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Bhargav Bhatt

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Derived splinters in positive characteristic

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Abstract

This paper introduces the notion of a derived splinter. Roughly speaking, a scheme is a derived splinter if it splits off from the coherent cohomology of any proper cover. Over a field of characteristic 0, this condition characterises rational singularities, as suggested by the work of Kovács. Our main theorem asserts that over a field of characteristic p, derived splinters are the same as (underived) splinters, i.e. schemes that split off from any finite cover. Using this result, we answer some questions of Karen Smith concerning the extension of Serre/Kodaira-type vanishing results beyond the class of ample line bundles in positive characteristic; these are purely projective geometric statements independent of singularity considerations. In fact, we can prove 'up to finite cover' analogues in characteristic p of many vanishing theorems known in characteristic 0. All these results fit naturally in the study of F-singularities, and are motivated by a desire to understand the direct summand conjecture.

1. Introduction

The work presented in this paper is directly inspired by Hochster's *direct summand conjecture*, which has taunted commutative algebraists for about forty years. It says the following.

Conjecture 1.1. Every module-finite extension of a regular ring R splits as a map of R-modules.

Hochster himself settled in 1973 the case where R contains a field [Hoc73]. The only open case is thus arithmetic in nature, i.e. where R has mixed characteristic. The most significant progress has been Heitmann's 2002 work [Hei02], which settled the conjecture in dimension 3. In the intervening years, mathematicians have uncovered an intricate web linking Conjecture 1.1 to various seemingly unrelated problems. For example, Raynaud and Gruson [RG71, p. 67, Question 2] asked in 1971 whether injective integral ring homomorphisms (with a noetherian base) descend flatness, which Ohi showed in 1996 is equivalent to the direct summand conjecture [Ohi96]. For other equivalent formulations, see [Hoc07].

The goal of this paper is to understand the severity of the direct summand condition, i.e. to understand the constraints imposed on the geometry of $\mathrm{Spec}(R)$ if R is an $\mathbb{F}_p$-algebra that satisfies the conclusion of Conjecture 1.1. In order to explain our results, it is convenient to recall the following definition.

Definition 1.2. A scheme S is called a *splinter* if for any finite surjective map $f : X \to S$, the pullback map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is split in the category $\mathrm{Coh}(S)$ of coherent sheaves on S.

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This term was coined in [Sin99], though the idea is much older. In characteristic 0, splinters are exactly normal schemes (see Example 2.1). On the other hand, Conjecture 1.1 may be reformulated as the assertion that all regular affine schemes are splinters. The philosophy informing our investigations is that the geometry of splinters is best understood by first understanding the relationship with a suitable derived analogue, which may then be related to geometry. Combining this idea with the *ansatz* that proper maps provide robust derived analogues of finite maps, at least for coherent cohomology, leads to the following definition.

Definition 1.3. A scheme S is called a *derived splinter*, or simply a *D-splinter*, if for any proper surjective map $f : X \to S$, the pullback map $\mathcal{O}_S \to \mathrm{R}\, f_*\mathcal{O}_X$ is split in derived category $\mathrm{D}(\mathrm{Coh}(S))$ of coherent sheaves on S.

D-splinters are, in fact, well-known to complex geometers, but under a different name: a theorem of Kovács [Kov00] identifies D-splinters in characteristic 0 with rational singularities (the proof given in [Kov00] is incomplete, so we provide a direct proof in Theorem 2.12). Since splinters in characteristic 0 are precisely normal schemes, one notices immediately that splinters and D-splinters define extremely different classes of singularities in characteristic 0. In characteristic $p > 0$, however, we discover a remarkably different picture; one of the main theorems of this paper is the following.

Theorem 1.4. *A noetherian $\mathbb{F}_p$-scheme S is a splinter if and only if it is a D-splinter.*

The main tool used to prove Theorem 1.4 is a cohomology-annihilation result that is of independent interest: we show that the higher cohomology of the structure sheaf on a projective variety in characteristic p can always be killed by a finite cover. In fact, we prove the following stronger relative statement.

Theorem 1.5. *Let $f : X \to S$ be a proper morphism of noetherian $\mathbb{F}_p$-schemes. Then there exists a proper morphism $g : Y \to S$ and a finite surjective morphism $\pi : Y \to X$ such that the pullback map $\pi^* : \tau_{\geqslant 1}\, \mathrm{R}\, f_*\mathcal{O}_X \to \tau_{\geqslant 1}\, \mathrm{R}\, g_*\mathcal{O}_Y$ is 0.*

The proof of Theorem 1.5 is inspired by Hochster and Huneke's theorem [HH92] on the existence of big Cohen–Macaulay algebras in positive characteristic (and also [HL07]). The same paper led K. Smith to ask certain questions concerning extensions of the vanishing theorems of Serre and Kodaira beyond the ample cone (see § 7). Using our methods, we are able to answer these questions. The negative answers are recorded in the form of counterexamples at the end of § 7, while the affirmative answers are summarised in Theorem 1.6. We refer the reader to Propositions 7.2 and 7.3 for more precise statements.

Theorem 1.6. *Let X be a proper variety over a field k of positive characteristic, and let $\mathcal{L}$ be a semiample line bundle on X. Then $H^i(X, \mathcal{L})$ can be killed by finite covers for $i > 0$. If $\mathcal{L}$ is big as well, then $H^i(X, \mathcal{L}^{-1})$ can be killed by finite covers for $i < \dim(X)$.*

It is worthwhile to remark that the results mentioned above, in conjunction with those proven in § 7, have applications unrelated to splinters or D-splinters: these results suggest that numerous vanishing theorems that are true in characteristic 0 have analogues in characteristic p provided one works 'up to finite covers'. This idea has been pursued in much more depth in the recent work [BST11], where 'up to finite cover' analogues of the Nadel vanishing have been established. Moreover, we can also prove a weak mixed-characteristic analogue of Theorem 1.5 (see [Bha11b]), and this analogue has surprising applications, including some in p-adic Hodge theory [Bei11].

