cases} R^{(1)}(q) = q \text{ with multiplicity } \rho_1 \\ R^{(2)}(q) = 2q \text{ with multiplicity } \rho_2. \end{cases}$$

We give below the possible lists of multiplicities and corresponding symmetric spaces.

b) *Real hyperbolic spaces*. The real hyperbolic space is

$$SO(n, 1)/SO(n).$$

Its compact dual is the sphere $SO(n + 1)/SO(n) \times SO(1)$. In this case

$$\begin{cases} \rho_1 = n - 1 \\ \rho_2 = 0. \end{cases}$$

c) *Hermitian hyperbolic spaces*. The hermitian hyperbolic space is

$$SU(n, 1)/SU(n).$$

Its compact dual is the complex projective space $SU(n + 1)/SU(n) \times SO(1)$. In this case the multiplicities are

$$\begin{cases} \rho_1 = 2(n - 1) \\ \rho_2 = 1. \end{cases}$$

d) *Quaternionian spaces*. This is $Sp(n, 1)/Sp(n) \times Sp(1)$ with compact dual the quaternionian projective space $Sp(n + 1)/Sp(n) \times Sp(1)$. The multiplicities are

$$\begin{cases} \rho_1 = 4(n - 1) \\ \rho_2 = 3. \end{cases}$$

The projective Weyl chamber is, in all these cases, $q > 0$. The Weyl group is just $q \to q$ and $q \to -q$.

Remark. We leave aside the exceptional space of rank 1.

2. *Volume element and Laplace operator*. We can treat these spaces as particular cases of BC_p spaces with $p = 1$ and we have just to look at the computations of Section 4, 2.

a) *In q coordinates*. The volume element on the euclidean maximal algebra E is

$$(7.2) \qquad d\hat{V}^{(\rho_1,\rho_2)} = \left(\sin \frac{q}{2}\right)^{\rho_1} (\sin q)^{\rho_2} dq$$

in the compact case and

$$dV^{(\rho_1,\rho_2)} = \left(\sinh \frac{q}{2}\right)^{\rho_1} (\sinh q)^{\rho_2}\, dq.$$

Calling $W^{(\rho_1,\rho_2)}$ the function appearing in front of dq, we have for the radial part of the Laplace operator

$$\hat{\Delta}^{(\rho_1,\rho_2)} = \frac{1}{\hat{W}^{(\rho_1,\rho_2)}} \frac{\partial}{\partial q}\left(\hat{W}^{(\rho_1,\rho_2)} \frac{\partial}{\partial q}\right)$$

in the compact case and

$$(7.3) \qquad \Delta^{(\rho_1,\rho_2)} = \frac{1}{W^{(\rho_1,\rho_2)}} \frac{\partial}{\partial q}\left(W^{(\rho_1,\rho_2)} \frac{\partial}{\partial q}\right)$$

in the non compact case.

b) *In algebraic coordinates*. Let us define $x = \cos q$ in the compact case and $x = \cosh q$ in the non compact case. Then $-1 < x < +1$ or $x > 1$ in the compact case or non compact case respectively. We define as in (4.6)

$$\rho_1 = 2\alpha - 2\beta$$

$$\rho_2 = 2\beta + 1.$$

Then we have as in Section 4, 3

$$d\hat{V}^{(\rho_1,\rho_2)} = (1 - x)^{\alpha}(1 + x)^{\beta} dx$$

in the compact case and in the non compact case we have just to change $1 - x$ to $x - 1$. Call $\hat{\mu}^{(\alpha,\beta)}$ this function $(1 - x)^{\alpha}(1 + x)^{\beta}$. Then the Laplace operator in this coordinate is in the compact case

$$\hat{L}^{(\alpha,\beta)} = \frac{1}{\hat{\mu}^{(\alpha,\beta)}} \frac{\partial}{\partial x}\left((1 - x^2)\hat{\mu}^{(\alpha,\beta)} \frac{\partial}{\partial x}\right)$$

(7.4) and in the non compact case;

$$\mu^{(\alpha,\beta)} = (x - 1)^{\alpha}(x + 1)^{\beta}$$

and:

$$L^{(\alpha,\beta)} = \frac{1}{\mu^{(\alpha,\beta)}} \frac{\partial}{\partial x}\left((x^2 - 1)\mu^{(\alpha,\beta)} \frac{\partial}{\partial x}\right) = -\hat{L}^{(\alpha,\beta)}.$$

If we develop this computation, we obtain

$$(7.5) \qquad L^{(\alpha,\beta)} = \pm\left[(1 - x^2)\frac{\partial^2}{\partial x^2} + (\beta - \alpha - (\alpha + \beta + 2)x)\frac{\partial}{\partial x}\right]$$

with the sign + in the compact case and − in the non compact case (remember also that the interval of definitions are $[-1, +1]$ and $[1, +\infty[$ respectively). We can summarize this by the table

(7.6)

spaces of rank 1	ρ_1	ρ_2	α	β
$SO(n, 1)/SO(n)$	$n - 1$	0	$\dfrac{n-2}{2}$	$-\dfrac{1}{2}$
$SU(n, 1)/SU(n) \times SO(1)$	$2(n - 1)$	1	$n - 1$	0
$Sp(n, 1)/Sp(n) \times Sp(1)$	$4(n - 1)$	3	$2n - 1$	1

Remark 1. With these conventions, the hyperbolic distance from the origin to a point q is

$$r = \frac{q}{2}.$$

The Laplace-Beltrami operator acting on function of r is

$$\frac{d^2}{dr^2} + (n - 1) \coth r \frac{d}{dr}$$

for the hyperbolic real space

$$\frac{d^2}{dr^2} + 2((n - 1) \coth r + \coth 2r)\frac{d}{dr}$$

for the hyperbolic hermitian space.

Remark 2. The spaces $SO(2, 1)/SO(2)$ and $SU(1, 1)/SU(1) \times SO(1)$ coincide with the hyperbolic space of real dimension 2. Their multiplicities are $(1, 0)$ and $(0, 1)$ respectively, but it is clear that this means a change of coordinate $q \rightarrow 2q$ to identify them at the level of their radial parts.

3. *Action of* $(1 + x)^\rho$ *on* $L^{(\alpha,\beta)}$. Let us denote by $\hat{L}^{(\alpha,\beta)}$ the hypergeometric operator

$$\begin{aligned}\hat{L}^{(\alpha,\beta)} &= \frac{1}{\hat{\mu}^{(\alpha,\beta)}} \frac{d}{dx}\left((1 - x^2)\hat{\mu}^{(\alpha,\beta)}\frac{d}{dx}\right) \\ &= (1 - x^2)\frac{d^2}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x)\frac{d}{dx}.\end{aligned}$$

Let $u(x)$ be some test function and $v(x) = (1 + x)^\rho u(x)$; we have

$$(1 + x)^\rho\frac{du}{dx} = \frac{dv}{dx} - \frac{\rho}{1 + x}v$$

$$(1 + x)^\rho\frac{d^2u}{dx^2} = \frac{d^2v}{dx^2} - \frac{2\rho}{1 + x}\frac{dv}{dx} + \frac{\rho^2 + \rho}{(1 + x)^2}v$$

so that

$$(1 + x)^\rho \hat{L}^{(\alpha,\beta)}u$$
$$= (1 - x^2)\frac{d^2v}{dx^2} + (\beta - \alpha - (\alpha + \beta + 2)x - 2\rho(1 - x))\frac{dv}{dx}$$
$$+ \left[\rho(\rho + 1)\frac{1 - x}{1 + x} + \rho(\alpha + \beta + 2)\frac{x}{1 + x} - \rho\frac{\beta - \alpha}{1 + x}\right]v.$$

The term in v can be rewritten as

$$\left[\frac{1}{1 + x}(2\rho(\rho + 1) - \rho(\alpha + \beta + 2) - \rho(\beta - \alpha)) \right.$$
$$\left. + ((\alpha + \beta + 2)\rho - \rho(\rho + 1))\right]v.$$

Choosing $\rho = \beta$ makes disappear the term $v/(1 + x)$ and gives

$$(1 + x)^\beta \hat{L}^{(\alpha,\beta)}u = (1 - x^2)\frac{d^2v}{dx^2}$$
$$+ (-\beta - \alpha - (\alpha - \beta + 2)x)\frac{dv}{dx}$$
$$+ \beta(\alpha + 1)v.$$

LEMMA 1. *We have the exchange property*

(7.7) $$(1 + x)^\beta \hat{L}^{(\alpha,\beta)}u$$
$$= \hat{L}^{(\alpha,-\beta)}((1 + x)^\beta u) + \beta(\alpha + 1)((1 + x)^\beta u).$$

4. *The Riemann-Liouville integral.*

LEMMA 2. *We have the exchange property*

(7.8) $$\left(\frac{d}{dx}\right)^n o\hat{L}^{(\alpha,\beta)}$$
$$= [\hat{L}^{(\alpha+n,\beta+n)} - n(\alpha + \beta + 2) - n(n - 1)]o\frac{d^n}{dx^n}.$$

To see this it is sufficient to prove it for $n = 1$ where the property is almost obviously by direct computation.

Let us now introduce the Riemann-Liouville integral (see [**27**])

(7.9) $$(I^{(\rho)}f)(x) = \frac{1}{\Gamma(\rho)}\int_a^x f(t)(x - t)^{\rho - 1}dt.$$

If f is a C^∞ function, $I^{(\rho)}f$ can be analytically continued in all the complex plane in ρ and it is well known that

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$$\begin{cases} \text{Identity} = I^{(0)} \\ \dfrac{d}{dx} = I^{(-1)} \\ I^{(\rho)}I^{(\rho')} = I^{(\rho+\rho')}. \end{cases}$$

LEMMA 3. *We have the exchange property*

$$(7.10) \qquad I^{(\rho)}o\hat{L}^{(\alpha,\beta)} = \hat{L}^{(\alpha-\rho,\beta-\rho)}oI^{(\rho)} + \rho(\alpha + \beta - \rho - 1)I^{(\rho)}.$$

Proof. For $\rho = -n$ (n a positive integer) this is Lemma 2; so Lemma 3 is obtained by analytic continuation. Another way is to compute directly, replacing $\dfrac{d}{dx}$ by $I^{(-1)}$ in $L^{(\alpha,\beta)}$ and using the identity

$$I^{(\rho)}(xg(x)) = xI^{(\rho)}(g(x)) - I^{(\rho+1)}(g(x))\frac{\Gamma(\rho + 1)}{\Gamma(\rho)}.$$

We can now combine Lemmas 1 and 3: let us compute firstly:

$$(1 + x)^\rho I^{(\theta)}\hat{L}^{(\alpha,\beta)} = (1 + x)^\rho \hat{L}^{(\alpha-\theta,\beta-\theta)}I^{(\theta)} + (1 + x)^\rho\theta(\alpha + \beta - \theta - 1)I^{(\theta)}.$$

Then using Lemma 1 with $\rho = \beta - \theta$, we have that this is

$$\hat{L}^{(\alpha-\theta,\theta-\beta)}(1 + x)^{\beta-\theta}I^{(\theta)} + (\beta - \theta)(\alpha - \theta + 1)(1 + x)^{\beta-\theta}I^{(\theta)} + (1 + x)^{\beta-\theta}\theta(\alpha + \beta - \theta - 1)I^{(\theta)}.$$

In particular, if we now want that β stays constant after these operations, we must choose $\beta = \theta/2$, $\rho = -\theta/2$, and we obtain

LEMMA 4. *We have*

$$(7.11) \qquad (1 + x)^{-\beta}I^{(2\beta)}\hat{L}^{(\alpha,\beta)} = \hat{L}^{(\alpha-2\beta,\beta)}(1 + x)^{-\beta}I^{(2\beta)} + \beta(\alpha - 3)(1 + x)^{-\beta}I^{(2\beta)}$$

$$(1 + x)^{\beta-\theta}I^{(\theta)}\hat{L}^{(\alpha,\beta)} = \hat{L}^{(\alpha-\theta,\theta-\beta)}(1 + x)^{\beta-\theta}I^{(\theta)} + [\alpha\beta + \beta - 2\theta](1 + x)^{\beta-\theta}I^{(\theta)}.$$

COROLLARY 1. *When* $\beta = -1/2$, *we obtain*

$$(7.12) \qquad (1 + x)^{1/2}\frac{d}{dx}\hat{L}^{(\alpha,-1/2)} = \hat{L}^{(\alpha+1,-1/2)}(1 + x)^{1/2}\frac{d}{dx} - \frac{1}{2}(\alpha - 3)(1 + x)^{1/2}\frac{d}{dx}.$$

This relation means that, when we put $x = \cosh 2r$, we obtain

COROLLARY 2.

$$\frac{1}{\sinh r}\frac{d}{dr}\left(\frac{d^2}{dr^2} + (n-1)\coth r\frac{d}{dr}\right) \tag{7.13}$$
$$= \left(\frac{d^2}{dr^2} + (n+1)\coth r\frac{d}{dr}\right)\frac{1}{\sinh r}\frac{d}{dr} - n\frac{1}{\sinh r}\frac{d}{dr}.$$

5. *Application to the heat kernel of hypergeometric equations.* Let us now consider the hypergeometric operator $L^{(\alpha,\beta)}$ on $[1, +\infty[$; it is formally self-adjoint with respect to $\mu^{(\alpha,\beta)}$. We want to study

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\hat{L}^{(\alpha-\rho,\beta-\rho)}u \\ u|_{t=0} = \psi. \end{cases} \tag{7.14}$$

(Recall $L^{(\alpha,\beta)} = -\hat{L}^{(\alpha,\beta)}$. Define

$$v = e^{-\rho(\alpha+\beta-\rho+1)t}u(t, x).$$

Then

$$\begin{cases} \dfrac{\partial v}{\partial t} = -\hat{L}^{(\alpha-\rho,\beta-\rho)}v - \rho(\alpha + \beta - \rho + 1)v \\ v|_{t=0} = \psi. \end{cases}$$

Let $p_t^{(\alpha,\beta)}(x_0|x)$ be the heat kernel for $+L^{(\alpha,\beta)}$ with respect to the volume element $\mu^{(\alpha,\beta)}(x)dx$. Lemma 3 proves that

$$\begin{cases} v(t, x_0) = I_{x_0}^{(\rho)}\left(\int p_t^{(\alpha,\beta)}(x_0|x)\varphi(x)m^{(\alpha,\beta)}(x)dx\right) \\ \varphi(x) = I_x^{(-\rho)}\psi(x) \end{cases}$$

(here $I_x^{(\rho)}$ denotes $I^{(\rho)}$ acting on the variable x), so that we obtain:

$$u(t, x_0) = I_{x_0}^{(\rho)}\left(\int p_t^{(\alpha,\beta)}(x_0|x)(I_x^{(-\rho)}\psi)(x)\mu^{(\alpha,\beta)}(x)dx\right) \times e^{+\rho(\alpha+\beta-\rho+1)t}. \tag{7.15}$$

Now the kernel $I^{(\rho)}$ depends on the origin a of the integral (7.9) (except when $p = -n$, n positive integer). But if ψ is compactly supported in $[1, +\infty[$, the solution of the heat equation tends to 0 at ∞. This forces us to choose $a = \infty$ in the definition of $I^{(\rho)}$. Finally we obtain the following theorem.

THEOREM 1. *Suppose that we know the heat kernel $p_t^{(\alpha,\beta)}(x_0|x)$ of $L^{(\alpha,\beta)}$ on $[1, +\infty[$ (with respect to the volume element $\mu^{(\alpha,\beta)}(x)dx$) with boundary conditions 0 at ∞. The solution of the problem (7.14) vanishing at ∞ for*

$L^{(\alpha-\rho,\beta-\rho)}$ *is given by formula* (7.15) *with* $I^{(\rho)}$ *given by* (7.9) *where we have done* $a = \infty$ *for the origin of integrals.*

Let us also use Lemma 4: we obtain

THEOREM 2. *The solution of the heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\hat{L}^{(\alpha-\theta,\theta-\beta)}u \\ u|_{t=0} = \psi \end{cases}$$

is given by

$$(7.16) \qquad u(t, x_0) = (1 + x_0)^{\beta-\theta} I^{(\theta)}_{x_0}\left(\int p_t^{(\alpha,\beta)}(x_0|x) \times I^{(-\theta)}_x((1 + x)^{\theta-\beta}\psi(x))\mu^{(\alpha,\beta)}(x)dx\right) e^{(\alpha\beta+\beta-2\theta)t}.$$

We can also prove the following result when the source is at point 1.

THEOREM 3. *The heat kernel*

$$p_t^{(\alpha+1,-1/2)}(1|x)$$

with source at point 1 *is given by*

$$(7.17) \qquad p_t^{(\alpha+1,-1/2)}(1|x) = \frac{-(\sqrt{2})^{-1}}{\alpha + 1}(1 + x)^{1/2}\frac{d}{dx}(p_t^{(\alpha,-1/2)}(1|x))e^{+(1/2)(\alpha-3)t}.$$

Proof. The heat kernel is symmetric (because we take it with respect to the volume element

$$\mu^{(\alpha+1,-1/2)}(x)dx).$$

So it has to satisfy the heat equation in x for $-\hat{L}^{(\alpha+1,-1/2)}$. Using Corollary 1 of Lemma 4 it is easy to see that the second member of (7.17) satisfies the heat equation. We have then to check that for any function $\psi \in C^\infty$ with compact support around 1,

$$-\frac{(\sqrt{2})^{-1}}{\alpha + 1}e^{+(1/2)(\alpha-3)t}\int_1^\infty (1 + x)^{1/2}\frac{d}{dx}(p_t^{(\alpha,-1/2)}(1|x)) \times \psi(x)(x - 1)^{\alpha+1}(1 + x)^{-1/2}dx$$

tends to $\psi(1)$ if $t \to 0^+$; but by integration by parts this means that

$$+\frac{1}{(\alpha + 1)\sqrt{2}}\int_1^\infty p_t^{(\alpha,-1/2)}(1|x)\frac{d}{dx}(\psi(x)(x - 1)^{\alpha+1})dx \to \psi(1).$$

The integral can be rewritten as

$$+\frac{1}{(\alpha + 1)\sqrt{2}} \int_1^\infty p_t^{(\alpha, -1/2)}(1|x)\mu^{(\alpha, -1/2)}(x)$$
$$\times [\psi'(x)(1 + x)^{1/2}(x - 1) + (\alpha + 1)\psi(x)(1 + x)^{1/2}]dx.$$

But

$$p_t^{(\alpha, -1/2)}(1|x)\mu^{(\alpha, -1/2)}(x)dx \to \delta(x - 1),$$

which is the result claimed.

6. *Heat kernels for the real hyperbolic spaces.*

a) *Some notations.* We want to find, as in Section 3, 1, the heat kernel

$$P_t^{(X)}(mm')dv(m')$$

($dv(m')$ = volume element of X). Taking for m the origin 0 of X, we are reduced to computing the heat kernel of the radial part $\Delta^{(\rho_1,\rho_2)}$ (see (7.3)) with source at $q_0 = 0$ and with respect to $dV^{(\rho_1,\rho_2)}(q)$. We shall denote by

$$P_t^{(\rho_1,\rho_2)}(0|q)dV^{(\rho_1,\rho_2)}(q)$$

this kernel. We now change the coordinates putting

$$x = \cosh \quad q \geqq 1$$

and we define

$$p_t^{(\alpha,\beta)}(1|x)\mu^{(\alpha,\beta)}(x)dx = P_t^{(\rho_1,\rho_2)}(0|q)dV^{(\rho_1,\rho_2)}(q).$$

Then $p_t^{(\alpha,\beta)}(1|x)$ is the heat kernel (with respect to $m^{(\alpha,\beta)}(x)dx$) with source at point 1 in $[1, +\infty[$, vanishing at ∞ and (α, β) are related to the multiplicities by

$$\rho_1 = 2(\alpha - \beta) \quad \rho_2 = 2\beta + 1.$$

b) We also remark, that one can define $P_t^{(\rho_1,\rho_2)}(q|q')$ for any $q, q' > 0$. This kernel has a meaning in the symmetric space X (see Section 3, 2): in fact, if we have at time 0 a uniform distribution of heat on the sphere $S(0, q)$ of centre 0 and radius q, then

$$P_t^{(\rho_1,\rho_2)}(q|q')dV^{(\rho_1,\rho_2)}(q')$$

will be the amount of heat obtained by diffusion at time t on the sphere $S(0, q')$; clearly $P_t^{(\rho_1,\rho_2)}(q|q')$ is the fundamental solution of $\Delta^{(\rho_1,\rho_2)}$ with pole q; it is also clear that if we know $P_t^{(\rho_1,\rho_2)}(0|q')$, we are in principle able to compute $P_t^{(\rho_1,\rho_2)}(q|q')$ for any $q, q' > 0$: in fact if we know $P_t^{(\rho_1,\rho_2)}(0|q')$ we know the heat kernel of X completely and so we know $P_t^{(\rho_1,\rho_2)}(q|q')$ using its interpretation given above in term of heat diffusion in X.

c) *The case* $\alpha = \beta = -1/2$. In this case $\rho_1 = \rho_2 = 0$, $dV^{(0,0)}(q) = dq$: this is the degenerate case of $\mathbf{R}$ considered as a symmetric space with its euclidean structure. The heat kernel of $\mathbf{R}$ is

$$P_t^{\mathbf{R}}(m|m')dm' = \frac{e^{-(|m-m'|^2)/4t}}{\sqrt{4\pi t}}dm'.$$

Now,

$$\Delta^{(0,0)} = \frac{d^2}{dq^2},$$

q being twice the distance to the origin, say O. It is then clear that

$$P_t^{(0,0)}(0|q')dq' = 2P_t^{\mathbf{R}}(0|q')dq'$$

(because we restrict ourselves to $q' > 0$ in the trivial Weyl chamber $\mathbf{R}^+$). The sphere $S(0, q')$ is the set $\{q', -q'\}$. Looking at the interpretation we see that

$$(7.18) \qquad P_t^{(0,0)}(q|q')dq' = \left(\frac{e^{-(|q-q'|^2)/16t}}{\sqrt{4\pi t}} + \frac{e^{-(|q+q'|^2)/16t}}{\sqrt{4\pi t}}\right)dq'$$

which is the heat kernel on $\mathbf{R}^+$ with Neumann condition at O. ($q, q' \geqq 0$). Then

$$(7.19) \qquad p_t^{(-1/2,-1/2)}(x|x')\mu^{(-1/2,-1/2)}(x')dx' = P_t^{(0,0)}(q|q')dq'$$

with

$$x = \sinh q, \; x' = \sinh q'.$$

d) *The case* $\alpha = 1/2$, $\beta = -1/2$. By the table (7.6), this is the case of the three dimensional real hyperbolic space $\mathbf{H}_3$.

This case was treated in [**8**] using a series expansion; we found there the following formula

$$(7.20) \qquad P_t^{\mathbf{H}_3}(r) = \frac{1}{(4\pi t)^{3/2}}e^{-t}e^{-(r^2)/4t}\left(\frac{r}{\sinh r}\right)$$

in terms of the radial hyperbolic distance. This formula can also be checked directly. Because $r = q/2$ the formula gives

$$(7.21) \qquad P_t^{(2,0)}(q) = \frac{1}{(4\pi t)^{3/2}}e^{-t}e^{-(q^2)/16t}\left(\frac{q/2}{\sinh q/2}\right)$$

with respect to the volume element

$$dV^{(2,0)}(q) = \left(\sinh\frac{q}{2}\right)dq;$$

going to algebraic coordinates, we obtain

(7.22) $$p_t^{(1/2,-1/2)}(x)\mu^{(1/2,-1/2)}(x)dx = P_t^{(2,0)}(q)dV^{(2,0)}(q)$$

with $x = \cosh q$.

e) *The case* $\alpha = 0$, $\beta = -1/2$. This is the case of the two dimensional hyperbolic space $\mathbf{H}_2$ with multiplicities (1, 0). But we can also consider it as having multiplicities (0, 1) and with $\alpha = 0$, $\beta = 0$. Then we can apply Theorem 1 with $\rho = -1/2$, $\alpha = \beta = -1/2$. The solution of the heat equation $-\hat{L}^{(0,0)}$ with initial data ψ is by formula (7.15) specialized to this case,

$$u(t, x_0) = I_{x_0}^{(-1/2)}\left(\int p_t^{(-1/2,-1/2)}(x_0|x)(I_x^{(1/2)}\psi)(x) \times \mu^{(-1/2,-1/2)}(x)dx\right)e^{-t/4}.$$

Now we know that

$$I^{(-1/2)} = I^{(1/2)}I^{(-1)} = I^{(1/2)}\frac{d}{dx};$$

we can rewrite the preceding equality as

$$\begin{aligned} u(t, x_0) &= \frac{e^{-t/4}}{\Gamma\left(\frac{1}{2}\right)^2}\int_{x_0}^{\infty}\frac{d\xi}{\sqrt{\xi - x_0}}\int_1^{\infty}\frac{d}{d\xi}p_t^{(-1/2,-1/2)}(\xi|x) \\ &\quad\times\int_x^{+\infty}\frac{\psi(\eta)}{\sqrt{\eta - x}}d\eta\mu^{(-1/2,-1/2)}(x)dx \\ &= \frac{e^{-t/4}}{\Gamma\left(\frac{1}{2}\right)^2}\int_1^{\infty}\psi(\eta)d\eta\int_{x_0}^{+\infty}d\xi \\ &\quad\times\int_1^{\eta}\frac{d}{d\xi}(p_t^{(-1/2,-1/2)}(\xi|n)) \\ &\quad\times\frac{\mu^{(-1/2,-1/2)}(x)dx}{\sqrt{(\xi - x_0)(\eta - x)}} \end{aligned}$$

and the heat kernel is then

(7.23) $$p_t^{(0,0)}(x_0|x) = \frac{e^{-t/4}}{\pi}\int_{x_0}^{+\infty}d\xi\int_1^x d\eta\frac{d}{d\xi}(p_t^{(-1/2,-1/2)}(\xi|\eta)) \times\frac{\mu^{(-1/2,-1/2)}(\eta)}{\sqrt{(\xi - x_0)(x - \eta)}}$$

where $p_t^{(-1/2,-1/2)}(\xi|\eta)$ is given by the formula (7.19). We can take the limit when x or x_0 tends to 1; in that case, we see that the interval of the first integration in (7.23) is $[1, x]$; moreover we also have inside this integral

$$(x - \eta)^{-1/2}(\eta - 1)^{-1/2};$$

so the intermediary integral in (7.23) is of the type

$$\int_1^{1+\epsilon} \frac{\alpha(\eta)}{\sqrt{(\eta - 1)}\sqrt{1 + \epsilon - \eta}} d\eta$$

where $x = 1 + \epsilon$ and α is a C^1 function. This is

$$\int_1^{1+\epsilon} \frac{\alpha(\eta) - \alpha(1)}{\sqrt{(\eta - 1)(1 + \epsilon - \eta)}} d\eta + \alpha(1)$$
$$\times \int_1^{1+\epsilon} \frac{1}{\sqrt{(\eta - 1)(1 + \epsilon - \eta)}} d\eta.$$

The first term tends to 0 and the second term is $\alpha(1)\pi/3$; so we obtain in (7.23)

(7.24) $$p_t^{(0,0)}(1|x) = \frac{e^{-t/4}}{3\sqrt{2}} \int_x^\infty \frac{d\xi}{\sqrt{\xi - x}} \frac{d}{d\xi} p_t^{(-1/2,-1/2)}(\xi|1).$$

Let us write $x = \cosh q$, $\xi = \cosh q'$ and use (7.19) and (7.18) to get

(7.25) $$p_t^{(0,1)}(q) = \frac{e^{-t/4} 2\pi}{(4\pi t)^{3/2}} \int_q^\infty \frac{e^{-(q'^2)/16t} q'}{\sqrt{\cosh q' - \cosh q}} dq'$$

which is the formula given (without proof) by McKean in [**23**]. Then we obtain

(7.26) $$P_t^{(1,0)}(q) = P_t^{(0,1)}(q/2)$$

(see remark 2 in 2). In particular with the radial distance r,

(7.27) $$P_t^{\mathbf{H}_2}(r) = P_t^{(0,1)}(r).$$

f) *The general Lobatchevski space.* We shall obtain now the heat kernel of $L^{(k,-1/2)}$ and $L^{(k+1/2,-1/2)}$ for k integer by applying recursively Theorem 3 formula (7.17) to $p_t^{(1/2,-1/2)}$ and $p_t^{(0,-1/2)}$ to get

(7.28) $$p_t^{((n+2-2)/2,-1/2)}(1|x)$$
$$= \frac{-(\sqrt{2})^{-1}}{\frac{n}{2}} (1 + x)^{1/2} \frac{d}{dx} (p_t^{((n-2)/2,-1/2)}(1|x))e^{-(1/4)(n-8)t}.$$

This formula gives the passage from a real hyperbolic space $\mathbf{H}_n$ of dimension n to the one $\mathbf{H}_{n+2}$ of dimension $n + 2$. Moreover, using (7.13), we can obtain directly the recursion formula for the heat kernel in the variable r (hyperbolic instance) by

(7.29) $$P_t^{\mathbf{H}_{n+2}}(r) = -e^{-nt}\frac{1}{\sinh r}\frac{d}{dr}P_t^{\mathbf{H}_n}(r).$$

Remark. If we want

$$p_t^{(k,-1/2)}(x|x') \quad \text{and} \quad p_t^{(k+(1/2),-1/2)}(x|x'),$$

we can obtain them by two methods:

1[st] method. We start with

$$p_t^{(-1/2,-1/2)}(x|x')$$

given by (7.19) and (7.18) and apply repeatedly (7.16) with $\beta = -1/2$, $\theta = -1$, $\alpha = -1/2$, to get

$$p_t^{(1/2,-1/2)}(x|x')$$

and then again $\beta = -1/2$, $\theta = -1$, $\alpha = 1/2$ to get

$$p_t^{(3/2,-1/2)}(x|x').$$

Then we start from $p_t^{(0,0)}(x|x')$ (which is equivalent to

$$p_t^{(0,-1/2)}(x_1|x_1')$$

up to a trivial change of variable of the type

$$x_1 = \cosh\left(\frac{1}{2}\operatorname{Arg}\cosh x\right));$$

then we apply to

$$p_t^{(0,-1/2)}(x|x')$$

the formula (7.16) respectedly with $\beta = -1/2$, $\theta = -1$, $\alpha = 0$ to get

$$p_t^{(1,-1/2)}(x|x') \ldots$$

2[nd] method. Knowing

$$p_t^{(k,-1/2)}(1|x),$$

we know the heat kernel of the corresponding Lobatchevski space and then we know (in principle) how to compute

$$P_t^{\mathbf{H}_n}(q|q')$$

and also

$$p_t^{(k,-1/2)}(x|x')$$

for k "integer" or "integer $+\frac{1}{2}$".

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7. *The heat kernels of hermitian and quaternionian hyperbolic spaces.*

a) *The hermitian hyperbolic spaces.* In the table (7.6) of 1, they correspond to $\alpha = n - 1$, $\beta = 0$ ($\rho_1 = 2(n - 1)$, $\rho_2 = 1$) and $n \geqq 2$. We start with the real hyperbolic space $\alpha = n - 3/2$, $\beta = -1/2$ and apply Theorem 1 with $\rho = -1/2$: we obtain

$$p_t^{(n-1,0)}(x|x')$$

in terms of

$$p_t^{(n-3/2,-1/2)}(x|x')$$

through $I^{(-1/2)}$ and by the same reasoning as the one leading to formula (7.23), we get

$$(7.30)\quad p_t^{(n-1,0)}(x_0|x) = \frac{e^{-(n-5/2)(1/2)t}}{\pi}\int_{x_0}^{+\infty} d\xi \times \int_1^x d\eta \frac{d}{d\xi} p_t^{(n-3/2,-1/2)}(\xi|\eta)\frac{\mu^{(-1/2,-1/2)}(\eta)}{\sqrt{(\xi - x_0)(x - \eta)}}.$$

If we want $p_t^{(n-1,0)}(1|x)$ we obtain by the same line of reasoning as the one leading to (7.24)

$$(7.31)\quad p_t^{(n-1,0)}(1|x) = \frac{e^{-(n-5/4)(1/2)t}}{3\sqrt{2}}\int_x^{+\infty}\frac{d\xi}{\sqrt{\xi - x}}\frac{d}{d\xi}p_t^{(n-3/2,-1/2)}(\xi|1).$$

b) *The quaternionian hyperbolic spaces.* These correspond to $\alpha = 2n - 1$, $\beta = 1$. We can then apply Theorem 1 with $\rho = -1$ to $p_t^{(2n-2,0)}(x|x')$. But

$$I^{(-1)} = \frac{d}{dx},$$

so

$$p_t^{(2n-1,1)}(x|x') = e^{2nt}\left(\int_1^{x'}\frac{d}{dx}(p_t^{(2n-2,0)}(x|\xi))(x' - \xi)(1 - \xi)^{2n-2}\right) \times \frac{1}{(1 + x')(1 - x')^{2n-1}}$$

the extra factor

$$\frac{1}{(1 + x')(1 - x')^{2n-1}}$$

coming from the fact that the kernel is taken with respect to the weight $\mu^{(2n-1,1)}(x')dx'$. If we want to find the kernel with source at $x = 1$, we see that

$$e^{2nt}\frac{d}{dx'}p_t^{(2n-2,0)}(1|x')$$

satisfy the heat equation $L^{(2n-1,1)}$ by Lemma 3 (7.10); moreover if we consider

$$\frac{e^{2nt}}{2(2n-1)}\int_1^\infty \frac{d}{dx'}(p_t^{(2n-2,0)}(1|x'))\psi(x')(1+x')(1-x')^{2n-1}dx'$$

by integration by parts we get

$$e^{2nt}\int_1^\infty p_t^{(2n-2,0)}(1|x')\Big(\psi(x')(1-x')^{2n-2}\frac{(1+x')}{2} + \frac{(1-x')^{2n-1}}{2(2n-1)}\frac{d}{dx'}(\psi(x')(1+x'))\Big)dx'$$

which tends to $\psi(1)$ if $t \to 0^+$ because

$$p_t^{(2n-2,0)}(1|x')(1-x')^{2n-2}dx' \to \delta(x'-1)$$

so that

$$p_t^{(2n-1,1)}(1|x) = \frac{e^{2nt}}{2(2n-1)}\frac{d}{dx}p_t^{(2n-2,0)}(1|x). \tag{7.32}$$

References. The heat kernels for Rank 1 symmetric spaces have been obtained in [**21**]. Our method gives the general hypergeometric equation. See also [**23**] and [**8**].

8. The heat kernels on certain symmetric spaces with the root system BC_p.

1. *Preliminary notations on the Laplace operators.* a) We consider, in this chapter, symmetric spaces with the root systems BC_p. This root system has already been described in Section 4, 1. We recall that E is a euclidean space of dimension p, q a point in E with coordinates $(q_1, \dots, q_p)$; the roots are

$$\begin{cases} R_i^{(1)}(q) = q_i \quad \text{with equal multiplicities } \rho_1 \\ R_i^{(2)}(q) = 2q_i \quad \text{with equal multiplicities } \rho_2 \\ R_{ij}^{(1)}(q) = q_i - q_j \quad R_{ij}^{(2)}(q) = q_i + q_j \\ \qquad\qquad\qquad \text{with equal multiplicities } \rho_3. \end{cases} \tag{8.1}$$

b) The positive roots are q_i, $2q_i$, $q_i - q_j$ $(i < j)$ and $q_i + q_j$ $(i < j)$. We have also define to Weyl chamber and the Weyl alcove

$$\Lambda_q = \{q \in E_p \quad 0 < q_p < q_{p-1} < \ldots < q_1\}$$

and the volume element

$$(8.2) \qquad d\hat{V}^{(\rho_1,\rho_2,\rho_3)} = \prod_{i=1}^{p} \left(\sin \frac{q_i}{2}\right)^{\rho_1} (\sin q_i)^{\rho_2}$$
$$\times \prod_{1 \leqq i \leqq j \leqq p} \left(\sin\left(\frac{q_i - q_j}{2}\right) \sin\left(\frac{q_i + q_j}{2}\right)\right)^{\rho_3} \prod_{i=1}^{p} dq_i$$

and in the non compact case

$$(8.3) \qquad dV^{(\rho_1,\rho_2,\rho_3)} = \prod_{i=1}^{p} \left(\sinh \frac{q_i}{2}\right)^{\rho_1} (\sinh q_i)^{\rho_2}$$
$$\times \prod_{1 \leqq i < j \leqq p} \left(\sinh\left(\frac{q_i - q_j}{2}\right) \sinh\left(\frac{q_i + q_j}{2}\right)\right)^{\rho_3} \prod_{i=1}^{p} dq_i.$$

c) Calling $\hat{W}^{(\rho_1,\rho_2,\rho_3)}$ and $W^{(\rho_1,\rho_2,\rho_3)}$ the density of the volume

$$d\hat{V}^{(\rho_1,\rho_2,\rho_3)} \quad \text{and} \quad dV^{(\rho_1,\rho_2,\rho_3)}$$

with respect to Lebesgue measure, we define the Laplace-Beltrami operators

$$(8.4) \qquad \begin{cases} \hat{\Delta}^{(\rho_1,\rho_2\rho_3)} = \dfrac{1}{\hat{W}^{(\rho_1,\rho_2,\rho_3)}} \displaystyle\sum_{i=1}^{p} \frac{\partial}{\partial q_i} \left(\hat{W}^{(\rho_1,\rho_2,\rho_3)} \frac{\partial}{\partial q_j} \right) \\ \Delta^{(\rho_1,\rho_2,\rho_3)} = \dfrac{1}{W^{(\rho_1,\rho_2,\rho_3)}} \displaystyle\sum_{i=1}^{p} \frac{\partial}{\partial q_i} \left(W^{(\rho_1,\rho_2,\rho_3)} \frac{\partial}{\partial q_i} \right). \end{cases}$$

d) We also defined in the compact case the algebraic coordinates $x_i = \cos q_i$, the Weyl alcove becoming

$$C_x = \{x \in \mathbf{R}^p \quad -1 < x_1 < x_2 < \ldots < x_p < 1\}.$$

The volume element becomes

$$(8.5) \qquad dV^{(\alpha,\beta,\gamma)} = C \prod_{i=1}^{p} (1 - x_i)^{\alpha} (1 + x_i)^{\beta}$$
$$\times \prod_{1 \leqq j < i \leqq p} (x_i - x_j)^{2\gamma+1} dx_1 \ldots dx_p$$

where C is a constant (see (4.7)) and

$$(8.6) \qquad \rho_1 = 2\alpha - 2\beta \quad \rho_2 = 2\beta + 1 \quad \rho_3 = 2\gamma + 1.$$

Call $\hat{m}^{(\alpha,\beta,\gamma)}$ the density of $dV^{(\alpha,\beta,\gamma)}$ with respect to Lebesgue measure. The Laplace operator is

$$(8.7)\qquad \hat{\Delta}^{(\alpha,\beta,\gamma)} = \frac{1}{\hat{m}^{(\alpha,\beta,\gamma)}} \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\left((1 - x_i^2)\hat{m}^{(\alpha,\beta,\gamma)}\frac{\partial}{\partial x_i}\right)$$

$$= \sum_{i=1}^{p} \left\{(1 - x_i^2)\frac{\partial^2}{\partial x_i^2} + \left[\beta - \alpha - (\alpha + \beta + 2)x_i\right.\right.$$

$$\left.\left. + (2\gamma + 1)(1 - x_i^2) \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{1}{x_i - x_j}\right]\frac{\partial}{\partial x_i}\right\}.$$

e) In the non compact case the algebraic coordinates are

$$x_i = \cosh q_i.$$

The Weyl chamber becomes

$$C_x = \{x \in \mathbf{R}^p \quad 1 < x_p < x_{p-1} < \ldots < x_1\}.$$

The volume element is

$$(8.8)\qquad dV^{(\alpha,\beta,\gamma)} = \prod_{i=1}^{p} (x_i - 1)^{\alpha}(x_i + 1)^{\beta}$$

$$\times \prod_{1\leqq j<i\leqq p} (x_i - x_j)^{2\gamma+1} dx_1 \ldots dx_p$$

with the convention (8.6) on α, β, γ and we call $m^{(\alpha,\beta,\gamma)}$ the density of $dV^{(\alpha,\beta,\gamma)}$ with respect to Lebesgue measure. The Laplace operator becomes

$$(8.9)\qquad \Delta^{(\alpha,\beta,\gamma)} = \frac{1}{m^{(\alpha,\beta,\gamma)}} \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\left((x_i^2 - 1)m^{(\alpha,\beta,\gamma)}\frac{\partial}{\partial x_i}\right)$$

and formally we have

$$(8.10)\qquad \Delta^{(\alpha,\beta,\gamma)} = -\hat{\Delta}^{(\alpha,\beta,\gamma)}.$$

f) Let us also recall that we have obtained in Section 4 the eigenfunctions of $\hat{\Delta}^{(\alpha,\beta,\pm 1/2)}$ and the heat kernels of these operators.