Returning to affine D-splinters, we note that in positive characteristic p, by Theorem 1.4 this class of singularities is closely related to other classes of singularities, the so-called F-singularities, defined using the Frobenius action. For example, locally excellent affine $\mathbb{Q}$-Gorenstein splinters are F-regular by [Sin99], which builds on the Gorenstein case of [HH94]; see also Example 2.4 below. In contrast, in mixed characteristic, our knowledge about either splinters or D-splinters is minimal, primarily because the direct summand conjecture is unknown. For progress towards establishing an analogue of Theorem 1.4, we refer the reader to [Bha11b].

Organisation of the paper

The purpose of § 2 is to collect some examples and non-examples of splinters and D-splinters; the goal is to bring out the geometry underlying these definitions. In § 3 we set up notation concerning derived categories, and also prove a lemma which allows passage from conclusions at the level of cohomology groups to those at the level of complexes. Theorems 1.4 and 1.5 are proven in § 4, and some refinements are proven in § 5; our method is inspired by that of [HL07] and, owing to transitivity, by that of [HH92]. Moving to applications, we discuss some purely algebraic applications of the preceding theorems in § 6. In § 7, we review some questions raised by Karen Smith in the wake of [HH92], and then discuss both positive and negative answers that we can provide; the highlights here are the 'up to finite covers' version of Kodaira vanishing in Proposition 7.3 and some of the counterexamples, especially Example 7.11. Finally, in § 8 we use Theorem 1.5 to show that the complete flag variety for GL_n is a D-splinter, thereby providing the first non-toric projective example of one.

2. Examples of splinters and D-splinters

We provide some examples of splinters and D-splinters in this section. Since the notions in characteristic 0 are quite well understood, we focus mainly on the case of characteristic p. Moreover, it is typically non-trivial to prove that any given ring is a splinter or D-splinter. Hence, we freely use results from the literature or elsewhere in this paper in our proofs; we hope that despite the resulting non-elementary nature of the examples, the reader will be convinced that splinters and D-splinters are geometrically interesting.

First, we dispose of the characteristic 0 case.

Example 2.1 (Splinters in characteristic 0). A connected noetherian $\mathbb{Q}$-scheme S is a splinter if and only if it is normal. For the forward direction, note that the map from the disjoint union of the irreducible components of S to S immediately shows that S is forced to be a domain if it is a splinter. The desired claim now follows from the following ring-theoretic fact: if R is an integral domain with $a/b \in \mathrm{Frac}(R)$ integral over R, then $R \to R[a/b]$ is not split unless $a/b \in R$. To prove this, we simply observe that if $R \to R[a/b]$ were split, then the quotient would be a torsion-free R-module with generic rank 0, which can only happen when the quotient is trivial.

For the converse implication, we need to show that if $f: X \to S$ is a finite surjective morphism and S is normal and connected, then $f^*: \mathcal{O}_S \to f_*\mathcal{O}_X$ has a section in $\mathrm{Coh}(S)$. After replacing X with an irreducible component dominating S, we may assume that X is integral. Let d denote the degree of the map induced by f at the level of function fields. Then the map $(1/d)\,\mathrm{Tr}_{X/S}$ provides a canonical splitting for the map $f^*: \mathcal{O}_S \to f_*\mathcal{O}_X$ (here we use the fact that the trace map on function fields preserves integrality).

Example 2.2 (D-splinters in characteristic 0). Let X be a variety over $\mathbb{C}$. Then X is a D-splinter if and only if X has rational singularities, i.e. if $\mathrm{R}\, f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ for some (equivalently, every) resolution of singularities $f : Y \to X$. This assertion is contained in [Kov00, Theorem 3]. However, as indicated by the referee, the proof in [Kov00] is incomplete as it only tests morphisms $Y \to X$ with connected fibres. Hence, we give a direct proof in Theorem 2.12 at the end of this section.

A splinter in positive characteristic p is subtler than its characteristic 0 avatar, as being a splinter imposes some kind of positivity (both local and global) on the variety. In fact, in view of Theorem 1.4, over $\mathbb{F}_p$, being a splinter is equivalent to being a D-splinter, a condition that is *a priori* much more restrictive. Nevertheless, large classes of examples of splinters (or, equivalently, D-splinters) over $\mathbb{F}_p$ do exist, and are catalogued below. The intuition informing most of these examples is that splinters should be analogous to rational singularities in characteristic 0.

Example 2.3 (Smooth affines are splinters). All regular affine $\mathbb{F}_p$-schemes are splinters; this is a result of Hochster (see [Hoc73]), and we record a proof of Hochster's theorem below for the convenience of the reader. The proof given below is cohomological in nature and different from Hochster's.

We first explain the idea informally. Let $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be a finite surjective map. Using the fact that R is Gorenstein, an elementary duality argument will reduce us to showing that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$ is injective. The kernel of this map is a Frobenius-stable proper submodule of $H^d_{\mathfrak{m}}(R)$ of finite length by an inductive argument due to Grothendieck (see [Gro68, Exposé VIII, Théorème 2.1]). The regularity of R will then imply that this is impossible for length reasons.

Now for the details. After localising and completing, we may assume that $(R, \mathfrak{m})$ is a complete regular local $\mathbb{F}_p$-algebra of dimension d. By the Cohen structure theorem (see [Mat80, § 28, Theorem 28.J and Corollary 2]), we know that $R \simeq k[\![x_1, \ldots, x_d]\!]$. Since field extensions $k \to L$ split as k-modules, we may pass to the algebraic closure of the coefficient field to assume that k is algebraically closed. In particular, the Frobenius map $F : R \to R$ is finite. Given a finite extension $f : R \to S$, we need to show that the evaluation map $\mathrm{ev}_f : \mathrm{Hom}(S, R) \to \mathrm{Hom}(R, R)$ is surjective. By induction, we may assume that the cokernel $\mathrm{coker}(\mathrm{ev}_f)$ is supported only at the closed point $\{\mathfrak{m}\} \subset \mathrm{Spec}(R)$ and so has finite length. Since R is Gorenstein, we have $\omega_R \simeq R$. Thus, ev_f can be identified with the trace map $\mathrm{Hom}(S, \omega_R) \to \omega_R$, which is dual to the canonical pullback map $H^d_{\mathfrak{m}}(f) : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$. As local duality interchanges kernels and cokernels while preserving lengths, the kernel $M := \ker(H^d_{\mathfrak{m}}(f))$ also has finite length; this kernel is also Frobenius-stable by construction. Now consider the following diagram.

$$\begin{tikzcd} F^*M \arrow[r, "b"] \arrow[d, "a"] & M \arrow[d, "c"] \\ F^*H^d_{\mathfrak{m}}(R) \arrow[r, "d"] & H^d_{\mathfrak{m}}(R) \end{tikzcd}$$

The map a is injective since F^* is exact (by regularity of R), while the map d is an isomorphism by the flat base-change isomorphism $\mathrm{R}\Gamma_{\mathfrak{m}}(R) \otimes_{R,F} R \simeq \mathrm{R}\Gamma_{\mathfrak{m}}(R)$ (see [BS98, § 4.3.2]). The diagram then shows that b is also injective, and thus the length of F^*M is bounded above by that of M. The claim now follows from the elementary observation that F^* multiplies length by $p^d > 0$.

Following the proof of Example 2.3 leads to a much larger class of splinters, defined in terms of F-rational rings. We remind the reader that a noetherian local $\mathbb{F}_p$-algebra $(R, \mathfrak{m})$ of dimension d is said to be *F-rational* if it is Cohen–Macaulay and normal, with the property that $H^d_{\mathfrak{m}}(R)$ has no

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proper Frobenius-stable submodules except 0. This is not the original definition of F-rationality, but is equivalent to it by the work of Smith; see [Smi94, Smi97b].

Example 2.4 (F-rational Gorenstein rings are splinters). Let $(R, \mathfrak{m})$ be a noetherian excellent local $\mathbb{F}_p$-algebra admitting a dualising complex. Assume that R is Gorenstein. If R is F-rational, then R is a splinter. This follows from the proof given in Example 2.3. In more detail, we may assume without loss of generality that $(R, \mathfrak{m})$ is an F-rational Gorenstein complete noetherian local ring of dimension $d > 0$. Given a finite extension $f : R \to S$, we need to verify that $\mathrm{ev}_f : \mathrm{Hom}(S, R) \to R$ is surjective. Since R is Gorenstein, we can identify ev_f with the trace map $\mathrm{Tr} : \omega_S \simeq \mathrm{Hom}(S, \omega_R) \to \omega_R$. The image $\mathrm{Tr}(\omega_S)$ is a Frobenius-stable submodule $M \subset \omega_R$. Since the formation of M commutes with localisation, we know that M is generically non-zero. Dualising, we obtain a non-zero Frobenius-stable submodule $D(M)$ of $H^d_{\mathfrak{m}}(R)$. By the definition of F-rationality, we have $D(M) = H^d_{\mathfrak{m}}(R)$, and so $M = \omega_R$.

Remark 2.5. One can show a converse to Example 2.4 as follows: any excellent splinter is F-rational. To see this, note that the argument in Example 2.1 shows that R is normal, while Corollary 6.4 below shows that R is Cohen–Macaulay. To show that R is F-rational, one can then use [Smi97b, Theorem 2.6] and [Smi94, Theorem 5.4]. Together, these theorems imply that it is enough to check that for all ideals I generated by a system of parameters, we have $IS \cap R = I$ for all finite extensions $R \to S$. The splinter property implies that $IS = I \oplus Q$, which easily shows that $IS \cap R = I$.

We work out a special case of Example 2.4, to give an idea of the relevant geometry.

Example 2.6 (The quadric cone). We claim that $R = k[\![x_1, \ldots, x_n]\!]/(\sum_i x_i^2)$ is a splinter for $n \geqslant 3$ provided $\mathrm{char}(k) > 2$. By Example 2.4, it suffices to show that R is F-rational. By [Hun96, Theorem 4.2], it suffices to show that $R/(x_n)$ is F-rational. Thus, we can set up an induction once we settle the $n = 3$ case. This case follows from [Hoc73, Example 3]. Alternatively, in the $n = 3$ case, we may identify R with (completion at the origin of) the affine cone on a smooth conic $C \subset \mathbb{P}^2$. Since C is a hypersurface, the scheme $\mathrm{Spec}(R)$ has an isolated hypersurface singularity at 0, and is thus Cohen–Macaulay and normal. Moreover, identifying $\mathrm{Spec}(R) - \{\mathfrak{m}\}$ with the total space of the complement of the 0 section in $\mathcal{O}_{\mathbb{P}^2}(-1)|_C$ shows that

$$H^2_{\mathfrak{m}}(R) \simeq \bigoplus_{n \in \mathbb{Z}} H^1(C, \mathcal{O}_{\mathbb{P}^2}(-n)|_C) \simeq \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}(2n)).$$

The preceding presentation is Frobenius-equivariant, where Frobenius acts on the grading on the right by multiplying the weights by p. By inspection, it easily follows then that $H^2_{\mathfrak{m}}(R)$ has no Frobenius-stable proper non-zero submodules, proving F-rationality.

Next, we show that certain quotient singularities are splinters.

Example 2.7 (Quotient singularities are often splinters). Let k be a field, and let R be a regular k-algebra. Let G be a linearly reductive group acting on R. Then $\mathrm{Spec}(R^G)$ is a splinter. Indeed, the inclusion $R^G \to R$ has an R^G-linear section given by the Reynolds operator, and so the splinter property of R^G follows from that of R (base change a finite extension of R^G to that of R). More generally, the same argument shows that any subring A of a regular ring R that splits off as an A-linear summand is a splinter. In particular, if G is a reductive group over $\mathbb{C}$ acting on an affine algebraic $\mathbb{C}$-scheme $\mathrm{Spec}(R)$, then almost all positive-characteristic reductions of $\mathrm{Spec}(R^G)$ are splinters. We prove in Corollary 6.4 that such rings are Cohen–Macaulay.

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We next list a large class of non-examples.