2. *Root systems* B_p, C_p, BC_p. The root system BC_p is a mixture of two simpler root systems B and C. We refer to Araki for the following classification.

a) *System* B_p. The positive fundamental roots are

$$\alpha_1 = q_1 - q_2, \alpha_2 = q_2 - q_3, \ldots, \alpha_{p-1}$$
$$= q_{p-1} - q_p, \alpha_p = q_p$$

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with the Dynkin diagram of fundamental roots

$$\alpha_1 - \alpha_2 - \ldots - \alpha_{p-1} \Rightarrow \alpha_p.$$

Let us recall that in a Dynkin diagram of fundamental roots, if α, β are two roots, the notation $\alpha - \beta$ means that $||\alpha|| = ||\beta||$ and $\alpha \Rightarrow \beta$ means that $||\alpha||^2 = 2||\beta||^2$.

b) *System C_p*. The positive fundamental roots are

$$\alpha_1 = q_1 - q_2, \alpha_2 = q_2 - q_3, \ldots, \alpha_{p-1} = q_{p-1} - q_p, \alpha_p = 2q_p$$

and the Dynkin diagram of the fundamental roots is

$$\alpha_1 - \alpha_2 - \alpha_3 - \ldots - \alpha_{p-1} \Leftrightarrow \alpha_p.$$

c) *System BC_p*. BC_p is a mixture of B_p and C_p; formally it is B_p but $2\alpha_p$ has also a positive multiplicity, so that it is a non reduced root system.

d) *System D_p*. By definition D_p has only the roots $\pm(q_i \pm q_j)$ with equal multiplicities: then it can be considered as a special BC_p (or B_p and C_p) with $\rho_1 = \rho_2 = 0$ and $\alpha = \beta = -1/2$. The positive fundamental roots are

$$\alpha_1 = q_1 - q_2, \alpha_2 = q_2 - q_3, \ldots, \alpha_{p-1} = q_{p-1} - q_p, \alpha_p = q_{p-1} + q_p.$$

D_p exists only for $p \geqq 3$; for $p = 2$, it degenerates in a product.

e) Recall also that in general

$$\rho_1 = \text{multiplicity of } q_i = 2\alpha - 2\beta$$

$$\rho_2 = \text{multiplicity of } 2q_i = 2\beta + 1$$

$$\rho_3 = \text{multiplicity of } q_i \pm q_j = 2\gamma + 1$$

$$\rho_1 = 0 \text{ for a pure } C_p \text{ system}$$

$$\rho_2 = 0 \text{ for a pure } B_p \text{ system}$$

$$\rho_1\rho_2 \neq 0 \text{ for a genuine } BC_p \text{ system.}$$

3. *Classification of the symmetric spaces with the BC_p root systems.* We list below, for clarity, the Cartan's classification only for BC_p root systems. (see [**1**], [**15**]).

a) *Notations.* $\mathfrak{G}$ is a semi simple real Lie algebra, $g = \mathfrak{K} \oplus \mathfrak{P}$ is its Cartan decomposition and $\mathfrak{G}^{\mathbf{C}}$ its complexification. $\mathfrak{X}$ is the Cartan subalgebra in $\mathfrak{G}^{\mathbf{C}}$, $\mathfrak{X}_{\mathfrak{P}}$ (or sometimes $\mathfrak{A}_{\mathfrak{P}}$ or $\mathfrak{X}^-$) is the subalgebra of $\mathfrak{X}$ contained in $\mathfrak{P}$.

b) *Type* BDI *or* $SO_0(p, q)/SO(p) \times SO(q)$. We suppose here $p \leqq q$. The rank is p; the space dimension is $2pq$; $\operatorname{Dim} \mathfrak{X} = [\,(1/2)(p + q)\,]$. This type splits in 2 subtypes:

Subtype DI. $p + q = 2l$ even.

$\alpha)$ $p = q = l$.

$$\mathfrak{X} = \mathfrak{A}_{\mathfrak{P}}$$

root system D_p

all multiplicities are 1

$$\rho_1 = \rho_2 = 0, \rho_3 = 1$$

$$\alpha = 0, \beta = -\frac{1}{2}, \gamma = 0$$

$\beta)$ $p < q, p = q - 2k, l > k > 0$.

root system B_p

$$\rho_1 = 2(l - p), \rho_2 = 0, \rho_3 = 1$$

$$\alpha = l - p - \frac{1}{2}, \beta = -\frac{1}{2}, \gamma = 0.$$

Subtype BI. $p + q = 2l + 1$ odd.

root system B_p

$$\rho_1 = 2(l - p) + 1, \rho_2 = 0, \rho_3 = 1$$

$$\alpha = l - p, \beta = -\frac{1}{2}, \gamma = 0.$$

c) *Type* A III *or* $SU(p, q)/S(U(p) \times U(q))$. We suppose $p \leqq q$. The rank is p; the space dimension $2pq$

$$\operatorname{Dim} \mathfrak{X} = p + q - 1.$$

Subtype A III-1. $p \leqq q - 1$.

genuine BC_p system

$$\rho_1 = 2(q - p), \rho_2 = 1, \rho_3 = 2$$

$$\alpha = l - 2p + 1, \beta = 0, \gamma = \frac{1}{2}.$$

Subtype A III-2. $p = q$.

C_p type system

$$\rho_1 = 0, \rho_2 = 1, \rho_3 = 2$$

$$\alpha = 0, \rho_2 = 0, \gamma = \frac{1}{2}.$$

d) *Type* D III. $SO^*(2n)/U(n)$. (Compact $SO(2n)/U(n)$). The rank is $[(1/2)n]$, the dimension $n(n - 1)$

$$\text{Dim } \mathfrak{X} = n.$$

Subtype D III-1. $n = 2p$.

C_p type system

$$\rho_1 = 0, \rho_2 = 1, \rho_3 = 4$$

$$\alpha = 0, \beta = 0, \gamma = \frac{3}{2}.$$

Subtype D III-2. $n = 2p + 1$.

genuine BC_p system

$$\rho_1 = 4, \rho_2 = 1, \rho_3 = 4$$

$$\alpha = 2, \beta = 0, \gamma = \frac{3}{2}.$$

e) *Type* C II. $Sp(p, q)/Sp(q) \times Sp(q)$. We assume $p \leqq q$.

$$\text{Rank} = p \quad \dim = 4pq$$
$$\dim \mathfrak{X}_0 = p + q.$$

Subtype C II-2. $p = q$.

C_p type system

$$\rho_1 = 0, \rho_2 = 3, \rho_3 = 4$$

$$\alpha = 1, \beta = 1, \gamma = \frac{3}{2}.$$

Subtype C II-1. $p < q$.

genuine BC_p type

$$\rho_1 = 4(l - 2p), \rho_2 = 3, \rho_3 = 4$$

$$\alpha = 2(l - 2p) + 1, \beta = 1, \gamma = \frac{3}{2}.$$

f) *Type* CI. $Sp(p, R)/U(p)$.

Rank $= p$

type C_p $\rho_1 = 0, \rho_2 = 1, \rho_3 = 1$

$\alpha = \beta = \gamma = 0.$

Remark. See the table at the end of this section.

4. *Heat kernels for* $\gamma = 1/2$ *and general* p.

a) *Root systems for* $\gamma = 1/2$. According to Araki-Helgason classification, the BC_p root system with $\gamma = 1/2$ are

$$\alpha = 0, \beta = 0, \gamma = \frac{1}{2} SU(p, p)/S(U(p) \times U(p))$$

$$\alpha = l - 2p + 1, \beta = 0, \gamma = \frac{1}{2}$$

$$SU(p, q)/S(U(p) \times U(q))(p \leqq q).$$

b) *Heat kernels of* $\Delta^{(\alpha,\beta,1/2)}$. Call

$$\varphi(x) = \prod_{1 \leqq j < i \leqq p} (x_i - x_j).$$

THEOREM 1. *The heat kernel*

$$p_t^{(\alpha,\beta,1/2)}(x|x')$$

(*with respect to the volume element* $m^{(\alpha,\beta,1/2)}(x')dx'$) *is given in the non compact Weyl chamber* C_x *by the formula*

$$p_t^{(\alpha,\beta,1/2)}(x|x') = \frac{e^{K_p t}}{p!\varphi(x)\varphi(x')} A_x A_{x'} \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x_j|x'_j) \tag{8.10}$$

where

(i) $K_p = (\alpha + \beta) + (\alpha + \beta - 1)\dfrac{p(p + 1)}{2} + \sum_{j=1}^{p} j^2.$

(ii) $p_t^{(\alpha,\beta)}(x_j|x'_j)$ *is the heat kernel of the one dimensional operator* $L_{x_j}^{(\alpha,\beta)}$.

(iii) A_x *is the operation of antisymmetry on the variable* x. *In the case where* $x' \to \mathbf{1}$, *we obtain*

$$p_t^{(\alpha,\beta,1/2)}(1|x) = \frac{e^{K_p t}}{\varphi(x)} \left[A_x L_{x'} \prod_{j=1}^{p} p_t^{(\alpha,\beta)}(x_j|x'_j) \right]_{x'_j = 1} \tag{8.11}$$

where

(iv) $L_{x'} = \prod_{1 \leqq j < i \leqq p} \left(\dfrac{\partial}{\partial x'_j} - \dfrac{\partial}{\partial x'_i} \right)$

(*simplest antisymmetric operator*).

Proof. The proof is easy; in the non compact case we cannot use the eigenfunction expansion. But we know by Section 4, 7 what is the heat kernel of $\hat{\Delta}^{(\alpha,\beta,1/2)}$ (formulas (4.32) and (4.33)) for the corresponding compact cases. If we change all the $\sin\frac{(R, q)}{2}$ (R root) to $\frac{1}{a}\sin\frac{(R, aq)}{2}$ we can compute easily the heat kernel for real a and everything will be holomorphic in a. Changing $a = 1$ into $a = i$, $\hat{\Delta}^{(\alpha,\beta,1/2)}$ is turned into its non compact version $\Delta^{(\alpha,\beta,1/2)}$ and the heat kernel becomes the one for the non compact case. But the heat kernel of $\hat{\Delta}^{(\alpha,\beta,1/2)}$ in the compact case was an antisymmetric combination of the one-dimensional heat kernel $\hat{p}_t^{(\alpha,\beta)}$ of $\hat{L}^{(\alpha,\beta)}$. By analytic continuation, $\hat{L}^{(\alpha,\beta)}$ changes into $L^{(\alpha,\beta)}$ and $\hat{p}_t^{(\alpha,\beta)}$ into $p_t^{(\alpha,\beta)}$ which gives the formulas described.

References. The above classification is taken from **[1]** and **[15]**, and **[31]**.

space	rank	Dimension of the space	root system	ρ_1	ρ_2	ρ_3	α	β	γ
$SO_0(p, q)/SO(p) \times SO(q)$ $p = q$	p	$2p^2$	D_p	0	0	1	0	$-\frac{1}{2}$	0
$SO_0(p, q)/SO(p) \times SO(q)$ $p < q, p + q = 2l$	p	$2pq$	B_p	$2(l - p)$	0	1	$l - p - \frac{1}{2}$	$-\frac{1}{2}$	0
$SO_0(p, q)/SO(p) \times SO(q)$ $p < q \quad p + q = 2l + 1$	p	$2pq$	B_p	$2(l - p) + 1$	0	1	$l - p$	$-\frac{1}{2}$	0
$SU(p, q)/S(U(p) \times U(q))$ $p < q$	p	$2pq$	BC_p	$2(q - p)$	1	2	$l - 2p + 1$	0	$\frac{1}{2}$
$SU(p, p)/S(U(p) \times U(p))$	p	$2p^2$	C_p	0	1	2	0	0	$\frac{1}{2}$
$SO^*(4p)/U(2p)$	$2p$	$4p(4p - 1)$	C_p	0	1	4	0	0	$\frac{3}{2}$
$SO^*(4p + 2)/U(2p + 1)$	$2p + 1$	$(2p + 1)2p$	BC_p	4	1	4	2	0	$\frac{3}{2}$
$Sp(p, R)/U(p)$	p		C_p	0	1	1	0	0	0
$Sp(p, p)/Sp(p) \times Sp(p)$	p	$4p^2$	C_p	0	3	4	1	1	$\frac{3}{2}$
$Sp(p, q)/Sp(p) \times Sp(q)$ $p < q$	p	$4pq$	BC_p	$4(l - 2p)$	3	4	$2(l - 2p) + 1$	1	$\frac{3}{2}$

9. The heat kernel on the symmetric spaces of type B_2 and C_2.

1. *Notations.* We shall consider below only rank 2 spaces of type B_2 and C_2 (pure type). Our main result is to obtain an exact formula for the heat kernels of all these spaces. We shall denote by (x, y) the algebraic coordinates. They satisfy

$$1 \leqq y \leqq x.$$

We shall also denote briefly by x a point (x, y). Apart from that, the notations are exactly as in Section 8, 1 for the Laplace operators, multiplicities of roots, etc.

2. *The particular cases of rank* 2 *spaces*: *raising operators*. We present below several intertwining computations. Let us consider with Koornwinder ([**19**] formula (5-1)) the operator

$$D_{-}^{(\gamma)} = \frac{1}{2(x-y)^{2\gamma+1}}\left(\frac{\partial}{\partial x}(x-y)^{2\gamma+1}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}(x-y)^{2\gamma+1}\frac{\partial}{\partial x}\right). \tag{9.1}$$

More explicitly we have

$$D_{-}^{(\gamma)} = \frac{\partial^2}{\partial x \partial y} + \frac{2\gamma+1}{2(x-y)}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right). \tag{9.2}$$

LEMMA 1. *We have the following formula*

$$D_{-}^{(\gamma)}\hat{L}^{(\alpha,\beta,\gamma)} = \hat{L}^{(\alpha+1,\beta+1,\gamma)}D_{-}^{(\gamma)} - (2\alpha + 2\beta + 2\gamma + 5)D_{-}^{(\gamma)}. \tag{9.3}$$

Proof. There are two ways to prove this formula. The first one is to commute $D_{-}^{(\gamma)}$ with $\hat{L}^{(\alpha,\beta,\gamma)}$ by direct algebraic computations (in the same spirit as in Section 7); the second way is to use Koornwinder results; because we are in the compact situation, we can use the Koornwinder's Jacobi polynomials $p_{n,k}^{(\alpha,\beta,\gamma)}$ of degree (n, k) where $n > k$. These polynomials satisfy, according to Koornwinder,

$$\hat{L}^{(\alpha,\beta,\gamma)}p_{n,k}^{(\alpha,\beta,\gamma)} = (-n(n+\alpha+\beta+2\gamma+2) - k(\alpha+\beta+k+1)) \times p_{n,k}^{(\alpha,\beta,\gamma)}$$

and

$$D_{-}^{(\gamma)}p_{n,k}^{(\alpha,\beta,\gamma)} = k(n+\gamma+1/2)p_{n-1,k-1}^{(\alpha+1,\beta+1,\gamma)}.$$

Using these two formulas it is easy to see that (9.3) is valid for all $p_{n,k}^{(\alpha,\beta,\gamma)}$ and so on all functions.

LEMMA 2. *We have*

$$(D_{-}^{(\gamma)})^r\hat{L}^{(\alpha,\beta,\gamma)} = \hat{L}^{(\alpha+r,\beta+r,\gamma)}(D_{-}^{(\gamma)})^r - (2r(\alpha+\beta+\gamma+r+1)+r)(D_{-}^{(\gamma)})^r. \tag{9.4}$$

The particular case $\gamma = \dfrac{n-2}{2}$. In that case

$$D_{-}^{((n-2)/2)} = \frac{\partial^2}{\partial x \partial y} + \frac{n-1}{2(x-y)}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right). \tag{9.5}$$

Let us denote $x - y = u$, $x + y = v$: then

$$D_{-}^{((n-2)/2)} = \frac{\partial^2}{\partial v^2} - \left(\frac{\partial^2}{\partial u^2} + \frac{n-1}{u}\frac{\partial}{\partial u}\right). \tag{9.5'}$$

Let us also recall that $u > 0$ because we are working in the Weyl chamber $x \geqq y$ and $u = x - y$. Then we can interpret $D_{-}^{((n-2)/2)}$ as the 1-time, n space variables wave operator

$$\square_{n+1} = \frac{\partial^2}{\partial v^2} - \Delta_{\mathbf{R}^n} \tag{9.6}$$

acting on radial functions $f(v, u)$, v being the time variable, u the radial distance to the origin in $\mathbf{R}^n$, ($n \geqq 2$). Moreover $v \geqq 2$ because x and y are greater than 1.

3. *The fractional powers of the wave operator.* Following [**27**], let us now define the fractional powers of the wave operator $\square_{n+1}$: these are

$$(I_{n+1}^{(\theta)}f)(P) = \frac{1}{H_{n+1}(\theta)}\int_{D_P} f(Q)r_{PQ}^{\theta-n-1}dQ \tag{9.7}$$

where

$$H_{n+1}(\theta) = \pi^{(n-1)/2}2^{\theta-1}\Gamma\left(\frac{\theta}{2}\right)\Gamma\left(\frac{\theta+1-n}{2}\right).$$

D_p denotes the forward light cone of vertex at $P \in \mathbf{R}^+ \times \mathbf{R}^n$, r_{PQ} is the classical Minkowski distance from P to Q.

Remark. We work here with the forward light cone although Riesz works with the backward light cone.

This integral converges (for f with compact support) for $\theta > n - 1$ and satisfies

$$I_{n+1}^{(\theta)}I_{n+1}^{(\theta')} = I_{n+1}^{(\theta+\theta')} \tag{9.8}$$

for such values of θ, θ'. Moreover

$$I_{n+1}^{(\theta)} = I_{n+1}^{(\theta+2r)}\square_{n+1}^r$$

which gives an analytic continuation in θ and it is proved that

$$I_{n+1}^{(0)} = \text{Identity}$$

and with this analytic continuation (9.8) holds for any θ, θ'. Moreover

$$\begin{cases} I_{n+1}^{(-2r)} = \square_{n+1}^r \\ I_{n+1}^{(2r)} = \square_{n+1}^{-r}. \end{cases} \tag{9.9}$$

It is also clear that for f with compact support in the forward light cone of vertex 0, $I_{n+1}^{(\theta)} f$ has compact support in the same cone. Moreover, if f

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is a function of the time v and the radial distance u in $\mathbf{R}^n$, $I_{n+1}^{(\theta)} f$ is also a function of v, u (and not of polar angles in $\mathbf{R}^n$). In particular using the change of variables

$$(9.10) \quad x - y = u, x + y = v$$

we can define $J_{n+1}^{(\theta)}$ as the restriction of the Riesz operator $I_{n+1}^{(\theta)}$ to functions of v, u, but read as functions of x, y for $x \geqq y$.

Remark. v is evidently not the time of the heat kernel $p_t^{(\alpha,\beta,\gamma)}$ that we want to construct; but it is the time-like variable in the Weyl chamber of the abelian subalgebra.