Example 2.8 (General-type cones are not splinters). Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > (n+1)$ over a perfect field k of characteristic p, and let S be the affine cone on X. Then S is not a splinter. To see this, note that as in Example 2.6 we have an identification

$$H^n_{\mathfrak{m}}(S) \simeq \bigoplus_{i \in \mathbb{Z}} H^{n-1}(X, \mathcal{O}_X(i))$$

that is Frobenius-equivariant, where Frobenius acts on the right by scaling the weights by p. Now $\omega_X \simeq \mathcal{O}(d-n-1)|_X$ by adjunction. One then easily computes that $H^{n-1}(X, \omega^p) = H^{n-1}(X, \mathrm{Frob}_X^* \omega_X) = 0$, and thus $\mathrm{Frob}_X^* : H^{n-1}(X, \omega_X) \to H^{n-1}(X, \mathrm{Frob}_X^* \omega_X)$ has a non-trivial kernel. It follows that $\mathrm{Frob}_S^* : H^n_{\mathfrak{m}}(S) \to H^n_{\mathfrak{m}}(S)$ also has a non-trivial kernel, and so $\mathrm{Frob}_S : S \to S$ is not split.

Lastly, we discuss a non-example due to Hochster: a hypersurface singularity of dimension 2 in characteristic 2 that is not a splinter. Aside from its intrinsic interest, this example is meant to caution the reader, as the standard lift of this hypersurface to characteristic 0 has rational singularities.

Example 2.9. Let k be a field of characteristic 2. Let $S = k[u, v]$ be a polynomial ring, and let $R = k[u^2, v^2, u^3 + v^3] \hookrightarrow S$. Since $\mathrm{char}(k) = 2$, R admits the presentation $R = k[x, y, z]/(x^3 + y^3 + z^2)$ where $x = u^2$, $y = v^2$, and $z = u^3 + v^3$. In particular, $\mathrm{Spec}(R)$ is a hypersurface singularity of dimension 2. Since the singularity is isolated, R is even normal. On the other hand, $\mathrm{Spec}(R)$ is not a splinter because the natural map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a finite surjective map such that $\mathcal{O}_{\mathrm{Spec}(R)} \to f_* \mathcal{O}_{\mathrm{Spec}(S)}$ has no section: identifying sheaves with modules and applying such a section s to $u^3 + v^3 = u \cdot u^2 + v \cdot v^2$ would give us $u^3 + v^3 = s(u^3 + v^3) = s(u)u^2 + s(v)v^2 \in (u^2, v^2)R$, which is false. The same example can be adapted to arbitrary positive characteristic p by setting $R = k[u^p, v^p, u^a + v^a]$ for some $p < a < 2p$.

The examples discussed hitherto have all been affine. Requiring a projective variety X over a positive-characteristic field k to be a splinter leads to questions of a very different flavour, as the geometry of X is heavily constrained. For example, Theorem 1.5 shows that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. In fact, the same theorem applied to a high iterate of Frobenius shows that $H^i(X, \mathcal{L}) = 0$ for $i > 0$ whenever $\mathcal{L}$ is an ample line bundle. Thus, projective examples are harder to find; nevertheless, they do exist, as we show below. We will discuss such examples further in § 8.

Example 2.10 (Toric varieties are often splinters). Any toric variety X that is projective over an affine is a splinter. To see this, note that any such X can be obtained as a quotient U/G (see [MS05, Theorem 10.27]), where $U \subset \mathbb{A}^n$ is an open subscheme and $G \subset \mathbb{G}_m^n$ is an algebraic subgroup preserving U. As $\mathbb{G}_m^n$ is linearly reductive, so is G (see [AOV08, Proposition 2.5]). In particular, we see that $\mathcal{O}_X \to \pi_* \mathcal{O}_U \simeq \mathrm{R}\, \pi_* \mathcal{O}_U$ is a direct summand. The result now follows from the fact that U is a splinter, which in turn follows from Example 2.3 and the fact that any finite cover of U comes from a finite cover of $\mathbb{A}^n$ (by normalisation, for example).

Our last example exhibits the ubiquity of non-splinters amongst smooth projective varieties.

Example 2.11 (Projective varieties are rarely splinters). Let E be an elliptic curve over a field k of characteristic p. We will show that E is not a splinter. Consider the multiplication-by-p map $[p] : E \to E$. The induced map on $H^1(E, \mathcal{O}_E)$ is 0: one can show that $[n]^*$ induces multiplication by n on $H^1(A, \mathcal{O}_A)$ for any abelian variety A. It follows that $\mathcal{O}_E \to [p]_* \mathcal{O}_E$ is not split (as the map on H^1 is not split).

We end by giving a complete and direct proof of [Kov00, Theorem 3] as promised in Example 2.2.

THEOREM 2.12. *Let S be a scheme of finite type over a field k of characteristic 0. Then S is a D-splinter if and only if it has rational singularities.*

Proof. Let us prove first that if S is a D-splinter, then S has rational singularities. By Example 2.1, we may assume that S is normal of dimension d. By Hironaka's theorem (see [Hir64]) or even the weaker results of Abramovich and de Jong [AdJ97], we may assume that there exists a proper birational map $f: X \to S$ with X smooth. The natural map $\mathcal{O}_S \to \mathrm{R}\, f_*\mathcal{O}_X$ has a section by assumption. Thus, we have a diagram

$$\mathcal{O}_S \to \mathrm{R}\, f_*\mathcal{O}_X \to \mathcal{O}_S$$

with the composite map being the identity. Applying $\mathrm{R}\,\mathcal{H}\mathrm{om}(-, \omega_S^\bullet)$ where $\omega_S^\bullet$ is the dualising complex on S (normalised so that the dualising sheaf sits in homological degree d), we obtain a diagram

$$\omega_S^\bullet \to \mathrm{R}\,\mathcal{H}\mathrm{om}(\mathrm{R}\, f_*\mathcal{O}_X, \omega_S^\bullet) \to \omega_S^\bullet$$

with the composite map being the identity. By Grothendieck duality, the middle term is identified with $\mathrm{R}\, f_*\omega_X^\bullet$ where $\omega_X^\bullet$ is the dualising complex on X normalised as above. Thus, we obtain a diagram

$$\omega_S^\bullet \to \mathrm{R}\, f_*\omega_X^\bullet \to \omega_S^\bullet$$

with the composite map being the identity. As X is smooth, $\omega_X^\bullet \simeq \omega_X[d]$ where $\omega_X = \det(\Omega_X^1)$ is the canonical bundle and $d = \dim(X) = \dim(S)$. Grauert–Riemenschneider vanishing (see [Laz04a, Theorem 4.3.9]) tells us that $\mathrm{R}\, f_*\omega_X$ is concentrated in degree 0. Thus, the complex $\omega_S^\bullet$ is also concentrated in degree d. In particular, S is Cohen–Macaulay with dualising complex $\omega_S[d]$, where ω_S is the dualising sheaf. Moreover, the preceding diagram tells us that we have a diagram