Now $n = 2\gamma + 2$, so that we obtain

LEMMA 3. *We have the intertwining property*

$$(9.11) \quad J_{2\gamma+3}^{(\theta)} L^{(\alpha,\beta,\gamma)} = L^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)} J_{2\gamma+3}^{(\theta)} + \left(-\theta\left(\alpha + \beta + \gamma - \frac{\theta}{2} + 1\right) - \frac{\theta}{2}\right) J_{2\gamma+3}^{(\theta)}.$$

Proof. By Lemma 2 this formula is valid for $\theta = -2r$, in which case

$$J_{2\gamma+3}^{(-2r)} = (\square_{-}^{(\gamma)})^r$$

by (9.9) and the definition of $J_{2\gamma+3}^{(\theta)}$. Recall also that

$$L^{(\alpha,\beta,\gamma)} = -\hat{L}^{(\alpha,\beta,\gamma)}$$

(because of the passage of compact to non compact). By analytic continuation we obtain the lemma.

4. *The exchange property between B_2 and C_2 root systems.* Our next step is a very peculiar property of the root systems in a two dimensional space. Consider two examples E and E' of a 2-dimensional space $\mathbf{R}^2$, E with coordinates (q_1, q_2) and E' with coordinates (Q_1, Q_2) and let us consider the linear transformations inverse of each other:

$$(9.12) \quad \begin{array}{ll} Q_1 = q_1 - q_2 & 2q_1 = Q_1 + Q_2 \\ Q_2 = q_1 - q_2 & 2q_2 = Q_1 + Q_2. \end{array}$$

Let us now consider on E a C_2 root system with multiplicities (ρ_1, ρ_2, ρ_3), so that $\rho_1 = 0$. The Weyl chamber $\{0 \leqq q_2 \leqq q_1\}$ becomes the Weyl chamber $\{0 \leqq Q_1 \leqq Q_2\}$.

We then see that the system C_2 transferred to E' by the transformation (9.12) becomes a system B_2 according to the following table $C_2 \to B_2$

(9.13)

C_2 system on E $\longrightarrow$		B_2 system on E'	
q_i	$\rho_1 = 0$	$Q_i = q_1 \pm q_2$	$\rho_1' = \rho_3$
$2q_i$	ρ_2	$2Q_i$	$\rho_2' = 0$
$q_1 + q_2$	ρ_3	$Q_1 \pm Q_2$	$\rho_3' = \rho_2$

the multiplicities becoming $\rho_1' = \rho_3$, $\rho_3' = \rho_2$. Coming back to the α, β, γ and α', β', γ', we obtain

$$(9.14) \qquad \begin{cases} \alpha' = \gamma \\ \beta' = -\dfrac{1}{2} \quad \text{(recall that } \alpha = \beta \text{ for } C_2) \\ \gamma' = \beta. \end{cases}$$

Conversely we can go from B_2 to C_2

(9.15)

C_2 system in E' $\longleftarrow$		B_2 system in E	
q_i	$\rho_1' = 0$	Q_i	ρ_1
$2q_i$	$\rho_2' = \rho_3$	$2Q_i$	$\rho_2 = 0$
$q_1 + q_2$	$\rho_3' = \rho_1$	$Q_1 \pm Q_2$	ρ_3

$$(9.16) \qquad \begin{cases} \alpha' = \beta' = \gamma \quad \left(\text{recall that } \beta = -\dfrac{1}{2} \text{ for } B_2\right) \\ \gamma' = \alpha. \end{cases}$$

Let us consider these transformation properties on the specific root systems described in 2 of Section 8 for the multiplicities corresponding to symmetric spaces:

(9.17)

C_2 system		corresponds to	B_2 system	
$\alpha = \beta = 0$	$\gamma = \frac{1}{2}$	$\alpha' = \frac{1}{2}$	$\beta' = -\frac{1}{2}$	$\gamma' = 0$
$\alpha = \beta = 1$	$\gamma = \frac{3}{2}$	$\alpha' = \frac{3}{2}$	$\beta' = -\frac{1}{2}$	$\gamma' = 1$
$\alpha = \beta = 0$	$\gamma = l - 2 - \frac{1}{2}$	$\alpha' = l - 2 - \frac{1}{2}$	$\beta' = -\frac{1}{2}$	$\gamma' = 0$
$\alpha = \beta = \frac{1}{2}$	$\gamma = \frac{1}{2}$	$\alpha' = \frac{1}{2}$	$\beta' = -\frac{1}{2}$	$\gamma' = \frac{1}{2}$
$\alpha = \beta = 0$	$\gamma = 0$	$\alpha' = 0$	$\beta' = -\frac{1}{2}$	$\gamma' = 0$

Now, we have to do the change of coordinates (9.12) on the Laplace operators. We start with the C_2 root system in variable (q_1, q_2), multiplicities $(\rho_1 = 0, \rho_2, \rho_3)$, so that

$$\Delta^{(0,\rho_2,\rho_3)} = \frac{1}{W^{(0,\rho_2,\rho_3)}} \sum_{i=1}^{2} \frac{\partial}{\partial q_i} W^{(0,\rho_2,\rho_3)} \frac{\partial}{\partial q_i}.$$

But

$$\frac{\partial}{\partial q_1} = \frac{\partial}{\partial Q_1} + \frac{\partial}{\partial Q_2} \quad \frac{\partial}{\partial q_2} = \frac{\partial}{\partial Q_2} - \frac{\partial}{\partial Q_1}.$$

The Laplace operator becomes

$$\frac{1}{W}\left[\left(\frac{\partial}{\partial Q_1} + \frac{\partial}{\partial Q_2}\right)\left(\left\{\sinh\left(\frac{Q_1 + Q_2}{2}\right)\sinh\left(\frac{Q_1 - Q_2}{2}\right)\right\}^{\rho_2}\right.\right.$$
$$\times \left\{\sinh\frac{Q_1}{2}\sinh\frac{Q_2}{2}\right\}^{\rho_3}\left.\left(\frac{\partial}{\partial Q_1} + \frac{\partial}{\partial Q_2}\right)\right)$$
$$+ \left(\frac{\partial}{\partial Q_2} - \frac{\partial}{\partial Q_1}\right)\left(\left\{\sinh\left(\frac{Q_1 + Q_2}{2}\right)\sinh\left(\frac{Q_1 - Q_2}{2}\right)\right\}^{\rho_2}\right.$$
$$\times \left\{\sinh\frac{Q_1}{2}\sinh\frac{Q_2}{2}\right\}^{\rho_3}\left.\left.\left(\frac{\partial}{\partial Q_1} - \frac{\partial}{\partial Q_2}\right)\right)\right]$$

but this is exactly twice the Laplace operator of the B_2 root system in coordinates (Q_1, Q_2) and multiplicities $(\rho_1' = \rho_3, \rho_2' = 0, \rho_3' = \rho_2)$. We can summarize all this by the following lemma.

LEMMA 4. *By the transformation* (9.12), *the C_2 root system transforms to B_2 root system*; *the multiplicities transform according to table* (9.13); *the* (α, β, γ) *transform according to* (9.14) *and the Laplace operator becomes twice the Laplace operator of the B_2* system.

5. *Deduction of certain heat kernels in rank* 2: *formal schemes.*

a) *What we already know.* We know how to compute the heat kernel of $\Delta^{(\alpha,\beta,\gamma)}$ in the case of rank 2 and $\gamma = -1/2$ (in this case, we know that $\Delta^{(\alpha,\beta,-1/2)}$ is just the sum $L_x^{(\alpha,\beta)} + L_y^{(\alpha,\beta)}$ of two independant operators and the heat kernel $e^{t\Delta}$ is just the product of two heat kernels in the independent variables x and y). We also know how to compute the heat kernel for $\gamma = +1/2$ by Section 8, Theorem 1. So we basically know $(\alpha, \beta, \pm 1/2)$.

b) *Deduction by change of variables.* 1°) If $\alpha = \beta$, in which case we have a C_2 root system, we can treat by (9.14)

$$(9.18) \qquad \left(\alpha' = \pm\frac{1}{2}, \beta' = -\frac{1}{2}, \gamma' = \alpha\right)$$

from $\left(\alpha = \beta, \gamma = \pm\frac{1}{2}\right)$ by the change of variables

$$Q_1 = q_1 - q_2$$
$$Q_2 = q_1 + q_2.$$

2°) If $\beta = -1/2$, in which case we start from a B_2 root system, we can treat by (9.15)

$$\left(\alpha' = \beta' = \pm\frac{1}{2}, \gamma' = \alpha\right) \tag{9.19}$$

from (α, $\beta = -1/2$, $\gamma = \pm 1/2$) by the change of variables

$$q_1 = \frac{Q_1 + Q_2}{2}$$

$$q_2 = \frac{Q_1 + Q_2}{2}.$$

c) *Deduction by* $J^{\theta}_{2\gamma'+3}$. If γ' is an integer or an integer $+1/2$, we can apply the operator $J^{\theta}_{2\gamma'+3}$ to the case (9.17) and (9.18) to obtain

$$\left(-\frac{\theta}{2} \pm \frac{1}{2}, -\frac{\theta}{2} - \frac{1}{2}, \gamma'\right) \tag{9.20}$$

from (9.17) by $J^{\theta}_{2\gamma'+3}$ or

$$\left(\alpha'' = \beta'' = -\frac{\theta}{2} \pm \frac{1}{2}, \gamma'\right) \tag{9.21}$$

from (9.18) by $J^{\theta}_{2\gamma'+3}$ whatever θ is. But the case (9.20) corresponds also to a C_2 system and so we can apply a change of variable to (9.20)

$$Q_1 = q_1 - q_2$$
$$Q_2 = q_1 + q_2$$

to obtain

$$\left(\gamma', \beta = -\frac{1}{2}, -\frac{\theta}{2} \pm \frac{1}{2}\right) \tag{9.22}$$

from (9.20).

d) Finally we can summarize this by the table

$$\left(\alpha, \alpha, \pm\frac{1}{2}\right) \xrightarrow{C_2 \to B_2} \left(\pm\frac{1}{2}, -\frac{1}{2}, \alpha\right)$$

$$\xrightarrow{J^{\theta}} \left(-\frac{\theta}{2} \pm \frac{1}{2}, -\frac{\theta}{2} - \frac{1}{2}, \alpha\right)$$

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$$(9.23)\quad \left(-\frac{\theta}{2}-\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}, \alpha\right) \xrightarrow{C_2 \to B_2} \left(\alpha, -\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}\right)$$

$$\left(\alpha, -\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}\right) \overset{J^\rho}{\to} \left(\alpha-\frac{\rho}{2}, -\frac{1}{2}-\frac{\rho}{2}, -\frac{\theta}{2}-\frac{1}{2}\right)$$

the applications of J are submitted to the conditions that α and $-\theta/2 - 1/2$ are integers or integers $+1/2$. The arrows

$$C_2 \to B_2 \text{ or } \overset{J}{\mapsto}$$

mean the operation by which we can pass from the left to the right of the arrow.

e) *The cases of symmetric spaces*. We come back to the table (9.16); in that table, we can find the heat kernel of any symmetric space with $\gamma = 1/2$. If we can find on a given line the (α, β, γ) of the column, we can find also the one of the other column by the operation $C_2 \rightleftarrows B_2$.

1st line. This can be treated easily because one of the elements in that line contains $\gamma = 1/2$.

2nd line. The left element $\alpha = \beta = 1$, $\gamma = 3/2$ can be obtained as

$$\left(-\frac{\theta}{2}-\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}, \alpha\right)$$

with $\alpha = 3/2$, $\theta = -3$, applying $J^{(-3)}$ to $(-1/2, -1/2, \alpha)$.

3rd line. $\alpha = 0 = \beta$, $\gamma = l - 2 - \dfrac{1}{2}$ can be obtained as

$$\left(-\frac{\theta}{2}-\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}, \alpha\right)$$

with $\alpha = l - 2 - 1/2$ and $\theta = -1$ applying $J^{(-1)}$ to $(-1/2, -1/2, \alpha)$.

4th line. This is trivial because it contains an element with $\gamma = 1/2$.

5th line. $\alpha = \beta = \gamma = 0$ can be obtained as

$$\left(-\frac{\theta}{2}-\frac{1}{2}, -\frac{\theta}{2}-\frac{1}{2}, \alpha\right)$$

with $\theta = -1$, $\alpha = 0$ applying $J^{(-1)}$ to $(-1/2, -1/2, \alpha)$. We can summarize all this by the lemma following:

LEMMA 5. *It is possible to obtain an explicit form for the heat kernel $\Delta^{(\alpha,\beta,\gamma)}$ in the cases given by the schemes* (9.22). *In particular, all heat kernels of non compact symmetric spaces of pure types B_2 or C_2 can be obtained explicitly.*

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6. *Explicit expressions for the heat kernels: analytic formulas.*

a) *Transformation $C_2 \to B_2$ in algebraic coordinates.* We denote (q_1, q_2) the coordinates of a C_2 system, (Q_1, Q_2) the coordinates of the B_2 system corresponding to (q_1, q_2) by

$$(9.24) \qquad Q_1 = q_1 - q_2 \quad Q_2 = q_1 + q_2.$$

We denote $x_i = \cosh q_i$, $X_i = \cosh Q_i$ the corresponding algebraic coordinates; they are related to each other by (9.24). We proved in Lemma 4 that the Laplace operator $\Delta_x^{(\alpha,\beta,\gamma)}$ of the C_2 system becomes $2\Delta_X^{(\alpha',\beta',\gamma')}$ of the B_2 system by (9.24); the indices x, X refer to variables in which these operators are written. The jacobian of (9.24) is 2. This is also the jacobian of $x \to X$, so that the volume element transforms according to

$$dV^{(\rho_1',\rho_2',\rho_3')}(Q) = 2dV^{(\rho_1,\rho_2,\rho_3)}(q)$$
$$dV^{(\alpha',\beta',\gamma')}(X) = 2dV^{(\alpha,\beta,\gamma)}(x).$$

The correspondences

$$\rho \to \rho' \quad \text{and} \quad (\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$$

are described in 4 and 5. Let us start with the heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} p_t^{(\alpha,\beta,\gamma)}(x|x') = \Delta_x^{(\alpha\beta\gamma)} p_t^{(\alpha,\beta,\gamma)}(x|x') \\ p_t^{(\alpha,\beta,\gamma)}(x|x') dV^{(\alpha\beta\gamma)}(x') \to \delta(x - x'). \end{cases}$$

If we perform the change of coordinates $x \to X$ we obtain

$$\begin{cases} \dfrac{\partial}{\partial t} p_t^{(\alpha,\beta,\gamma)}(x(X)\,|x'(X')\,) = 2\Delta_X^{(\alpha'\beta'\gamma')} p_t^{(\alpha,\beta,\gamma)}(x(X)\,|x'(X')\,) \\ p_t^{(\alpha,\beta,\gamma)}(x|X)x'(X')\,)\dfrac{1}{2} dV^{(\alpha'\beta'\gamma')}(X') \to \delta(X - X') \end{cases}$$

so we deduce

$$(9.25) \qquad p_t^{(\alpha'\beta'\gamma')}(X|X') = \frac{1}{2} p_{2t}^{(\alpha\beta\gamma)}(x|x').$$

Remark. $p_t^{(\alpha\beta\gamma)}$ is always taken with respect to the volume element $dV^{(\alpha\beta\gamma)}(x')$; in the same manner, $p_t^{(\alpha',\beta',\gamma')}(X|X')$ is taken with respect to the volume element $dV^{(\alpha'\beta'\gamma')}(X')$. As a corollary we deduce

THEOREM 1. *We have*

$$(9.26) \qquad p_t^{(\pm 1/2, -1/2, \alpha)}(X|X') = \frac{1}{2} p_{2t}^{(\alpha,\alpha,\pm 1/2)}(x|x')$$

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with the usual correspondence $X = X(x)$, $X' = X'(x')$ *given through the* q_i *and* Q_i *coordinates by* (9.24).

We have also

$$(9.27)\quad p_t^{(\alpha,-1/2,(-(\theta/2))-(1/2))}(X|X') = \frac{1}{2}p_{2t}^{(-(\theta/2)-(1/2),(-(\theta/2))-(1/2),\alpha)}(x|x').$$

Proof. This is the first line and second line of (9.23) and the formula (9.25).

b) *Application of the J operator.* Let us consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} = L_x^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}u \\ u|_{t=0} = u_0. \end{cases}$$

Define

$$v = e^{(-\theta(\alpha+\beta+\gamma-(\theta/2)+1)-(\theta/2))t}u.$$

Then

$$\begin{cases} \dfrac{\partial v}{\partial t} = L^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}v - \left(\theta\left(\alpha+\beta+\gamma-\dfrac{\theta}{2}+1\right)+\dfrac{\theta}{2}\right)v \\ v|_{t=0} = u_0. \end{cases}$$

Define w by

$$v = J_{2\gamma+3}^{(\theta)}w:$$

then we have

$$\begin{cases} \dfrac{\partial w}{\partial t} = L^{(\alpha,\beta,\gamma)}w \\ w|_{t=0} = J_{2\gamma+3}^{(-\theta)}\,u_0 \end{cases}$$

using Lemma 3 (9.11); this means that

$$(9.29)\quad u(t,x) = e^{(\theta(\alpha+\beta+\gamma-(\theta/2)+1)+(\theta/2))t} \times J_{2\gamma+3,x}^{(\theta)}\int p_t^{(\alpha\beta\gamma)}(x|x')(J_{2\gamma+3,x'}^{(-\theta)}\,u_0)(x')m^{(\alpha\beta\gamma)}(x')dx'$$

(this is true provided γ is integer or integer $+1/2$). The notation $J_{2\gamma+3,x}^{(\theta)}$ means the operator $J_{2\gamma+3}^{(\theta)}$ acting on x variable. We shall denote by

$$j_{2\gamma+3}^{(-\theta)}(x'|x'')dx''$$

the kernel of $J_{2\gamma+3,x'}^{(-\theta)}$ (with respect to the Lebesgue measure). The domain of integration is (in the space $\mathbf{R}^{2\gamma+3}$) the light cone with vertex at a point corresponding to x' by the rule given in 3 (which point precisely we choose

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does not matter because of rotational invariance in the space like direction). When we write

$$j_{2\gamma+3}^{(-\theta)}(x'|x'')dx''$$

the situation is here asymmetric; in fact by that notation we mean that

$$\int j_{2\gamma+3}^{(-\theta)}(x'|x'')f(x'')dx'' = (I_{2\gamma+3}^{(-\theta)}\tilde{f})(v', u')$$

where $\tilde{f}$ is the space-time function rotationally invariant in the space-like direction,

$$u' = x_1' - x_2' \text{ and } v' = x_1' + x_2'$$

and $I_{2\gamma+3}^{(-\theta)}$ has been previously defined. In the previous integral, x' and x'' are both in the Weyl chamber with vertex at 1 in $\mathbf{R}^2$. With these conventions made, it is obvious by (9.28) that we have the following theorem.

THEOREM 2. *For γ integer or integer $+1/2$, the heat kernel of*

$$L_x^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}$$

is given by the formula

$$(9.29) \quad p_t^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x|x') = \frac{e^{\theta(\alpha+\beta+\gamma-(\theta/2)+(3/2))t}}{m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x')} \int\int j_{2\gamma+3}^{(\theta)}(x|\xi)p_t^{(\alpha,\beta,\gamma)}(\xi|\eta)x \times j_{2\gamma+3}^{(-\theta)}(\eta|x')m^{(\alpha\beta\gamma)}(\eta)d\xi d\eta.$$

c) *Remarks on the integrations in* (9.29). Consider the light cone of vertex $(v', \vec{\xi}') \in \mathbf{R}^{2\gamma+3}$ corresponding to x'. Let $(v'', \vec{\xi}'')$ be a generic point in this light cone,

$$r^2 = (v' - v'')^2 - |\vec{\xi}' - \vec{\xi}''|^2$$

the square of the Minkowski distance, so that $r^2 \geqq 0$ and $v'' - v' > 0$. Let $u' = |\vec{\xi}'|$, $u'' = |\vec{\xi}''|$ and α the angle between $\vec{\xi}'$ and $\vec{\xi}''$; then

$$-1 \leqq \cos\alpha \equiv \frac{r^2 - (v' - v'')^2 + u'^2 + u''^2}{2u'u''} \leqq +1$$

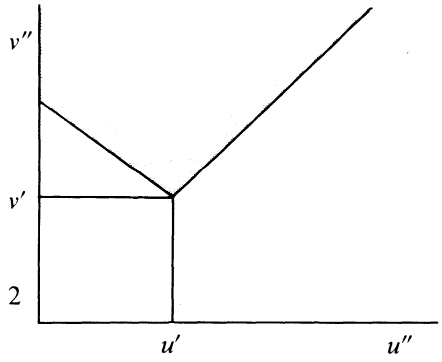
which implies in particular

$$r^2 - (v' - v'')^2 + (u' - u'')^2 \leqq 0$$

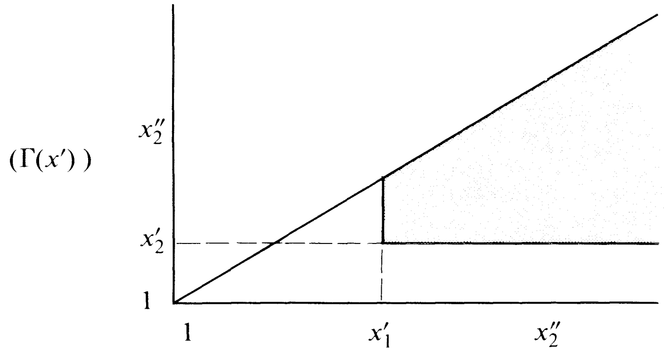
and because $r^2 > 0$, we get

$$v'' - v' \geqq |u' - u''|.$$

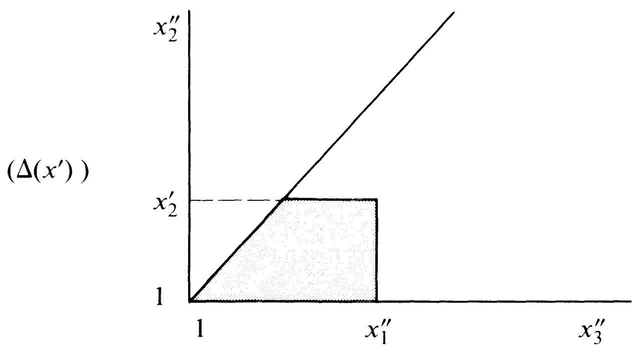
This, in turn, implies that the domain of integration in (u'', v'') variable is contained in the shadowed domain:



This implies that the domain of integration in the x'' variables is obtained by rotating by $-\pi/4$ and is contained in the domain $\Gamma(x')$:



Conversely the point x'' corresponding to a backward light cone with vertex at x' will be contained in the following domain $\Delta(x')$:



This means that in (9.29) the double integral is taken
1°) in ξ, on a domain of the type $\Gamma(x')$ with vertex at x
2•) in η, on a domain of the type $\Delta(x')$ with vertex at x'.

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d) *We want now to obtain more explicit expressions for these formulas.* We shall begin in the next two sections by the case when $\theta = -2r$ with r a positive integer. In that case,

$$J_{2\gamma+3}^{(\theta)} = (D_{-}^{(\gamma)})^r$$

and we shall have a simplification similar to the one found in Section 7, 6 and 7, if we put the source of heat at point 1.

7. *The adjoint of* $(D_{-}^{(\gamma)})^r$. All functions are defined on the Weyl chamber

$$C(1) = \{ (x_1, x_2) \,|\, 1 \leqq x_2 \leqq x_1 \}$$

but they extend smoothly to $[1, +\infty[\times [1, +\infty[$ as invariant functions by the action of the Weyl group, which, in the algebraic coordinates $x_i = \cosh q_i$ reduces to the permutation of x_1 and x_2; so the functions we consider are smooth symmetric functions on $[1, +\infty[\times [1, +\infty[$. The following lemma was proved by Koornwinder [**18**] in the compact case:

LEMMA 6. *Let*

$$\mu^{(\alpha,\beta)}(X) = ((1 - x_1)(1 - x_2))^\alpha((1 + x_1)(1 + x_2))^\beta$$

and let

$$D_{+}^{(\alpha\beta\gamma)} = (\mu^{(\alpha,\beta)}(x))^{-1} \circ D_{-}^{(\gamma)} \circ \mu^{(\alpha+1,\beta+1)}(x). \tag{9.30}$$

Then we have the following formula for φ, ψ, C^∞ *and defined on* Λ_x, *with compact support*

$$\int_{\Lambda_x} (D_{-}^{(\gamma)}\psi)(x_1, x_2)\varphi(x_1, x_2)m^{(\alpha+1,\beta+1,\gamma)}(x_1, x_2)dx_1dx_2 \tag{9.31}$$
$$= \int_{\Lambda_x} \psi(x_1, x_2)(D_{+}^{(\alpha,\beta,\gamma)}\varphi)(x_1, x_2)m^{(\alpha,\beta,\gamma)}(x_1, x_2)dx_1dx_2$$

for $\gamma > -1/2$, $\alpha > -1$.

Proof. By definition of $m^{(\alpha+1,\beta+1,\gamma)}$ and of $D_{-}^{(\gamma)}$ we are reduced to proving

$$\int_{\Lambda_x} \left(\frac{\partial}{\partial x_1}\left((x_1 - x_2)^{2\gamma+1}\frac{\partial\psi}{\partial x_2}\right)\right. \tag{9.32}$$
$$\left. + \frac{\partial}{\partial x_2}\left((x_1 - x_2)^{2\gamma+1}\frac{\partial\psi}{\partial x_1}\right)\right)\Phi dx_1dx_2$$
$$= \int_{\Lambda_x} \psi(x_1, x_2)\left(\frac{\partial}{\partial x_1}\left((x_1 - x_2)^{2\gamma+1}\frac{\partial\Phi}{\partial x_2}\right)\right.$$

$$+ \frac{\partial}{\partial x_2}\left((x_1 - x_2)^{2\gamma+1}\frac{\partial \Phi}{\partial x_1}\right)\bigg) dx_1 dx_2$$

where

$$\Phi(x_1, x_2) = \mu^{(\alpha+1,\beta+1)}(x_1, x_2)\varphi(x_1, x_2).$$

Consider the integral

$$I = -\int_{\Lambda_x}\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\Phi}{\partial x_2} + \frac{\partial\psi}{\partial x_2}\frac{\partial\Phi}{\partial x_1}\right)(x_1 - x_2)^{2\gamma+1}dx_1dx_2.$$

Then the two members of (9.32) are equal to I up to boundary terms. But for $\alpha > -1$, the boundary term on $x_2 = 1$ will be 0 and for $\gamma > -1/2$, the boundary term on $x_1 = x_2$ will also be 0. By iteration of Lemma 6, we obtain

COROLLARY. *Under the hypothesis of Lemma* 6, *we have also for any integer* r

$$(9.33) \quad \int_{\Lambda_x} ((D_-^{(\gamma)})^r\psi)(x_1, x_2)\varphi(x_1, x_2)m^{(\alpha+r,\beta+r,\gamma)}(x_1, x_2)dx_1dx_2$$
$$= \int_{\Lambda_x} \psi(x_1, x_2)(D_+^{(\alpha,\beta,\gamma)} \circ D_+^{(\alpha+1,\beta+1,\gamma)} \circ$$
$$\ldots \circ D_+^{(\alpha+r-1,\beta+r-1,\gamma)}\varphi)(x_1, x_2)m^{(\alpha,\beta,\gamma)}(x_1, x_2)dx_1dx_2.$$

8. *Reduction of the analytic expressions of the heat kernel for θ a negative even integer.* Let us now come back to Lemma 2; this can be rewritten as

$$(D_-^{(\gamma)})^r L^{(\alpha,\beta,\gamma)} = L^{(\alpha+r,\beta+r,\gamma)}(D_-^{(\gamma)})^r$$
$$+ (2r(\alpha + \beta + \gamma + r + 1) + r)(D_-^{(\gamma)})^r.$$

It is clear that the function

$$(9.34) \quad e^{-(2r(\alpha+\beta+\gamma+r+1)+r)t}(D_-^{(\gamma)})_x^r p_t^{(\alpha,\beta,\gamma)}(1|x)$$

is a solution of the heat equation

$$\frac{\partial}{\partial t} = L_x^{(\alpha+r,\beta+r,\gamma)}.$$

We want to find the singularity for $t \to 0^+$; let φ be a C^∞ function with compact support around 1 in Λ_x and consider the integral:

$$\int_{\Lambda_x} (D_-^{(\gamma)})_x^r p_t^{(\alpha,\beta,\gamma)}(1|x)\varphi(x)m^{(\alpha+r,\beta+r,\gamma)}(x)dx.$$

By (9.33) this is

$$\int_{\Lambda_x} p_t^{(\alpha,\beta,\gamma)}(1|x)(D_+^{(\alpha,\beta,\gamma)} \circ D_+^{(\alpha+1,\beta+1,\gamma)} \circ \ldots \circ D_+^{(\alpha+r-1,\beta+r-1,\gamma)}\varphi)(x)m^{(\alpha,\beta,\gamma)}(x)dx.$$

But

$$p_t^{(\alpha,\beta,\gamma)}(1|x)m^{(\alpha,\beta,\gamma)}(x)dx$$

tends to $\delta(1 - x)$ if $t \to 0^+$, so this integral tends to

$$(D_+^{(\alpha,\beta,\gamma)} \circ D_+^{(\alpha+1,\beta+1,\gamma)} \circ \ldots \circ D_+^{(\alpha+r-1,\beta+r-1,\gamma)}\varphi)(1). \tag{9.35}$$

Let us compute the number (9.35). First we have:

LEMMA 7. *Let $f(x_1, x_2)$ be a C^∞ function on $[1, +\infty[\times [1, +\infty[$ symmetric in x_1, x_2. Then*

$$\begin{aligned}\frac{1}{\mu^{(\alpha',\beta',\gamma)}} D_-^{(\gamma)}(\mu^{(\alpha'+1,\beta'+1,\gamma)}f) \\ = (\alpha' + 1)(1 + x_1)(1 + x_2)(\alpha' + 2 + 2\gamma)f + g\end{aligned} \tag{9.36}$$

where g is a C^∞ symmetric function vanishing at $(1, 1)$ of the form

$$(1 - x_1)g_1 + (1 - x_2)g_2.$$

Proof. We compute

$$\begin{aligned}&\frac{1}{((1 - x_1)(1 - x_2))^{\alpha'}((1 + x_1)(1 + x_2))^{\beta'}} \\ &\times \left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{2\gamma + 1}{x_1 - x_2}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\right) \\ &\times (((1 - x_1)(1 - x_2))^{\alpha'+1}((1 + x_1)(1 + x_2))^{\beta'+1}f) \\ &= (\alpha' + 1)^2(1 + x_1)(1 + x_2)f \\ &+ (1 + x_1)(1 + x_2)(2\gamma + 1)(\alpha' + 1)f \\ &+ (2\gamma + 1)(1 - x_1)(1 - x_2)\frac{1}{x_1 - x_2} \\ &\times \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)((1 + x_1)(1 + x_2))^{\beta'+1}f) \\ &+ 0(1 - x_1, 1 - x_2).\end{aligned}$$

Here the first two terms give

$$(1 + x_1)(1 + x_2)(\alpha' + 1)(\alpha' + 2 + 2\gamma)f;$$

the third term is smooth in x_1, x_2 because

$$((1 + x_1)(1 + x_2))^{\beta'+1}f$$

is symmetric in x_1, x_2 and C^{∞}; the last term is also symmetric and C^{∞} in x_1, x_2 and comes from action of

$$\frac{\partial^2}{\partial x_1 \partial x_2} \quad \text{on } \mu^{(\alpha'+1,\beta'+1)}f$$

and is of the form

$$(1 - x_1)\varphi + (1 - x_2)\psi.$$

Now, to compute (9.35) we apply recursively Lemma 7 (9.36): we obtain

$$\frac{1}{\mu^{(\alpha+r-1,\beta+r-1)}} D_-^{(\gamma)} \mu^{(\alpha+r,\beta+r)} f$$
$$= (\alpha + r)(\alpha + r + 1 + 2\gamma)(1 + x_1)(1 + x_2)f + g_1$$

with g_1 as in Lemma 7; then

$$D_+^{(\alpha+r-2,\beta+r-2,\gamma)}(D_+^{(\alpha+r-1,\beta+r-1,\gamma)}f)$$
$$= (\alpha + r - 1)(\alpha + r + 2\gamma)(1 + x_1)(1 + x_2)$$
$$\times D_+^{(\alpha+r-1,\beta+r-1,\gamma)}f + g_2$$
$$= (\alpha + r)(\alpha + r - 1)(\alpha + r + 1 + 2\gamma)(\alpha + r + 2\gamma)$$
$$\times ((1 + x_1)(1 + x_2))^2 f + g_3$$

where g_2 and g_3 are as in Lemma 7. So we finally obtain

$$(9.37) \quad D_+^{(\alpha\beta\gamma)} \circ \ldots \circ D_+^{(\alpha+r-1,\beta+r-1,\gamma)} \varphi|_1$$
$$= (\alpha + r)(\alpha + r - 1) \ldots (\alpha + 1)$$
$$\times (\alpha + r + 1 + 2\gamma) \ldots (\alpha + r + 2\gamma) \times 2^{2r}\varphi(1).$$

This implies, using (9.37), and the beginning of this part, the following theorem.

THEOREM 3. *The heat kernel of $L^{(\alpha+r,\beta+r,\gamma)}$ (with r integer) at pole* 1, *is*

$$(9.37) \quad \frac{e^{-(2r(\alpha+\beta+\gamma+r+1)+r)t}}{(\alpha + r) \ldots (\alpha + 1)(\alpha + 2\gamma + r + 1) \ldots (\alpha + 2\gamma + r)2^{2r}}$$
$$\times (D_-^{(\gamma)})_x^r p_t^{(\alpha,\beta,\gamma)}(1|x)$$
$$\equiv p_t^{(\alpha+r,\beta+r,\gamma)}(1|x).$$

Remark. Unfortunately, this formula is not sufficient to obtain heat kernels for symmetric spaces of rank 2, because we need the action of $J^{(\theta)}$ for θ integer (positive or negative) and Theorem 3 gives the action for θ even integer only in which case

$$J^{(-2r)}_{2\gamma+3} = (D^{(\gamma)}_{-})^r.$$

On the other hand, in Theorem 3, γ does not need to be integer or integer $+1/2$.

9. *Limit behaviour of the heat kernel for* $x' \to \mathbf{1}$.

a) *Simplifying* (9.29) *for* $x' \to \mathbf{1}$. We suppose below that γ is "integer" or "integer $+1/2$". We come back to Theorem 2 formula (9.29) and we want to examine the behaviour of

$$p_t^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x|x')$$

when $x' \to \mathbf{1} = (1, 1)$. We have an integral in η over the domain $\Delta(x')$ shown on the figure of part 6. This domain shrinks to $\mathbf{1}$ when $x' \to 1$ but the denominator

$$m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x')$$

tends to 0. We shall show below that the same phenomenon as in Section 7, 6 occurs, namely that when $-\theta > 2\gamma + 1$, the integral

$$\frac{1}{m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x')} \int_{\Delta(x')} p_t^{(\alpha,\beta,\gamma)}(\xi|\eta) \times j^{(-\theta)}_{2\gamma+3}(\eta|x') m^{(\alpha\beta\gamma)}(\eta) d\eta \tag{9.38}$$

has the limit $Cp_t^{(\alpha,\beta,\gamma)}(\xi|\mathbf{1})$ when $x' \to \mathbf{1}$ (C being a fixed constant independent of ξ). We assume below $-\theta > 2\gamma + 1 = n - 1$ (which is the condition under which $I^{(-\theta)}_{2\gamma+3}$ is absolutely convergent (see part 3).

b) *Computation of the integral* (9.38) *for* $x' \to \mathbf{1}$. Let us fix a point $x' = (x'_0, y'_0)$ $(x'_0, y'_0 > 1)$; by a translation of coordinates we shall assume that $x'_0, y'_0 > 0$ and tends to 0 and $x'_0 > y'_0$. Then

$$m^{(\alpha,\beta,\gamma)}(x') = (x'_0 y'_0)^\alpha ((2 + x'_0)(2 + y'_0))^\beta (x'_0 - y'_0)^{2\gamma+1}.$$

Call as usual

$$u'_0 = x'_0 - y'_0, \; v'_0 = x'_0 + y'_0$$

which is in a light cone; call $(\vec{\xi}'_0, v'_0)$ a space-time point with

$$|\vec{\xi}'_0| = u'_0.$$

Obviously $(\vec{\xi}'_0, v'_0)$ is in the forward light cone. Call $\Delta(\vec{\xi}'_0, v'_0)$ the intersection of the forward light cone with vertex $(\vec{0}, 0)$ with the backward light cone with vertex $(\vec{\xi}'_0, v'_0)$. We study the integral

$$I(\vec{\xi}'_0, v'_0) = \int_{\Delta(\vec{\xi}'_0, v'_0)} ((v_0 - v)^2 - |\vec{\xi}'_0 - \vec{\xi}|^2)^{(-\theta-n-1)/2}$$

$$\times\ m^{(\alpha,\beta,\gamma)}(\vec{\xi}, v)d\vec{\xi}dv$$

when

$$(\vec{\xi}_0', v_0') \to 0.$$

This integral depends only on $|\vec{\xi}_0'| = u_0'$ and v_0'. We shall make $(\vec{\xi}_0', v_0')$ tend to 0 according to a straight line and we shall study

$$I(\lambda\vec{\xi}_0', \lambda v_0') \quad \lambda \to 0^+.$$

It is clear that when $\lambda \to 0^+$

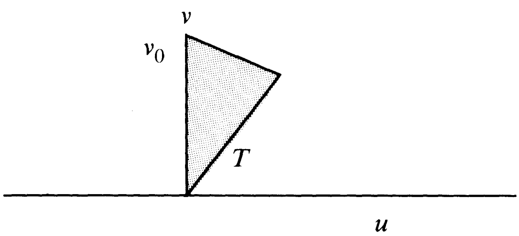
$$I(\lambda\vec{\xi}_0', \lambda v_0') \sim I(0, \lambda v_0').$$

But because

$$m^{(\alpha,\beta,\gamma)}(\vec{\xi}, v) \sim (u)^{2\gamma+1}(v^2 - u^2)^{\alpha},$$

$$I(\vec{0}, v_0') \sim \int_{\Delta(\vec{0},v_0')} ((v_0 - v)^2 - |\vec{\xi}|^2)^{\rho/2}u^{2\gamma+1}(v^2 - u^2)^{\alpha}d\vec{\xi}dv$$

where $\rho = -\theta - n - 1$ and so this last integral is the integral over the triangle



$$\int_T ((v_0 - v)^2 - u^2)^{\rho/2}u^{2\gamma+n}(v^2 - u^2)^{\alpha}dudv$$

which can be split into two integrals

$$\text{(9.39)} \quad \int_0^{v_0/2} dv \int_0^{v} ((v_0 - v)^2 - u^2)^{\rho}u^{2\gamma+n}(v^2 - u^2)^{\alpha}du$$
$$+ \int_{v_{0/2}}^{v_0} dv \int_0^{v_0-v} ((v_0 - v)^2 - u^2)^{\rho}u^{2\gamma+n}(v^2 - u^2)^{\alpha}du.$$

Both integrals in (9.39) are absolutely converging because $\rho > -2$ and are equivalent to

$$Cv_0^{2\alpha+2\gamma+2+n+\rho}$$

and so

$$I(\lambda\vec{\xi}_0', \lambda v_0') \sim C\lambda^{2\alpha+2\gamma+2+n+\rho} \quad \text{if } \lambda \to 0^+.$$

But we obviously have

$$m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(\lambda u_0', \lambda v_0') \sim C\lambda^{2\alpha-\theta+2\gamma+1}$$

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with

$$-\theta = \rho + n + 1$$

so that

$$I(\lambda\vec{\xi}_0', \lambda v_0') \quad \text{and} \quad m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(\lambda u_0', \lambda v_0')$$

have the sum equivalent when $\lambda \to 0$.

We thus obtain:

LEMMA 8. *The integral* (9.38) *tends to* $Cp_t^{(\alpha,\beta,\gamma)}(\xi|1)$ *when* $x' \to 1$ *when* $-\theta > 2\gamma + 1$.

c) *The heat kernel for* $-\theta > 2\gamma + 1$.