$$\omega_S \to f_*\omega_X \to \omega_S$$

with the composite map being the identity. As ω_X is a torsion-free sheaf of generic rank 1, the same is true of $f_*\omega_X$. In particular, it admits no non-trivial direct summands for rank reasons. Hence, we have $\omega_S \simeq f_*\omega_X$ and, therefore, $\omega_S^\bullet \simeq \mathrm{R}\, f_*\omega_X^\bullet$. Now we have the following sequence of canonical isomorphisms:

$$\mathcal{O}_S \simeq \mathrm{R}\,\mathcal{H}\mathrm{om}(\omega_S^\bullet, \omega_S^\bullet) \simeq \mathrm{R}\,\mathcal{H}\mathrm{om}(\mathrm{R}\, f_*\omega_X^\bullet, \omega_S^\bullet) \simeq \mathrm{R}\, f_*\,\mathrm{R}\,\mathcal{H}\mathrm{om}(\omega_X^\bullet, \omega_X^\bullet) \simeq \mathrm{R}\, f_*\mathcal{O}_X,$$

which implies that S has rational singularities.

For the reverse implication, suppose that S is a k-variety with rational singularities, i.e. there exists a resolution $f: X \to S$ such that $\mathcal{O}_S \simeq \mathrm{R}\, f_*\mathcal{O}_X$. Let $g: Y \to S$ be a proper surjective morphism. We need to show that $\mathcal{O}_S \to \mathrm{R}\, g_*\mathcal{O}_Y$ has a section. By Chow's lemma, we can assume that Y is projective. By repeatedly cutting Y by suitable hyperplane sections, we may assume that g is generically finite. By the Raynaud–Gruson flattening theorem (see [RG71, Théorème 5.2.2]) or a simple Hilbert scheme argument, we can find a diagram as follows.

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d, "h"] & Y \arrow[d, "g"] \\ Y' \arrow[r, "b"] & S \end{tikzcd}$$

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Here b is a normalised blowup of S, h is the strict transform of g along b, and h is finite flat. As we are in characteristic 0 and Y' is normal, $\mathcal{O}_{Y'} \to h_*\mathcal{O}_{Y''} \simeq \mathrm{R}\, h_*\mathcal{O}_{Y''}$ has a section coming from the trace map. Thus, to show that $\mathcal{O}_S \to \mathrm{R}\, g_*\mathcal{O}_Y$ has a section, it suffices to show that $\mathcal{O}_S \to \mathrm{R}\, b_*\mathcal{O}_{Y'}$ has a section. In other words, we may assume that g is a modification. By Hironaka's theorem or Abramovich and de Jong's results as above, we may even assume that Y is smooth. The rest now follows by a standard argument (see [KM98, Theorem 5.10]) showing that the definition of rational singularities is independent of the choice of resolution. □

3. A fact about derived categories

The purpose of this section is to record some notation and a basic result about triangulated categories for later use. As a general reference for triangulated categories and t-structures, we suggest [BBD82]. For the convenience of the reader, we first recall some notation regarding truncations.

Notation 3.1. Let $\mathcal{D}$ be a triangulated category with a t-structure given by a pair $(\mathcal{D}^{\geqslant 0}, \mathcal{D}^{\leqslant 0})$ of full subcategories satisfying the usual axioms. For each integer n, we let $\mathcal{D}^{\geqslant n} = \mathcal{D}^{\geqslant 0}[-n]$ (respectively, $\mathcal{D}^{\leqslant n} = \mathcal{D}^{\leqslant 0}[-n]$); this can be thought of as the full subcategory spanned by objects with cohomology only in degree at least (respectively, at most) n. Moreover, there exist truncation functors: for each integer n, there exist endofunctors $\tau_{\leqslant n}$ and $\tau_{\geqslant n}$ of $\mathcal{D}$ which are retractions of $\mathcal{D}$ onto the full subcategories $\mathcal{D}^{\leqslant n}$ and $\mathcal{D}^{\geqslant n}$. We let $\tau_{>n} = \tau_{\geqslant n+1}$ and $\tau_{<n} = \tau_{\leqslant n-1}$. These truncation functors are not exact, but they sit in an exact triangle $\tau_{\leqslant n} \to \mathrm{id} \to \tau_{>n} \to \tau_{\leqslant n}[1]$. Moreover, they satisfy the adjunctions

$$\mathrm{Hom}_{\mathcal{D}^{\leqslant n}}(K, \tau_{\leqslant n} L) \simeq \mathrm{Hom}_{\mathcal{D}}(K, \tau_{\leqslant n} L) \simeq \mathrm{Hom}_{\mathcal{D}}(K, L) \quad \text{for } K \in \mathcal{D}^{\leqslant n} \text{ and } L \in \mathcal{D}$$

and, dually,

$$\mathrm{Hom}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n} K, L) \simeq \mathrm{Hom}_{\mathcal{D}}(\tau_{\geqslant n} K, L) \simeq \mathrm{Hom}_{\mathcal{D}}(K, L) \quad \text{for } K \in \mathcal{D} \text{ and } L \in \mathcal{D}^{\geqslant n}.$$

These adjunctions can be remembered as algebraic analogues of the fact that all maps $X \to Y$ are nullhomotopic if X is an n-connected CW complex and Y is an $(n-1)$-truncated one.

Let us fix a triangulated category $\mathcal{D}$, with a t-structure $(\mathcal{D}^{\geqslant 0}, \mathcal{D}^{\leqslant 0})$. The main question that arises repeatedly in the following is: given a morphism $f : K \to L$ in $\mathcal{D}$ such that $H^*(f) = 0$, when can we conclude that $f = 0$? As the non-trivial extension $\mathbb{Z}/2 \to \mathbb{Z}/2[1]$ in the derived category $\mathrm{D}(\mathrm{Ab})$ of abelian groups shows, the short answer is 'not always'. To understand this phenomenon better, fix a test object $M \in \mathcal{D}$, and consider the associated map of abelian groups

$$\mathrm{Hom}(M, f) : \mathrm{Hom}(M, K) \to \mathrm{Hom}(M, L).$$

The chosen t-structure gives rise to a functorial filtration on the morphism spaces of $\mathcal{D}$ (via the filtration by cohomology groups of the target). Thus, the preceding map is a filtered map of filtered abelian groups. The assumption that $H^*(f) = 0$ implies that this filtered map induces the 0 map on the associated graded pieces. In other words, f moves the filtration one level down. This simple analysis suggests that under certain boundedness hypotheses, we may be able to salvage an implication of the form '$H^*(f) = 0 \Rightarrow f = 0$' at the expense of iterating a map like f a few times. This idea is formalised in the next lemma.