LEMMA 9. *Let us suppose that* $-\theta > 2\gamma + 1$ *and* γ *integer or integer* $+1/2$. *Then we have*

$$(9.40) \quad p_t^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x|1) = Ce^{\theta(\alpha+\beta+\gamma-(\theta/2)+(3/2))t} \int_{\Gamma(x)} j_{2\gamma+3}^{(\theta)}(x|\xi)p_t^{(\alpha,\beta,\gamma)}(\xi|1)d\xi$$

where C *is a constant depending only on* α, β, θ, γ. *When* $\theta = -2r$, *we find again* (9.37).

Proof. This is obvious from Lemma 8 and Theorem 2.

d) *The general case* $\theta < 0$. We are now able to treat the general case, i.e., when we remove the condition $-\theta > 2\gamma + 1$. We maintain $\theta < 0$.

THEOREM 4. *Let us suppose* θ *negative and* γ *integer or integer* $+1/2$. *Then*

$$(9.41) \quad p_t^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x|1) = Ce^{\theta(\alpha+\beta+\gamma-(\theta/2)+(3/2))t} \int_{\Gamma(x)} j_{2\gamma+3}^{(\theta)}(x|\xi)p_t^{(\alpha,\beta,\gamma)}(\xi|1)d\xi,$$

C *depending only on* α, β, θ, γ.

Proof. We know by Lemma 3, formula (9.21) that the second member of (9.41) is a solution of the next equation

$$\frac{\partial}{\partial t} = L^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}.$$

Let us prove that it has the correct singularity at $t = 0$ when $x \to 1$. Let f be a C^∞ function with compact support near 1. We study for $t \to 0$

$$\int_{\Lambda_x} f(x)m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx \int_{\Gamma(x)} j_{2\gamma+3}^{(\theta)}(x|\xi)p_t^{(\alpha,\beta,\gamma)}(\xi|1)d\xi.$$

This is also

(9.42) $$\int_{\Lambda_\xi} p_t^{(\alpha,\beta,\gamma)}(\xi|1)d\xi \int_{\Delta(\xi)} f(x) j^{(\theta)}_{2\gamma+3}(x|\xi) m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx$$

where the second integral is extended on a domain of type $\Delta(x)$. Now, the integral:

(9.43) $$\int_{\Delta(\xi)} j^{(\theta)}_{2\gamma+3}(x|\xi) m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx$$

is of the type studied above in (9.38) except that $\theta \to -\theta$ and

$$\alpha \to \alpha - \frac{\theta}{2} \quad \beta \to \beta - \frac{\theta}{2}$$

and γ is unchanged. It was shown that it is equivalent for $\xi \to \mathbf{1}$ to $m^{(\alpha,\beta,\gamma)}(\xi)$ provided that $\theta > 2\gamma + 1$. In particular we obtain that there exists a constant C depending only of α, β, γ, θ such that

$$\lim_{\xi \to 1} \frac{\int_{\Delta(\xi)} f(x) j^{(\theta)}_{2\gamma+3}(x|\xi) m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx}{m^{(\alpha,\beta,\gamma)}(\xi)} = Cf(1)$$

for $\theta > 2\gamma + 1$. But, by Riesz theory, the integral is analytic in θ for all θ if f is C^∞. In particular, the function Φ

$$\Phi:(\xi, \theta) \to \frac{\int_{\Delta(x)} f(x) j^{(\theta)}_{2\gamma+3}(x|\xi) m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx}{m^{(\alpha,\beta,\gamma)}(\xi)}$$

is analytic in θ, C^∞ in ξ and takes the value $Cf(1)$ for $\xi = 1$; we can rewrite (9.42) as

(9.44) $$\int_{\Lambda_\xi} p_t^{(\alpha,\beta,\gamma)}(\xi|1) m^{(\alpha,\beta,\gamma)}(\xi)\Phi(\xi, \theta)d\xi.$$

But

$$p_t^{(\alpha,\beta,\gamma)}(\xi|1) m^{(\alpha,\beta,\gamma)}(\xi)d\xi \to \delta(\xi - 1) \quad \text{if } t \to 0^+,$$

so that (9.44) tends to $\Phi(1, \theta) = Cf(1)$ which proves that

$$p_t^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x|1) m^{(\alpha-(\theta/2),\beta-(\theta/2),\gamma)}(x)dx \to \delta(x - 1)$$

so Theorem 4 is proved.

References. The definition of $D_-^{(\gamma)}$ comes from **[18]**; the definition and properties of the fractional wave operators come from **[27]**.

10. Kernels of elliptic invariant operators on certain solvable groups and applications to quantum mechanics. The purpose of this section is to apply the preceding analysis to obtain explicit constructions for heat kernels on certain solvable Lie groups. The case of nilpotent Lie groups has been treated in very special cases in [**12**], but apparently, few things are known for non nilpotent solvable Lie groups.

1. *A general construction*: *going from a symmetric space to a solvable group or its quotient.*

a) *The symmetric space in horospherical coordinates and its associated solvable group.* We consider a root system $(E, \mathfrak{R}, \rho)$ such that its multiplicities correspond to a non compact symmetric space $X = G/K$. Then, in horospherical coordinates, we obtain

$$X = NA \cdot o$$

where N is a nilpotent group, A is an abelian group (isomorphic to E) and $S_X = NA$ is a solvable group which is the semi direct product of A and N. It is clear that the Laplace-Beltrami operator of X written in horospherical coordinates (which are just the coordinates of S_X) induces a left invariant Laplace operator on S_X (see Section 5 for the analysis in horospherical coordinates).

We have seen that we can compute the heat kernel of X in a more or less explicit form in radial coordinates, in the following cases:

(i) $X = SL(p + 1, \mathbf{C})/SU(p + 1)$ in which case

$$(E, \mathfrak{R}, \rho) = (\mathbf{R}^p, A_p, \rho)$$

with all $\rho_\alpha = 2$ (see Section 3).

(ii) All symmetric spaces of rank 1, in which case $p = 1$, and there are at most two roots (see Section 7).

(iii) $(E, \mathfrak{R}, \rho) = (\mathbf{R}^p, BC_p, \rho)$ where $\gamma = 1/2$ (see Section 8), which includes in particular $SU(p, p)/S(U(p) \times U(p))$.

(iv) $(E, \mathfrak{R}, \rho) = (\mathbf{R}^2, B_2 \text{ or } C_2, \rho)$ (see Section 9).

In all these cases, the solution of the heat equation, the Green kernel, eigenfunctions . . . for the solvable group S_X is obtained by a change of variables in the radial expression for the heat kernel, Green kernel, eigenfunctions . . . of X.

b) *Generalization to other solvable groups of type* $S_{X,n-1}$. We now come back to the notations of Section 5. In this section, we obtained the following decomposition of the nilpotent algebra

$$(10.1) \quad \mathfrak{N}^{(+)} = \sum_{k \geqq 1} \mathfrak{V}^{(k)}$$

where $\mathfrak{V}^{(k)}$ is the sum of the root spaces $\mathfrak{G}_{(\alpha)}$ where α is a positive root which is the sum of exactly k fundamental roots; in particular $\mathfrak{V}^{(1)}$ is the sum of the fundamental root spaces. Let us now fix $n > 1$ and define

$$(10.2) \quad \mathfrak{N}^{(+)}_{n-1} = \sum_{k \geqq n} \mathfrak{V}^{(k)}$$

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which is an ideal of $\mathfrak{N}^{(+)}$ due to the root structure of $\mathfrak{N}^{(+)}$ (more precisely the "filtration" induced by the root system, see Section 5); in particular, the quotient algebra

(10.3) $\quad \mathfrak{Q}_{n-1} = \mathfrak{N}^{(+)}/\mathfrak{N}^{(+)}_{n-1}$

induces a solvable group $S_{X,n-1}$ with the Lie algebra $\mathfrak{A} + \mathfrak{Q}_{n-1}$.

Let us quickly prove this assertion; first $\mathfrak{Q}_{n-1}$ is naturally a Lie algebra because of (10.3) and the fact that $\mathfrak{N}^{(+)}_{n-1}$ is an ideal. Then if $A \in \mathfrak{A}$, we obtain that

$$[A, X_\alpha] = \alpha(A)X_\alpha$$

for $X_\alpha \in \mathfrak{N}^{(+)}_{n-1}$ by definition of a root so that

$$[\mathfrak{A}, \mathfrak{N}^{(+)}_{n-1}] \subset \mathfrak{N}^{(+)}_{n-1}$$

and the bracket by $\mathfrak{A}$ induces a natural structure of Lie algebra on $\mathfrak{A} + \mathfrak{Q}_{n-1}$ which is a quotient of $\mathfrak{A} + \mathfrak{N}^{(+)}/\mathfrak{N}^{(+)}_{n-1}$; in particular $S_{X,n-1}$ is a quotient of S_X by the nilpotent Lie group with Lie algebra $\mathfrak{N}^{(+)}_{n-1}$.

Example 1. Let us take $n = 2$; then $\mathfrak{Q}_1$ is an abelian algebra generated by the $\mathfrak{G}_{(\alpha)}$ for α fundamental roots and $S_{X,1}$ is a semi direct product of two abelian algebras. Its Laplace operator is

$$\Delta_{S_{X,1}} = \sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} + \sum_{\alpha \in \mathfrak{F}^+} \sum_{j=1}^{\rho_\alpha} e^{2(\alpha,q)} \frac{\partial^2}{\partial x_{\alpha,j}^2}. \tag{10.4}$$

Here the q_j are the coordinates of A (in horospherical decomposition), the ρ_α are the multiplicities of root α, $\mathfrak{F}^+$ is the set of fundamental roots and the $x_{\alpha,j}$, $j = 1 \dots \rho_\alpha$ are the variable in $\mathfrak{G}_{(\alpha)}$.

Example 2. Let us take $n = 3$ (if this is possible); then $\mathfrak{Q}_2$ is a nilpotent Lie algebra of rank 2 (this means that two or more bracket operations give 0) and $S_{X,2}$ is a solvable Lie group which is a semi direct product of an abelian group and a nilpotent Lie group of rank 2; its Laplace operator is

$$\Delta_{S_{X,2}} = \sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} + \sum_{\alpha \in \mathfrak{F}^+} \sum_{j=1}^{\rho_\alpha} e^{2(\alpha,q)} X_{\alpha,j}^2 \tag{10.5}$$
$$+ \sum_{\beta \in (\mathfrak{F}^+ + \mathfrak{F}^+)\cap \mathfrak{R}} \sum_{j=1}^{\rho_\beta} e^{2(\beta,q)} \frac{\partial^2}{\partial x_{\beta,j}^2}.$$

Here $x_{\beta,j}$ are the variables of $\mathfrak{G}_{(\beta)}$ and $(\mathfrak{F}^+ + \mathfrak{F}^+) \cap \mathfrak{R}$ is the set of positive roots which are the sum of exactly two fundamental roots; the $X_{\alpha,j}$ are vector fields on N/N_2 which are left invariant. They are first order operators, linear combinations of

$$\frac{\partial}{\partial x_{\alpha,j}} \quad \text{and} \quad x_{\alpha,k}\frac{\partial}{\partial x_{\beta,l}}$$

with

$$\alpha \in \mathfrak{F}^+ \quad \text{and} \quad \beta \in (\mathfrak{F}^+ + \mathfrak{F}^+) \cap \mathfrak{R}.$$

(We refer to Section 5 for the proofs of all these assertions.)

c) *Deduction of the kernels for* $S_{X,n-1}$. We shall start with the heat kernel of $S_X = N \cdot A$. An element $n \in N$ has the exponential coordinates

$$n = (x_{k,j})_{k,j}$$

where $(x_{k,j})_j \in \mathfrak{V}^{(k)}$ (see Section 5); then the Laplace-Beltrami operator Δ_X acts on functions f_0 which do not depend on the $(x_{k,j})_{k \geqq n,j}$ exactly as the operator $\Delta_{S_{X,n-1}}$; in particular it leaves this class of functions invariant. Then the heat kernel on $S_{X,n-1}$ is obtained by integrating out all these $(x_{k,j})_{k \geqq n,j}$

$$(10.6) \quad p_{S_{X,n-1}}(q^{(1)}, (x_{k,j}^{(1)})_{k<n}, t|q^{(0)}, (x_{k,j}^{(0)})_{k<n}) = \int p_X(q^{(1)}, (x_{k,j}^{(1)}), t|q^{(0)}, (x_{k,j}^{(0)})) \prod_{k \geqq n} \prod_j dx_{k,j}.$$

Remark. Our procedure here is exactly the opposite one of Karpelevic in [**18**]. In this article, Karpelevic defined the heat kernel on the abelian part of the Laplace operator of X in horospherical coordinates (which is a constant coefficient operator) to study the boundary behaviour on a symmetric space. Our procedure is to use information on X (in radial coordinates) to deduce the kernels on solvable groups.

2. *Generalization to other solvable groups*. a) Until now our construction applies only to solvable groups which are the quotient of the solvable Lie group S_X associated to a symmetric space X by a nilpotent Lie algebra. But in general, for a symmetric space, the multiplicities of the roots, and so the dimension of the vector spaces appearing in the horospherical decomposition of S_X and its quotient, are rather special numbers.

We shall now prove that it is rather easy to overcome that difficulty by using adapted Fourier-Bessel transforms.

b) Let us start with a root system $(E, \mathfrak{R}, \rho)$ which does not correspond to a symmetric space because its multiplicities ρ are not the correct ones for such a space. Moreover let us consider the operator

$$(10.7) \quad \Delta_1 = \sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} + \sum_{\alpha \in \mathfrak{F}^+} e^{2(\alpha,q)} \sum_{j=1}^{\rho_\alpha} \frac{\partial}{\partial y_{\alpha,j}^2}$$

(only restricted to the fundamental roots) and the heat problem with heat kernel

$$p_1(q^{(1)}, y^{(1)}_{\alpha,j}, t | q^{(0)}, y^{(0)}_{\alpha,j})$$

satisfying:

$$(10.8)\quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_1 u \\ u|_{t=0} = \delta(q^{(1)} - q^{(0)})\delta(y^{(1)} - y^{(0)}). \end{cases}$$

Let us also consider a symmetric space X with the root system $(E, \mathfrak{R}, \rho')$ such that $\rho'_\alpha \neq 0$ for all $\alpha \in \mathfrak{F}^+$; we have

$$(10.9)\quad \Delta_{S_{X,1}} = \sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} + \sum_{\alpha \in \mathfrak{F}^+} e^{2(\alpha,q)} \sum_{j=1}^{\rho'_\alpha} \frac{\partial^2}{\partial x^2_{\alpha,j}}$$

and we suppose that we know the heat kernel of $\Delta_{S_{X,1}}$:

$$p_{S_{X,1}}(q^{(1)}, (x^{(1)}_{\alpha,j}, t | q^{(0)}, x^{(0)}_{\alpha,j}).$$

We can integrate out all the $x^{(0)}_{\alpha,j}$ such that the corresponding ρ_α in (10.7) is 0; we obtain a heat kernel

$$p'_1(q^{(1)}, x^{(1)}_{\alpha,j} t | q^{(0)}, x^{(0)}_{\alpha,j})$$

for

$$\Delta'_1 = \sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} + \sum_{\alpha \in \mathfrak{F}^+} e^{2(\alpha,q)}(1 - \delta_{\rho_\alpha,0}) \sum_{j=1}^{\rho'_\alpha} \frac{\partial^2}{\partial x^2_{\alpha,j}}.$$

It satisfies

$$(10.9)\quad \begin{cases} \dfrac{\partial v}{\partial t} = \Delta'_1 v \\ v|_{t=0} = \delta(q^{(1)} - q^{(0)})\delta(x^{(1)} - x^{(0)}). \end{cases}$$

Now, both p_1 and p'_1 depend only on the euclidean distances

$$|y^{(1)}_\alpha - y^{(0)}_\alpha|^2 = \sum_{j=1}^{\rho_\alpha} |y^{(1)}_{\alpha,j} - y^{(0)}_{\alpha,j}|^2 \quad \text{and}$$

$$|x^{(1)}_\alpha - x^{(0)}_\alpha|^2 = \sum_{j=1}^{\rho'_\alpha} |x^{(1)}_{\alpha,j} - x^{(0)}_{\alpha,j}|^2.$$

Moreover if we perform a Fourier transform

$$\mathfrak{F}^{(1)}_\rho = \prod_{\rho_\alpha \neq 0} \mathfrak{F}^{(1)}_{\rho_\alpha}$$

in $y^{(1)}_{\alpha,j}$ variables on p_1 and a Fourier transform

$$\mathfrak{F}^{(1)}_{\rho'} = \prod_{\rho_\alpha \neq 0} \mathfrak{F}^{(1)}_{\rho'_\alpha}$$

in $x^{(1)}_{\alpha,j}$ on p'_1, they satisfy

$$(10.8) \quad \begin{cases} \dfrac{\partial}{\partial t} \mathfrak{F}^{(1)}_{\rho} p_1 = \left(\displaystyle\sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} - \sum_{\alpha \in \mathfrak{F}^+} e^{2(\alpha,q)}(1 - \delta_{\rho_\alpha,0}) \,|\hat{y}^{(1)}_\alpha|^2 \right) \mathfrak{F}^{(1)}_\rho p_1 \\ \mathfrak{F}^{(1)}_\rho p_1|_{t=0} = \delta(q^{(1)} - q^{(0)}) e^{i\Sigma y^{(0)}_{\alpha,j} \hat{y}^{(1)}_{\alpha,j}} \end{cases}$$

where the $\hat{y}^{(1)}_{\alpha,j}$ are conjugate variables of $y^{(1)}_{\alpha,j}$ and

$$|\hat{y}^{(1)}_\alpha|^2 = \sum_{j=1}^{\rho_\alpha} |\hat{y}^{(1)}_{\alpha,j}|^2,$$

and with the same kind of notations

$$(10.9) \quad \begin{cases} \dfrac{\partial}{\partial t} \mathfrak{F}^{(1)}_{\rho'} p'_1 = \left(\displaystyle\sum_{j=1}^{p} \frac{\partial^2}{\partial q_j^2} \right. \\ \qquad\qquad \left. - \displaystyle\sum_{\alpha \in \mathfrak{F}^+} e^{2(\alpha,q)}(1 - \delta_{\rho_\alpha,0}) \,|\hat{x}^{(1)}_\alpha|^2 \right) \mathfrak{F}^{(1)}_{\rho'} p'_1 \\ \mathfrak{F}^{(1)}_{\rho'} p'_1|_{t=0} = \delta(q^{(1)} - q^{(0)}) e^{i\Sigma x^{(0)}_{\alpha,j} \hat{x}^{(1)}_{\alpha,j}}. \end{cases}$$

At this level it is completely clear how to go from $(\widehat{10.9})$ to $(\widehat{10.8})$, in fact $\mathfrak{F}^{(1)}_{\rho'} p'_1$ is

$$(\mathfrak{F}^{(1)}_{\rho'} p'_1)(t, q^{(1)}, |\hat{x}^{(1)}_\alpha|, \theta_\alpha| \,|x^{(0)}_\alpha|, q^{(0)})$$

where θ_α is the angle between $(x^{(0)}_{\alpha,j})$ and $(\hat{x}^{(1)}_{\alpha,j})_j$ for every $\alpha \in \mathfrak{F}^+$ and in the same way we have

$$(10.10) \quad (\mathfrak{F}^{(1)}_\rho p_1)(t, q^{(1)}, |\hat{y}^{(1)}_\alpha|, \theta_\alpha| \,|y^{(0)}_\alpha|, q^{(0)})$$
$$= (\mathfrak{F}^{(1)}_{\rho'} p'_1)(t, q^{(1)}, |\hat{y}^{(1)}_\alpha|, \theta_\alpha| \,|y^{(0)}_\alpha|, q^{(0)}).$$

It is now sufficient to perform the inverse Fourier transform $(\mathfrak{F}^{(1)}_\rho)^{-1}$ to obtain the result. This can be expressed through Bessel functions. Explicit examples will be given in part 4.

3. *The easiest example*: *semi direct products of two abelian groups and the real hyperbolic spaces.*

a) *The groups S_p*. The groups S_p are the solvable Lie groups with the Lie algebra generated by the left invariant vector fields

$$\frac{\partial}{\partial y}, \quad \left(e^y \frac{\partial}{\partial x_j} \right) j = 1 \dots p$$

where y and x_j are real.

Example. In the case $p = 1$, S_1 is the group of affine transformations in the real line namely

$$(a, b):t \in \mathbf{R} \rightarrow at + b \quad a, b > 0.$$

If we define $a = e^{\xi}$ then we have the law

$$(\xi, b) \cdot (\xi', b') = (\xi + \xi', e^{\xi}b' + b)$$

with left invariant fields

$$\frac{\partial}{\partial \xi} + b\frac{\partial}{\partial b}, \frac{\partial}{\partial b}.$$

Defining $b = e^{\eta}$ and $\xi - \eta = x$, $\xi = y$ we obtain the realization

$$\frac{\partial}{\partial y}, e^{y}\frac{\partial}{\partial x}.$$

In the general case we define the left invariant operator

$$(10.11) \quad L_p = \frac{\partial^2}{\partial y^2} + e^{2y} \sum_{j=1}^{p} \frac{\partial^2}{\partial x_j^2}.$$

b) *The heat equation on* S_p. As usual we are interested in the Cauchy problem for the heat equation on S_p

$$(10.12) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial y^2} + e^{2y} \displaystyle\sum_{j=1}^{p} \dfrac{\partial^2 u}{\partial x_j^2} \\ u|_{t=0} = u_0. \end{cases}$$

We shall denote the heat kernel by

$$K(x^{(1)}, y^{(1)}, t|x^{(0)}, y^{(0)})$$

so that the solution of (10.12) is

$$(10.13) \quad u(x^{(1)}, y^{(1)}, t) = \int_{\mathbf{R}^{p+1}} K(x^{(1)}, y^{(1)}, t|x^{(0)}, y^{(0)}) \times u_0(x^{(0)}, y^{(0)})e^{-py^{(0)}}dx^{(0)}dy^{(0)}$$

with respect to the invariant measure $e^{py^{(0)}}dx^{(0)}dy^{(0)}$ on this group.

c) Consider now the hyperbolic space of dimension $n + 1$; we realize this space as the upper half space

$$H_{n+1} = \{ (y_1, \dots, y_{n+1}) \in \mathbf{R}^{n+1}, y_{n+1} > 0\}$$

with the metric

$$ds^2 = \frac{dy_1^2 + \dots + dy_{n+1}^2}{y_{n+1}^2}.$$

The volume element is

$$dv = y_{n+1}^{-(n+1)} dy_1 \dots dy_{n+1}.$$

If

$$\pi = \sum_{j=1}^{n+1} \pi_j dy_j$$

is a 1-form. The adjoint δ of the exterior differential d is given on π by

$$\int_{H_{n+1}} \delta\pi f dv = \int_{H_{n+1}} (\pi|df) dv$$
$$= \sum_{j=1}^{n+1} \int \pi_j \frac{\partial f}{\partial y_j} \frac{1}{y_{n+1}^{n-1}} dy_1 \dots dy_{n+1}$$

and so

$$\delta\pi = -\sum_{j=1}^{n+1} y_{n+1}^2 \frac{\partial \pi_j}{\partial y_j} + (n-1) y_{n+1} \pi_{n+1}.$$

The Laplace-Beltrami operator is then

$$\Delta f = -\delta df = y_{n+1}^2 \sum_{j=1}^{n+1} \frac{\partial^2 f}{\partial y_j^2} - (n-1) y_{n+1} \frac{\partial f}{\partial y_{n+1}}. \tag{10.14}$$

But y_{n+1} is positive on H_{n+1} and we can define

$$y_{n+1} = e^y. \tag{10.15}$$

With this change of coordinate we obtain

$$\Delta f = e^{2y} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial y_j^2} + \frac{\partial^2 f}{\partial y^2} - n \frac{\partial f}{\partial y}$$

and we rewrite

$$\frac{\partial^2 f}{\partial y^2} - n \frac{\partial f}{\partial y} = e^{(n/2)y} \frac{\partial^2}{\partial y^2} (e^{-(n/2)y} f) - \frac{n^2}{4} f.$$

Let us now define

$$g(y_1, \dots, y_n, y, t) = e^{-(n/2)y - (n^2/4)t} f(y_1, \dots, y_n, y, t). \tag{10.16}$$

Then the problem

$$\begin{cases} \dfrac{\partial f}{\partial t} = \Delta f \quad \text{on } H_{n+1} \\ f|_{t=0} = f_0 \end{cases} \tag{10.17}$$

is equivalent to the problem

$$(10.18)\quad \begin{cases} \dfrac{\partial q}{\partial t} = \dfrac{\partial^2 g}{\partial y^2} + e^{2y} \displaystyle\sum_{j=1}^{n} \dfrac{\partial^2 g}{\partial y_j^2} \\ g|_{t=0} = g_0. \end{cases}$$

For n even, the heat kernel of the space H_{n+1} is known (see Section 7). We call r the hyperbolic distance between two points of H_{n+1} and denote

$$p(m^{(1)}, t|m^{(0)}) \equiv p(r(m^{(1)}, m^{(0)}), t)$$

the heat kernel of (10.16), with respect to the volume element of H_{n+1}

$$f(m^{(1)}, t) = \int_{H_{n+1}} p(m^{(1)}), t(m^{(0)}) f_0(m^{(0)}) dv(m^{(0)}).$$

Using (10.16) we obtain the solution of (10.17) by the formula

$$(10.19)\quad g(y_1, \dots, y_n, y, t)$$
$$= e^{-(n/2)y-(n^2/4)t} \int p(y_1, \dots, y_n, e^y, t| y_1^0, \dots, y_n^0, e^{y^0})$$
$$\times g_0(y_1^0, \dots, y_n^0, y^0) e^{-(n/2)y^{(0)}} dy_1^{(0)} \dots dy_n^0 dy^0$$

because

$$dv(m^{(0)}) = \frac{1}{y_{n+1}^{n+1}} dy_1 \dots dy_{n+1} = e^{-ny} dy_1 \dots dy_n dy$$

and we obtain

THEOREM 1. *The heat kernel of problem* (10.12) *with* $p = n$ *is*

$$(10.20)\quad K(x_1^{(1)}, \dots, x_n^{(1)}, y^{(1)}, t|x_1^{(0)}, \dots, x_n^{(0)}, y^{(0)})$$
$$= e^{-(n/2)y^{(1)}+(n/2)y^{(0)}-(n^2/4)t} p(x_1^{(1)}, \dots, x_n^{(1)}, e^{y^{(1)}}, t|x_1^{(0)}, \dots, x_n^{(0)}, e^{y^{(0)}})$$

where p *is the heat kernel of* H_{n+1}.

4. *Another expression for the heat kernel of the preceding solvable group.*

a) We want to obtain a slightly more explicit computation of the heat kernel of problem (10.12). Let us first remark that in (10.12), we can restrict ourselves to the case when the initial data is a radial function of the x_j i.e.,

$$u_0(x_1, \dots, x_p, y) = u_0(\rho, y)$$

where

$$\rho = \left(\sum_{j=1}^{p} x_j^2\right)^{1/2}$$

because the heat kernel K is a function only of

$$|x^{(1)} - x^{(0)}|.$$

Moreover we have only a simple expression for the heat kernel of the Lobatchevski space H_3 for which

$$(10.21)\quad p(r, t) = (4\pi t)^{-3/2} e^{-t} e^{-(r^2/4)t} \frac{r}{\sinh r}.$$

Let us take the Fourier transform in $(x_j)_{j=1\dots p}$ of (10.12) and denote $\mathfrak{F}_p$ the Fourier transform in these p variables; we obtain

$$(\widehat{10.12})\quad \begin{cases} \dfrac{\partial \mathfrak{F}_p u}{\partial t} = \dfrac{\partial^2 \mathfrak{F}_p u}{\partial y^2} - e^{2y} \hat{\rho}^2 \mathfrak{F}_p u \\ \mathfrak{F}_p u|_{t=0} = \mathfrak{F}_p u_0 \end{cases}$$

and

$$\hat{\rho} = \left(\sum_{j=1}^{p} \hat{x}_j^2 \right)^{1/2}$$

where the $\hat{x}_j$ are the dual variables of the x_j.

Let us also take the Fourier transform of (10.17) for $n = 2$ and denote $\mathfrak{F}_2$ this Fourier transform with respect to the two variables y_1, y_2.

We obtain

$$(\widehat{10.17})\quad \begin{cases} \dfrac{\partial \mathfrak{F}_2 g}{\partial t} = \dfrac{\partial^2 \mathfrak{F}_2 g}{\partial y^2} - e^{2y} \hat{\sigma}^2 \mathfrak{F}_2 g \\ \mathfrak{F}_2 g|_{t=0} = \mathfrak{F}_2 g_0 \end{cases}$$

where

$$\hat{\sigma} = (\hat{y}_1^2 + \hat{y}_2^2)^{1/2}.$$

We also reduce ourselves to the case where g_0 is a function of

$$\sigma = (y_1^2 + y_2^2)^{1/2}$$

and y. The solution of $(\widehat{10.20})$ is

$$(10.22)\quad (\mathfrak{F}_2 g)(\hat{y}_1, \hat{y}_2, yt) = e^{-y-t} \int \hat{\hat{p}}(\hat{y}_1 \hat{y}_2 e^y t | \hat{y}_1^0 \hat{y}_2^0 e^{y^0}) \times (\mathfrak{F}_2 g_0)(\hat{\rho}_0, y^0) e^{-y^0} d\hat{y}_1^0 d\hat{y}_2^0 dy^0$$

where we have used (10.18), the Plancherel Formula and the definition

$$(10.22)\quad \hat{p}(\hat{y}_1 \hat{y}_2 e^y t | \hat{y}_1^0 \hat{y}_2^0, e^{y^0})$$

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$$= \frac{1}{(2\pi)^2} \int e^{i\Sigma_{j=1}^{2}(y_j\hat{y}_j - y_j^0\hat{y}_j^0)} p(y_1, y_2, e^y, t | y_1^0, y_2^0, e^{y^0}) dy_1^0 dy_2^0 dy_1 dy_2)$$
$$\equiv q(\hat{\sigma}, \theta, e^y, t | \theta^0, e^{y^0})$$

where

$$\hat{\sigma}^2 = \sum_{j=1}^{2} y_j^2(\hat{\sigma}^0)^2 = \sum_{j=1}^{2} (y_j^0)^2$$

and θ is the angle between the vectors (y_1, y_2) and $(y_1^0), y_2^0)$. Then

(10.23) $(\mathfrak{F}_2 g)(\hat{\sigma}, y, t)$

$$= e^{-y-t} \int (\mathfrak{F}_2 g_0)(\hat{\sigma}_0, y^0)\left(\int_0^{2\pi} q(\hat{\sigma}, \theta, e^y, t | \hat{\sigma}^0, e^{y^0}) d\theta\right)$$
$$\times e^{-y^0} \hat{\sigma}_0 d\hat{\sigma}_0 dy^0.$$

In (10.23), let us replace $\hat{\sigma}$ by $\hat{\rho}$, $\hat{\sigma}^0$ by $\hat{\rho}^0$, $\mathfrak{F}_2 g_0$ by $\mathfrak{F}_p u_0$ in the right hand side. On the left hand size, we obtain $(\mathfrak{F}_p u)(\hat{\rho}, y, t)$, solution of $(\widehat{10.12})$. Let us take the inverse Fourier transform $\mathfrak{F}_p^{-1}$ in the p variables $\hat{x}_1, \ldots, \hat{x}_p$; we obtain

(10.24) $u(\rho, y, t)$

$$= e^{-y-t} \int (\mathfrak{F}_p u_0)(\hat{\rho}_0, y^0) \mathfrak{F}_p^{-1} \int_0^{2\pi} q(\hat{\rho}, \theta, e^y, t | \hat{\rho}^0, e^{y^0}) d\theta x$$
$$\times e^{-y^0} \hat{\rho}_0 d\hat{\rho}_0 dy^0.$$

On radial functions, we obtain

$$(\mathfrak{F}_p^{-1} \varphi)(\rho) = \frac{1}{(2\pi)^p} \int_0^{+\infty} \hat{\rho}^{p-1} d\hat{\rho} \varphi(\hat{\rho})$$
$$\times \int_{S^{p-1}} e^{-i\rho\hat{\rho}\cos\theta} d\sigma(\theta, \alpha_1, \ldots, \alpha_{p-2})$$

where

$$d\sigma(\theta, \alpha_1, \ldots, \alpha_{p-2})$$
$$= \sin^{p-2}\theta \sin^{p-3}\alpha_1 \ldots \sin\alpha_{p-3} d\theta d\alpha_1 \ldots d\alpha_{p-2}$$

in the volume element of the unit sphere and

$$0 \leqq \theta, \alpha_1, \ldots, \alpha_{p-3} \leqq \pi \quad 0 \leqq \alpha_{p-2} \leqq 2\pi.$$

Introducing the Bessel function

$$J_{(p-2)/2}(\rho\hat{\rho}) = (\rho\hat{\rho})^{(p-2)/2} \int_{-1}^{+1} (1 - \xi^2)^{((p-2)/2)-(1/2)} e^{-i\rho\hat{\rho}\xi} d\xi$$

we see that

$$(10.24)\quad (\mathfrak{F}_p^{-1}\varphi)(\rho) = \frac{\sigma_{p-2}}{(2\pi)^p}\int_0^{+\infty} \hat{\rho}^{p-1}d\hat{\rho}\varphi(\hat{\rho})(\rho\hat{\rho})^{(2-p)/2}J_{(p-2)/2}(\rho\hat{\rho}).$$

On the other hand

$$(\mathfrak{F}_p u_0)(\hat{\rho}_0, y^0)$$
$$= \sigma_{p-2}\int_0^{+\infty} \rho_0^{p-1}d\rho_0 u_0(\rho_0, y^0)(\rho_0\hat{\rho}_0)^{(2-p)/2}J_{(p-2)/2}(\rho_0\hat{\rho}_0)$$

and the heat kernel of (10.12) is then given by

$$(10.25)\quad \frac{(\sigma_{p-2})^2}{(2\pi)^p}e^{-y-t}\int_0^{+\infty} \hat{\rho}_0 d\hat{\rho}_0(\rho_0\hat{\rho}_0)^{(2-p)/2}J_{(p-2)/2}(\rho_0\hat{\rho}_0)$$
$$\times \int_0^{+\infty} \hat{\rho}^{p-1}d\hat{\rho}(\rho\hat{\rho})^{(2-p)/2}J_{(p-2)/2}(\rho\hat{\rho})$$
$$\times \int_0^{2\pi} q(\hat{\rho}, \theta, e^y, t|\hat{\rho}^0, e^{y^0})d\theta$$

on the radial function $u_0(\rho_0, y^0)$ with q being defined as in (10.22) and p given as in (10.20).

5. *Quantum mechanics in the potential* e^{2y}. The Schrödinger equation for the one dimensional quantum problem in the potential e^{2x} is

$$(10.26)\quad \begin{cases} \dfrac{1}{i}\dfrac{\partial\psi}{\partial t} = \dfrac{\partial^2\psi}{\partial x^2} - \xi^2 e^{2y}\psi \\ \psi|_{t=0} = \psi_0 \end{cases}$$

where ξ^2 is some coupling constant. This is equivalent to $(\widehat{10.17})$. First we consider the problem in imaginary time $\tau = it$. In $(\widehat{10.17})$, $\psi_0 = \mathfrak{F}_2 g_0$ and does not depend on $\xi^2 = \hat{\sigma}^2$; this means that g_0 is a function of y^0 and a Dirac mass at the origin in y_1^0, y_2^0, and (10.22) becomes

$$(10.27)\quad \psi(y, t) = e^{-y-t}\int_{-\infty}^{+\infty} \psi_0(y^0)\hat{p}(\xi, e^y, t|0, 0, e^{y^0})e^{-y^0}dy^0$$

where

$$\hat{p}(\xi, e^y, t|0, 0, e^{y^0})$$
$$= \int_0^{2\pi}\int_0^{+\infty} e^{i\xi\rho\cos\theta}p(\rho e^{i\theta}, e^y, t|0, 0, e^{y^0})\rho d\rho d\theta$$

where $p(y_1, y_2, e^y, t|0, 0, e^{y^0})$ is the heat kernel of the Lobatchevski space H_3 and (ρ, θ) the polar coordinates of (y_1, y_2). It is clear that this function is independent of θ and we obtain

$$(10.28)\quad \hat{p}(\xi, e^y, t|0, 0, y^0) = 2\pi\int_0^{+\infty} J_0(\xi\rho)p(\rho, e^y, t|0, 0, e^{y^0})\rho d\rho$$

with

$$J_0(z) = \int_0^{2\pi} e^{iz\cos\theta}\frac{d\theta}{2\pi}.$$

The remaining problem is to express the variable r in (10.21) in terms of $(\rho, e^{y}, e^{y^0}$; r is the hyperbolic distance between the points

$$(0, 0, \ldots, 0, e^{y^0})$$

and

$$(\rho, 0, \ldots, 0, e^{y}).$$

If $\rho = 0$, the geodesic is the y-axis, the metrics is just dy_{n+1}/y_{n+1} and so it is dy and the distance is $|y^0 - y|$. If $\rho \neq 0$, we can consider that we are in the two dimensional Lobatchevski plane with coordinates (ξ, η); then if $\zeta = \xi + i\eta$, the group $SL(2, \mathbf{R})$ acts by the isometries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\zeta = \frac{a\zeta + b}{c\zeta + d} \quad ad - bc = 1.$$

The isotropy group of $i\eta_0$ is

$$\begin{pmatrix} \cos\theta & -\eta_0 \sin\theta \\ \eta_0^{-1}\sin\theta & \cos\theta \end{pmatrix}.$$

The point $\rho + i\eta$ can be put on the η-axis by such a transformation; it then becomes the point $i\eta'$ and θ must be chosen so that

$$i\eta' = \frac{(\rho + i\eta)\cos\theta - \eta_0\sin\theta}{(\rho + i\eta)\eta_0^{-1}\sin\theta + \cos\theta}$$

from which we deduce

$$\rho\eta' = (\eta - \eta')(\eta_0^2 - \eta'\eta)$$

and

$$\eta' = \frac{\eta_0^2 + \eta^2 + \rho^2 - \sqrt{(\eta_0^2 + \eta^2 + \rho^2)^2 - 4\eta^2\eta_0^2}}{2\eta}.$$

The distance between the two points is then

$$(10.29) \quad r = \Bigg| y^0 - \log\left(\frac{e^{2y_0} + e^{2y} + \rho^2 - \sqrt{(e^{2y_0} + e^{2y} + \rho^2)^2 - 4e^{2(y_0+y)}}}{2e^{y}}\right)\Bigg|.$$

THEOREM 2. *The kernel of the Schrödinger equation* (10.26) *is given by* (10.27), *where* $\hat{p}$ *is given by*

(10.30) $\hat{p}(\xi, e^y, t|0, 0, y^0) = 2\pi \int_0^{+\infty} J_0(\xi\hat{\rho})p(\rho, e^y, t|0, 0, e^{y^0})\rho d\rho$

where p is

(10.31) $p(\rho, e^y, t|0, 0, e^{y^0}) = (4\pi t)^{-3/2}e^{-t}e^{-r^2/4t}\dfrac{r}{\sinh r}$

and r is the function (10.29) *of* ρ, y *and* y^0.