Lemma 3.2. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\geqslant 0}, \mathcal{D}^{\leqslant 0})$ whose heart is $\mathcal{A}$. Assume that for a fixed integer $d > 0$, we are given objects $K_1, \ldots, K_{d+1} \in \mathcal{D}^{[1,d]}$ and*

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maps $f_i : K_i \to K_{i+1}$ such that $H^{d+1-i}(f_i) = 0$ for all i. Then the composite map $f_d \circ \cdots \circ f_2 \circ f_1 : K_1 \to K_d$ is the 0 map.

Proof. Consider the exact triangle

$$\tau_{\leqslant d-1} K_2 \to K_2 \to H^d(K_2)[-d] \to \tau_{\leqslant d-1} K_2[1].$$

Applying $\mathrm{Hom}_{\mathcal{D}}(K_1, -)$ and using the formula

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(K_1, H^d(K_2)[-d]) &= \mathrm{Hom}_{\mathcal{D}^{\geqslant d}}(\tau_{\geqslant d} K_1, H^d(K_2)[-d]) \\ &= \mathrm{Hom}_{\mathcal{D}^{\geqslant d}}(H^d(K_1)[-d], H^d(K_2)[-d]) \\ &= \mathrm{Hom}_{\mathcal{A}}(H^d(K_1), H^d(K_2))\end{aligned}$$

(coming from adjunction), we see that the map $K_1 \to H^d(K_2)[-d]$ factors through $H^d(f_1)$ and is thus 0 by hypothesis. We may therefore choose a (non-unique) factorisation of f_1 of the form $K_1 \to \tau_{\leqslant d-1} K_2 \to K_2$. The same method shows that the morphism $\tau_{\leqslant d-i} K_{i+1} \to \tau_{\leqslant d-i} K_{i+2}$ factors through $\tau_{\leqslant d-(i+1)} K_{i+2}$. Thus we obtain the following diagram of morphisms.

$$\begin{tikzcd} K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d, "f_1"] \\ \tau_{\leqslant d-1} K_2 \arrow[r] \arrow[d] & K_2 \arrow[d, "f_2"] \\ \cdots \arrow[r] \arrow[d] & \cdots \arrow[d, "f_d"] \\ \tau_{\leqslant 0} K_{d+1} \arrow[r] & K_{d+1} \end{tikzcd}$$

As $K_{d+1} \in \mathrm{D}^{\geqslant 1}(\mathcal{A})$, we see that $\tau_{\leqslant 0} K_{d+1} = 0$. Thus, the composite vertical morphism on the left is 0, which implies that the one on the right is 0 as well. □

4. The main theorem

This section is dedicated to the proof of Theorems 1.5 and 1.4. In fact, the bulk of the work involves proving Theorem 1.5, as Theorem 1.4 follows fairly easily. The proof given here draws on ideas going back to Hochster and Huneke's work [HH92] on big Cohen–Macaulay algebras in positive characteristic. We begin with a rather elementary result on extending covers of schemes.

Proposition 4.1. *Fix a noetherian scheme X. Given an open dense subscheme $U \to X$ and a finite (surjective) morphism $f : V \to U$, there exists a finite (surjective) morphism $\overline{f} : \overline{V} \to X$ such that $\overline{f}_U$ is isomorphic to f. Given a Zariski-open cover $\mathcal{U} = \{j_i : U_i \to X\}$ with a finite index set and finite (surjective) morphisms $f_i : V_i \to U_i$, there exists a finite (surjective) morphism $f : Z \to X$ such that f_{U_i} factors through f_i. The same claims hold if 'finite (surjective)' is replaced by 'proper (surjective)'.*

Proof. We first explain how to deal with the claims for finite morphisms. For the first part, Zariski's main theorem [Gro66, Théorème 8.12.6] applied to the morphism $V \to X$ gives a factorisation $V \hookrightarrow W \to X$ where $V \hookrightarrow W$ is an open immersion and $W \to X$ is a finite morphism. The scheme-theoretic closure $\overline{V}$ of V in W provides the required compactification as finite morphisms are closed.

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For the second part, by the first part we may extend each $j_i \circ f_i : V_i \to X$ to a finite surjective morphism $\overline{f_i} : \overline{V_i} \to X$ such that $\overline{f_i}$ restricts to f_i over $U_i \hookrightarrow X$. Setting W to be the fibre product over X of all the $\overline{V_i}$ is then seen to solve the problem.

To deal with the case of proper (surjective) morphisms instead of finite (surjective) ones, we repeat the same argument as above but replace the reference to Zariski's main theorem by one to Nagata's compactification theorem (see [Con07, Theorem 4.1]). $\square$

Next, we present the primary ingredient in our proof of Theorem 1.5: a general technique for constructing covers to annihilate coherent cohomology of $\mathbb{F}_p$-schemes under suitable finiteness assumptions. The method of construction is inspired by [HL07]. In fact, our key contribution is the observation that a suitable reformulation of the ideas of [HL07] apply in the *relative* setting of a proper morphism, rather than simply to a proper variety.

Proposition 4.2. *Let X be a noetherian $\mathbb{F}_p$-scheme with $H^0(X, \mathcal{O}_X)$ finite over a ring A. Given an A-finite Frobenius-stable submodule $M \subset H^i(X, \mathcal{O}_X)$ for $i > 0$, there exists a finite surjective morphism $\pi : Y \to X$ such that $\pi^*(M) = 0$.*

Proof. We first explain the idea informally. As M is A-finite, it suffices to work one cohomology class at a time. If $m \in M$, then the Frobenius stability of M gives us a monic additive polynomial $g(X^p)$ such that $g(m) = 0$ where X^p acts by Frobenius. After adjoining gth roots of certain local functions representing a coboundary, we can promote the preceding equation in cohomology to an equation of cocycles, i.e. we find $g(\overline{m}) = 0$ where $\overline{m}$ is a cocycle of local functions that represents m and the displayed equality is an equality of functions on the nose, not simply up to coboundaries. Since g is monic, such functions are forced to be globally defined (after normalisation), and this gives the desired result; the details follow.