6. *Second example*: *the semi direct product of an abelian group with a Heisenberg group.*

a) *The group* $\mathbf{R} \times H_{2n+1}$. In this section, we consider the semi direct product of the real line (with coordinate v), with the Heisenberg group H_{2n+1} of dimension $2n + 1$ with coordinates x_j, y_j ($j = 1 \dots n$) and u for its center. The multiplicative law on this group is defined as follows

1°) on $\mathbf{R}$, this is first the usual addition law

2•) on H_{2n+1}, this is just the law of the Heisenberg group (see [**12**])

3°) the action of $\mathbf{R}$ on H_{2n+1} is

$$e^v((z_j), u) = ((e^{v/2}z_j)_j, e^v u).$$

The basis of left invariant vector fields on H_{2n+1} is

(10.32) $X_i = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_i} + 2y_i\dfrac{\partial}{\partial u}\right)$, $Y_i = \dfrac{1}{2}\left(\dfrac{\partial}{\partial y_i} - 2x_i\dfrac{\partial}{\partial u}\right)$, $U = \dfrac{\partial}{\partial u}$

so that $[X_i, Y_i] = -U$ (see [**12**] with a different normalization). The basis of the Lie algebra of the semi direct product is then

(10.33) $\dfrac{\partial}{\partial v}$, $e^{v/2}X_i$, $e^{v/2}Y_i$, e^vU

and the left invariant laplacian is

(10.34) $L_n = \dfrac{\partial^2}{\partial v^2} + e^v \sum_{i=1}^{n} (X_i^2 + Y_i^2) + e^{2v}U^2$

and the heat equation is

(10.35) $\begin{cases} \dfrac{\partial f}{\partial t} = L_n f \\ f|_{t=0} = f_0. \end{cases}$

b) *The hermitian hyperbolic space*: *abstract realization.* We also consider the hermitian hyperbolic space of complex dimension $n + 1$; this is

$$\mathfrak{X}_{n+1} = SU(n + 1, 1)/U(n + 1);$$

its rank is 1 and its roots are $R^{(1)}(q) = q$ with multiplicity $\rho_1 = 2n$ and $R^{(2)}(q) = 2q$ with multiplicity $\rho_2 = 1$ (see Section 7, 1). The structure of the nilpotent Lie algebra of this space is an Heisenberg structure of real dimension $2n + 1$; in fact

$$\mathfrak{N}^+ = \mathfrak{G}_{R^{(1)}} \oplus \mathfrak{G}_{R^{(2)}}$$

where $\mathfrak{G}_{R^{(1)}}$ has dimension $2n$, $\mathfrak{G}_{R^{(2)}}$ has dimension 1 and moreover if X, Y are in $\mathfrak{G}_{R^{(1)}}$, then $[X, Y]$ is in $\mathfrak{G}_{R^{(2)}}$ and thus defines a non degenerate bilinear antisymmetric form on $\mathfrak{G}_{R^{(1)}}$ which can be reduced to a canonical form.

In particular, the expression of the Laplace-Beltrami operator in horospherical coordinate will be $L_n + Z$ where L_n is defined as in (10.34) and Z is the vector field on the abelian subalgebra which is the sum of the positive roots counted with their multiplicities (see Section 5).

c) *The hermitian hyperbolic space: half upper space realization.* Now, there is a well known realization of $\mathfrak{X}_{n+1}$ as the upper half space realization in $\mathbf{C}^{n+1}$. We refer to [**4**], [**5**] for the following computations. In $\mathbf{C}^{n+1}$ we define

$$h(x) = \operatorname{Im} z_0 - \sum_{k=1}^{n} |z_k|^2$$

(coordinates $z_0, \ldots, z_n$) and we define

$$\mathfrak{X}_{n+1} = \{z \in \mathbf{C}^{n+1} | h(z) > 0\}$$

and we define the Bergmann metric on $\mathfrak{X}_{n+1}$ (see [**4**], [**5**]); then $\mathfrak{X}_{n+1}$ is the space defined in the previous paragraph. If we define

$$(10.36) \quad \begin{cases} u = \Re e z_0 \\ e^v = h(z) \end{cases}$$

the Laplace-Beltrami operator of $\mathfrak{X}_{n+1}$ is exactly

$$(10.37) \quad \Delta_{\mathfrak{X}_{n+1}} = L_n - (n + 1)\frac{\partial}{\partial v}.$$

Moreover the invariant riemannian volume element is

$$(10.38) \quad e^{-(n+1)v} \left(\prod_{i=1}^{n} dx_i dy_i\right) du dv.$$

$\mathfrak{X}_{n+1}$ is invariant by the group $SU(n + 1, 1)$ of biholomorphic isometries. We shall only need the action of the elements of the solvable subgroup of $SU(n + 1, 1)$ (i.e., the semi direct product of $\mathbf{R}$, the abelian subgroup of $SU(n + 1, 1)$ with the Heisenberg subgroup H_{2n+1}).

Let $(z_0, z_1, \ldots, z_n)$ be a point in $\mathfrak{X}_{n+1}$; (so that $h(z) > 0$) and $v \in \mathbf{R}$: then v acts on $(z_0, \ldots, z_n)$ by

(10.39) $v \cdot (z_0, \ldots, z_n) = (e^v z_0, e^{v/2} z_1, \ldots, e^{v/2} z_n)$

(dilatations subgroup).

If $(u, \zeta_1, \ldots, \zeta_n)$ is in H_{2n+1}, its action on $(z_0, \ldots, z_n)$ is

(10.40) $$(u, \zeta_1, \ldots, \zeta_n) \cdot (z_0, \ldots, z_n) = \left(z_0 + u + i \sum_{k=1}^{n} |\zeta_k|^2 + 2i \sum_{k=1}^{n} z_k \zeta_k, z_1 + \zeta_1, \ldots, z_n + \zeta_n\right)$$

so that the function $h = e^v$ is conserved. Moreover, on $(\Re e z_0, z_1, \ldots, z_n)$ the action of $(u, \zeta_1, \ldots, \zeta_n)$ is just the left multiplication of the Heisenberg group H_{2n+1} namely

$$(u, \zeta_1, \ldots, \zeta_n)(\Re e z_0, z_1, \ldots, z_n) = \left(u + \Re e z_0 + 2\Im m \sum_{j=1}^{n} \zeta_j \bar{z}_j, z_1 + \zeta_1, \ldots, z_n + \zeta_n\right).$$

The heat kernel $p_{\mathfrak{X}_{n+1}}(m, t|m^{(0)})$ of $\Delta_{\mathfrak{X}_{n+1}}$ (with respect to the volume element (10.38)) generates the heat kernel $K_n(m, t|m^{(0)})$ of L_n as in part 3 because

$$\left(\frac{\partial^2}{\partial v^2} - (n + 1)\frac{\partial}{\partial v}\right)f = e^{((n+1)/2)v}\frac{\partial^2}{\partial v^2}(e^{-((n+1)/2)v}f) - \frac{(n + 1)^2}{4}f$$

so that

(10.41) $$K_n(v, u, x_i, y_i, t|v^{(0)}, u^{(0)}, x_i^{(0)}, y_i^{(0)}) = e^{-((n+1)/2)(v-v^{(0)})-((n+1)^2/4)t} p_{\mathfrak{X}_{n+1}} \times (v, u, x_i, y_i, t|v^{(0)}, u^{(0)}, x_i^{(0)}, y_i^{(0)})$$

is the heat kernel of (10.35) with respect to the invariant volume

$$e^{-(n+1)v} dv du \prod_{i=1}^{n} dx_i dy_i$$

on the solvable group $\mathbf{R} \times H_{2n+1}$.

d) *Expression of* $p_{\mathfrak{X}_{n+1}}$ *and of the distance between two points*. Now, we have to express $p_{\mathfrak{X}_{n+1}}$ in term of horospherical coordinates. But, we have obtained an expression for $p_{\mathfrak{X}_{n+1}}(m, t|m^{(0)})$ in Section 7, 7 in terms of a kernel $p(r(m, m^{(0)}), t)$ of the hyperbolic distance between m and $m^{(0)}$.

The only thing which remains to be computed is an explicit expression

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of the hyperbolic distance $(m, m^{(0)})$ between two points m and $m^{(0)}$ of $\mathfrak{X}_{n+1}$ in terms of their coordinates v, u, z_j and $v^{(0)}$, $u^{(0)}$, $z_j^{(0)}$ respectively. Let us write

$$m^{(0)} = n^{(0)} \cdot v^{(0)} \cdot O$$

$$m = n \cdot v \cdot O$$

where O is the origin of the symmetric space, here the point

$$O = (z_0 = i, z_1 = 0, \dots, z_n = 0)$$

in $\mathbf{C}^{n+1}$ and $n^{(0)}$ (resp. n) the unique element of H_{2n+1} which gives the nilpotent part of $m^{(0)}$ (resp. m) and $v^{(0)}$ (resp. v) the abelian part of $m^{(0)}$ (resp. m). The action of these respective elements are given by (10.39) and (10.40). It is clear that

$$r(m, m^{(0)}) = r(\,(n^{(0)-1}n) \cdot v \cdot 0, v^{(0)} \cdot 0).$$

Moreover

$$n^{(0)} = (u^{(0)}, z_1^{(0)}, \dots, z_n^{(0)})$$
$$(n^{(0)})^{-1} = (-u^{(0)}, -z_1^{(0)}, \dots, -z_n^{(0)})$$
$$(n^{(0)})^{-1}n = \Big(u - u^{(0)} - 2\Im m \sum_{j=1}^{n} z_j^{(0)}\bar{z}_j, z_1 - z_1^{(0)}, \dots, z_n - z_n^{(0)}\Big).$$

The action of a nilpotent element conserves the value of h or v, so

$$r(m, m^{(0)}) = r\Big(\Big(v, u - u^{(0)} - 2\Im m \sum_{j=1}^{n} z_j^{(0)}\bar{z}_j, z_1 - z_1^{(0)}, \dots, z_n - z_n^{(0)}\Big), (e^{v^{(0)}} \cdot i, 0 \dots 0)\Big)$$

and then using the isometric action of $-v^{(0)}$ given by (10.39)

$$= r\Big(\Big(v - v^{(0)}, e^{-v^{(0)}}\Big(u - u^{(0)} - \Sigma \Im m \sum_{j=1}^{n} z_j^{(0)}\,\bar{z}_j\Big), e^{-(v^{(0)})/2}(z_1 - z_1^{(0)}, \dots, e^{-(v^{(0)})/2}(z_n - z_n^{(0)})\,), (i, 0, \dots, 0)\Big).$$

We have then to compute the hyperbolic distance from a point m' of $\mathfrak{X}_{n+1}$ to the origin O of $\mathfrak{X}_{n+1}$. The best way is to use the Cayley transform; this is a mapping

$$C{:}z \in \mathfrak{X}_{n+1} \rightarrow Z(z) \in B_{n+1}$$

where B_{n+1} is the unit ball

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$$\sum_{j=0}^{n} |Z_j|^2 \leqq 1$$

of $\mathbf{C}^{n+1}$, which is a holomorphic isometry from $\mathfrak{X}_{n+1}$ into B_{n+1} when B_{n+1} has the Bergmann metric; moreover the origin $O \in \mathfrak{X}_{n+1}$ is sent by this mapping into the origin $Z = 0$ of B_{n+1} (see [**4**] for the precise definition of the Cayley transform). We have

$$r(m, 0) = r_{B_{n+1}}(C(m), 0).$$

But it is clear that the distance of Z to O in B_{n+1} is only a function of $|Z|$ and that the geodesics is the line segment joining 0 to Z with the clock given by the Bergmann metric. This distance is easily seen to be:

$$r_{B_{n+1}}(Z, 0) = \text{Arg tanh } |Z|.$$

Using this last formula, the exact value of C given in [**4**], and the coordinates v, u, z_j in $\mathfrak{X}_{n+1}$, we obtain

$$r((v, u, z_1 \dots z_n), 0)$$
$$= \text{Arg tanh}\left(\left[\frac{\left(1 + e^v + \sum_{k=1}^{n} |z_k|^2\right)^2 + u^2 - 4e^v}{\left(1 + e^v + \sum_{k=1}^{n} |z_k|^2\right)^2 + u^2}\right]^{1/2}\right)$$

so that the distance between m and $m^{(0)}$ is

$$(10.42) \quad r(m, m^{(0)}) = \text{Arg tanh}(R(m, m^{(0)}))^{1/2}$$

where

$$(10.43) \quad R(m, m^{(0)})$$
$$= 1 - 4e^{v - v^{(0)}}\Bigg[\left(1 + e^{v - v^{(0)}} + e^{-v^{(0)}} \sum_{j=1}^{m} |z_j - z_j^{(0)}|^2\right)^2$$
$$+ e^{-2v^{(0)}}\left(u - u^{(0)} - 2\mathfrak{I}m \sum_{j=1}^{n} z_j^{(0)}\bar{z}_j\right)^2\Bigg].$$

7. *Quantum mechanics in certain Morse potentials.*

a) *Schrödinger equation in a Morse potential.* The Cauchy problem for the Schrödinger equation in a Morse potential is

$$(10.44) \quad \begin{cases} \dfrac{1}{i}\dfrac{\partial \psi}{\partial t} = \left(\dfrac{\partial^2}{\partial v^2} + V(v)\right)\psi \\ \psi|_{t=0} = \psi_0 \end{cases}$$

where v denotes the spatial variable and

$$V(v) = -\lambda^2 e^v - \beta^2 e^{2v}$$

and $-\lambda^2$, β^2 are constants. ($V(v)$ is called a Morse potential [**20**]). We shall treat this problem by a similar method as the one used in 5; in part 5 we have related the quantum mechanics in the potential e^{2x} to the real hyperbolic space. Here we relate the quantum mechanics in the Morse potential to the hermitian hyperbolic space.

First, we come back to $\tau = it$ to work with the heat kernel and heat equation. We then look at the Fourier transform $\mathfrak{F}_u f$ of (10.35) where we take $n = 1$. This is

$$(\widehat{10.35}) \quad \begin{cases} \dfrac{\partial \mathfrak{F}_u f}{\partial t} = \dfrac{\partial^2}{\partial v^2} + e^v\left(\dfrac{1}{4}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) - yi\beta\dfrac{\partial}{\partial x} + xi\beta\dfrac{\partial}{\partial y}\right. \\ \qquad\qquad \left. - \beta^2(x^2 + y^2)\right)\mathfrak{F}_u f - e^{2v}\beta^2\mathfrak{F}_u f \\ \mathfrak{F}_u f|_{t=0} = \mathfrak{F}_u f_0 \end{cases}$$

where

$$\mathfrak{F}_u f = \int f(v, u, x, y) e^{i\beta u} du$$

(β is the conjugate variable of u).

b) *Reduction of* $(\widehat{10.35})$. $(\widehat{10.35})$ has a separation of coordinates. More precisely the coefficient of e^v in the second member of $(\widehat{10.35})$ is a differential operator in x, y which we shall denote by H_β. In mathematics it is the Fourier transform with respect to u of the subelliptic laplacian of the Heisenberg group (see [**20**] and [**12**]) and in physics, it is just the hamiltonian of a particle of charge β in a constant magnetic field in the direction z ([**20**], [**13**]). This operator has a spectrum (because we do not consider the z part of the motion). If $-\lambda_\beta^2$ is an eigenvalue and $\varphi_{\lambda_\beta}(\beta, x, y)$ is the corresponding eigenfunction of H_β, we have

$$(10.45) \quad H_\beta \varphi_{\lambda_\beta} = -\lambda_\beta^2 \varphi_{\lambda_\beta}.$$

Now, because of the separation of variables, if we suppose that, at time $t = 0$, we have

$$(\mathfrak{F}_u f_0)(v, \beta, x, y) = \varphi_{\lambda_\beta}(\beta, x, y) g_0(v)$$

then at any later time $t > 0$, we have

$$(10.46) \quad (\mathfrak{F}_u f)(v, \beta, x, y, t) = \varphi_{\lambda_\beta}(\beta, x, y) g(v, t, \beta)$$

where $g(v, t, \beta)$ satisfies

$$(10.47) \quad \begin{cases} \dfrac{\partial f(v, t, \beta)}{\partial t} = \left(\dfrac{\partial^2}{\partial v^2} - \lambda_\beta^2 e^v - \beta^2 e^{2v}\right) g(v, t, \beta) \\ g(v, t = 0, \beta) = g_0(v) \end{cases}$$

which is of the form (10.44) with the identifications $\psi_0 = g_0$, $\psi = g$. Now we shall define the solution of (10.47) by

$$(10.48) \quad g(v, t, \beta) = \int g_0(v_0) \pi(t, v|v_0, \beta, \lambda_\beta) e^{-2v_0} dv_0$$

(where we have stressed the β and λ_β dependence of the propagator π of (10.47)).

Now, we use the definition of K_1; it is clear by (10.41), (10.42) and (10.43), that K_1 is a function

$$K_1(t, v, u - u_0, x, y|v_0, x_0, y_0)$$

(so that it depends only on $u - u_0$), and we have

$$\begin{aligned} f(v, u, x, y, t) &= \int K_1(t, v, u - u_0, x, y|v_0, x_0, y_0) \\ &\quad \times \mathfrak{F}_{\beta_0}^{-1}[\varphi_{\lambda_{\beta_0}}(\beta_0, x_0, y_0) g_0(v_0)](u_0) \\ &\quad \times e^{-2v_0} dv_0 du_0 dx_0 dy_0 \\ &= \frac{1}{2\pi} \int e^{-i\beta_0 u_0} K_1(t, v, u - u_0, x, y|v_0, x_0, y_0) \\ &\quad \times \varphi_{\lambda_{\beta_0}}(\beta_0, x_0, y_0) g_0(v_0) e^{-2v_0} \\ &\quad \times d\beta_0 dv_0 du_0 dx_0 dy_0. \end{aligned}$$

Then

$$\begin{aligned} g(v, t, \beta) &= \frac{1}{\varphi_{\lambda_\beta}(\beta, x, y)} (\mathfrak{F}_u f(v, u, x, y, t))(\beta) \\ &= \frac{1}{2\pi} \frac{1}{\varphi_{\lambda_\beta}(\beta, x, y)} \int e^{i(\beta u - \beta_0 u_0)} \\ &\quad \times K_1(t, v, u - u_0, x, y|v_0, x_0, y_0) \\ &\quad \times \varphi_{\lambda_{\beta_0}}(\beta_0, x_0, y_0) g_0(v_0) e^{-2v_0} d\beta d\beta_0 du du_0 dx_0 dy_0 dv_0 \\ &= \frac{1}{2\pi} \frac{1}{\varphi_{\lambda_\beta}(\beta, x, y)} \int \mathfrak{F}_u K_1(t, v, u, x, y|v_0, x_0, y_0)(\beta) \\ &\quad \times \varphi_{\lambda_\beta}(\beta, x_0, y_0) g_0(v_0) e^{-2v_0} dx_0 dy_0 dv_0 \end{aligned}$$

and we finally obtain

$$\pi(t, v|v_0, \beta, \lambda_\beta) = \frac{1}{2\pi\varphi_{\lambda_\beta}(\beta, x, y)} \int e^{i\beta u} K_1(t, v, u, x, y|v_0, x_0, y_0) \times \varphi_{\lambda_\beta}(\beta, x_0, y_0)dx_0dy_0du$$

and the second member of this formula is independent of x and y. In particular we can choose for x, y the value 0, 0 and so

$$\pi(t, v|v_0, \beta, \lambda_\beta) = \frac{1}{2\pi\varphi_{\lambda_\beta}(\beta, 0, 0)} \int e^{i\beta u} K_1(t, v, u, 0, 0|v_0, x_0, y_0) \times \varphi_{\lambda_\beta}(\beta, x_0, y_0)dx_0dy_0du. \tag{10.49}$$

The kernel K_1 is, in principle, computable by (10.41)

$$K_1(t, v, u, 0, 0|v_0, x_0, y_0) = e^{-((n+1)/2)(v-v_0)-((n+1)^2)/4t} p_{\mathfrak{X}_2}(r)$$

where

$$r = \operatorname{Arg} \tanh(R^{1/2})$$

and

$$R = 1 - 4e^{v-v_0}[(1 + e^{+(v-v_0)} + e^{-v_0}|z_0|^2)^2 + e^{-2v_0}u^2]^{-1}.$$

The functions $\varphi_{\lambda_\beta}(\beta, x_0, y_0)$ are in principle known by the theory of harmonic oscillators and $p_{\mathfrak{X}_2}(r)$ is computable by the methods of Section 7 using a Riemann-Liouville integral.

References. The Heisenberg group was treated in [**12**] from the point of view of the heat equation for subelliptic operators. Explicit formulae for Schrödinger propagators in certain potentials (like $\cosh^{-2}x$ or $\delta(x)$) were obtained in [**14**].

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