Fix a finite affine open cover $\mathcal{U} = \{U_i\}$ of X, and consider the cosimplicial A-algebra $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ as a model for the A-algebra $\mathrm{R}\,\Gamma(X, \mathcal{O}_X)$. The Frobenius action $\mathrm{Frob}_X^* : \mathrm{R}\,\Gamma(X, \mathcal{O}_X) \to \mathrm{R}\,\Gamma(X, \mathcal{O}_X)$ is modelled by the actual Frobenius map $X^p : x \mapsto x^p$ on each term. This gives $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ the structure of an $A\{X^p\}$-module, where $A\{X^p\}$ is the non-commutative polynomial ring on one generator X^p over A satisfying the relation $r^p X^p = X^p r$ (see [Lau96, § 1.1] for more details on this ring). In more concrete terms, at the level of cohomology we see the following: for each polynomial $g \in A\{X^p\}$, classes $\alpha, \beta \in H^i(X, \mathcal{O}_X)$, and a scalar $r \in A$, we have $g(\alpha + \beta) = g(\alpha) + g(\beta)$ and $g(r\alpha) = r^p g(\alpha)$.

The A-finiteness of the Frobenius-stable module M ensures that for any class $m \in M$ there exists a *monic* polynomial $g \in A\{X^p\}$ such that $g(m) = 0$. If we pick representatives in $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ for this equation, we obtain an equation in $\mathcal{C}^i(\mathcal{U}, \mathcal{O}_X)$ of the form

$$g(\tilde{m}) = d(n)$$

where $\tilde{m} \in \mathcal{C}^i(\mathcal{U}, \mathcal{O}_X)$ is a cocycle lifting m and $n \in \mathcal{C}^{i-1}(\mathcal{U}, \mathcal{O}_X)$. As g is a monic equation, we can find a finite surjective morphism $\pi' : Y' \to X$ such that $n = g(n')$ for some $n' \in \mathcal{C}^i(\mathcal{U} \times_X Y', \mathcal{O}_{Y'})$. For example, we could do the following: for each component n_j of n (where j is a multi-index), the scheme $V_j = \mathrm{Spec}(\mathcal{O}(U_j)[T]/(g(T) - n_j))$ is a quasi-finite X-scheme such that the equation $g(n') = n$ admits a solution in $H^0(V_j, \mathcal{O}_{V_j})$. Using Proposition 4.1, we find Y' and n' with the desired properties. The additivity of Frobenius now tells us that we obtain an equation in $\mathcal{C}^i(\mathcal{U} \times_X Y', \mathcal{O}_{Y'})$ of the form

$$g(\tilde{m} - d(n')) = 0.$$

The monicity of g implies that the components of $\tilde{m} - d(n')$ are integral over A. Setting Y to be an irreducible component of $Y' \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[T]/(g(T)))$ that dominates Y' under the natural map, we find a finite surjective morphism $Y \to Y'$. The pullback of $\tilde{m} - d(n')$ in $\mathcal{C}^i(\mathcal{U} \times_X Y, \mathcal{O}_Y)$ is a vector of local functions whose components satisfy the monic polynomial g over A. As Y is integral and $H^0(Y, \mathcal{O}_Y)$ already contains roots of g, it follows that these functions are globally defined. Thus, they lie in the image of the natural map $H^0(Y, \mathcal{O}_Y) \to \mathcal{C}^\bullet(\mathcal{U} \times_X Y, \mathcal{O}_Y)$ where $H^0(Y, \mathcal{O}_Y)$ is viewed as a constant cosimplicial algebra. As the complex underlying the former cosimplicial algebra has cohomology only in degree 0, it follows that $\tilde{m} - d(n')$ is a coboundary, which then implies that $\tilde{m}$ is a coboundary on Y; this shows that Y satisfies the required conditions. □

Remark 4.3. One may wonder whether Proposition 4.2 can be refined to show the existence of generically separable finite surjective maps that kill the relevant cohomology groups. In the local algebra setting, one can indeed do so by [SS12, Theorem 1.3]. Globally, however, requiring separability is too strong. For example, if X is a smooth projective variety over a perfect field k with $\alpha \in H^1(X, \mathcal{O}_X)$ being a non-zero class killed by Frob_X, then for any finite surjective generically separable map $\pi : Y \to X$ one has $\pi^*\alpha \neq 0$; see [Mum67, Lemma 5] for a proof.

Remark 4.4. Proposition 4.2 was proven above by mimicking the cocycle-theoretic methods of [HL07]. It is also possible to give more conceptual proofs of this result. We refer the reader to [Bha11a] for a proof based on general results on finite flat group schemes; see also [Bha11b] for a geometric proof based on curve fibrations, which has the advantage of generalising to mixed characteristic.

A corollary of Proposition 4.2 and the finiteness properties enjoyed by proper morphisms is the following.

COROLLARY 4.5. *Let $f : X \to S$ be proper with S a noetherian affine $\mathbb{F}_p$-scheme. Then there exists a finite surjective morphism $\pi : Y \to X$ such that $\pi^* : H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y)$ is 0 for $i > 0$.*

Proof. The properness implies that $H^i(X, \mathcal{O}_X)$ is a finite $H^0(X, \mathcal{O}_X)$-module and that $H^i(X, \mathcal{O}_X) = 0$ for i sufficiently large (see [Gro61, Corollaire 3.2.3]). Proposition 4.2 then finishes the proof. □

We will now finish the proof of Theorem 1.5. To pass from the conclusion of Corollary 4.5 to the general statement of Theorem 1.5, the obvious strategy is to cover S with affines, construct covers that work over the affines, and take the normalisation of X in the fibre product of all of these. When carried out, this process produces a finite cover $\pi : Y \to X$ such that, with $g = f \circ \pi$, the maps $R^i f_* \mathcal{O}_X \to R^i g_* \mathcal{O}_Y$ are 0 for $i > 0$. This is not quite enough to prove the theorem: a map in $\mathrm{D}(\mathrm{Coh}(S))$ that induces the 0 map on cohomology sheaves is not necessarily zero. However, with the boundedness conditions enforced by properness, a sufficiently high iteration of this process turns out to be enough.

Proof of Theorem 1.5. Fix a finite affine covering $\mathcal{U} = \{U_i\}$ of S, and denote $X \times_S U_i$ by X_i. Using Corollary 4.5, we can find finite surjective maps $\phi_i : Z_i \to X_i$ such that the induced map $H^j(X_i, \mathcal{O}_{X_i}) \to H^j(Z_i, \mathcal{O}_{Z_i})$ is 0 for each $j > 0$. Using Proposition 4.1, we may find a finite surjective morphism $\phi : Z \to X$ such that ϕ_{U_i} factors through ϕ_i. This implies that $\mathrm{R}^j f_* \mathcal{O}_X \to \mathrm{R}^j (f \circ \phi)_* \mathcal{O}_Z$ is 0 for each j (as vanishing is a local statement on S). Iterating this construction $\dim(X)$ times and using Lemma 3.2, we obtain a proper S-scheme $g : Y \to S$ and a finite

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surjective S-morphism $\pi : Y \to X$ such the natural pullback map $\pi^* : \tau_{\geqslant 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geqslant 1} \mathrm{R} g_* \mathcal{O}_Y$ is 0, thereby proving the theorem. □

Finally, having proven Theorem 1.5, we point out how Theorem 1.4 follows.

Proof of Theorem 1.4. It is clear that all D-splinters are also splinters. Conversely, let S be splinter over $\mathbb{F}_p$, and let $f : X \to S$ be a proper surjective morphism. By Theorem 1.5, there exists a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback map $\tau_{\geqslant 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geqslant 1} \mathrm{R} g_* \mathcal{O}_Y$ is 0. By applying $\mathrm{Hom}(\mathrm{R} f_* \mathcal{O}_X, -)$ to the exact triangle

$$g_* \mathcal{O}_Y \to \mathrm{R} g_* \mathcal{O}_Y \to \tau_{\geqslant 1} \mathrm{R} g_* \mathcal{O}_Y \to g_* \mathcal{O}_Y[1]$$

we see that the natural pullback map $\mathrm{R} f_* \mathcal{O}_X \to \mathrm{R} g_* \mathcal{O}_Y$ factors through $g_* \mathcal{O}_Y \to \mathrm{R} g_* \mathcal{O}_Y$; choose some factorisation $s : \mathrm{R} f_* \mathcal{O}_X \to g_* \mathcal{O}_Y$. As $g : Y \to S$ is a proper surjective morphism, the algebra $g_* \mathcal{O}_Y$ is a coherent sheaf of algebras corresponding to the structure sheaf of a finite surjective morphism. By assumption, the natural map $\mathcal{O}_S \to g_* \mathcal{O}_Y$ has a splitting s, and thus the map $s \circ t$ splits $\mathcal{O}_S \to \mathrm{R} f_* \mathcal{O}_X$. □

5. Some refinements

Roughly speaking, Theorem 1.5 says that proper morphisms and finite morphisms behave very similarly 'in the limit', at least as far as coherent sheaf cohomology is concerned. In the following proposition, we formalise this intuition, extract a kind of 'converse' to this statement, and work with non-trivial coefficients. These results will be useful later when we prove vanishing results.

Proposition 5.1. *Let S be a noetherian $\mathbb{F}_p$-scheme, and let $f : X \to S$ be a proper surjective morphism. Then we can find a diagram*

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "a"] \arrow[dr, "g"] & X \arrow[d, "f"] \\ S' \arrow[r, "h"] & S \end{tikzcd}$$

with π and h being finite surjective maps such that for each locally free sheaf $\mathcal{M}$ on S and all $i \geqslant 0$, we have that:

(i) *the morphism $h^* : H^i(S, \mathcal{M}) \to H^i(S', h^*\mathcal{M})$ factors through $f^* : H^i(S, \mathcal{M}) \to H^i(X, f^*\mathcal{M})$;*

(ii) *the morphism $\pi^* : H^i(X, f^*\mathcal{M}) \to H^i(Y, g^*\mathcal{M})$ factors through $a^* : H^i(S', h^*\mathcal{M}) \to H^i(Y, g^*\mathcal{M})$.*

Proof. Theorem 1.5 gives a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have a map s and the following diagram.

$$\begin{tikzcd} \mathcal{O}_S \arrow[r] \arrow[d] & g_* \mathcal{O}_Y \arrow[d] \\ \mathrm{R} f_* \mathcal{O}_X \arrow[r] \arrow[ur, "s"] & \mathrm{R} g_* \mathcal{O}_Y \end{tikzcd}$$

We claim that this diagram commutes. The triangle based at $\mathrm{R} g_* \mathcal{O}_Y$ commutes by construction. To see that the triangle based at $\mathcal{O}_S$ commutes, it suffices to show that $\mathrm{Hom}(\mathcal{O}_S, g_* \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{O}_S, \mathrm{R} g_* \mathcal{O}_Y)$ is injective. This injectivity (and, in fact, bijectivity) follows from adjunction for $\tau_{\leqslant 0}$. Thus, the preceding diagram is a commutative diagram in $\mathrm{D}(\mathrm{Coh}(S))$. Applying $- \otimes \mathcal{M}$, setting S' to be the Stein factorisation of $Y \to S$, and using the projection formula now gives the desired result. □

We have not strived to find the most general setting for Theorem 1.5. For example, one can easily extend the theorem to algebraic spaces or even Deligne–Mumford stacks. On the other hand, the properness hypothesis seems essential, as the example below shows. In fact, the key property needed is that the relative cohomology classes of the structure sheaf for $f : X \to S$ are annihilated by a monic polynomial in Frobenius. We do not know whether there is a better characterisation of this class of maps.

Example 5.2. Fix a base field k. Let $X = \mathbb{A}^2$ and $U = \mathbb{A}^2 - \{0\}$. The quotient map $U \to U/\mathbb{G}_m = \mathbb{P}^1$ gives a natural identification $H^1(U, \mathcal{O}_U) = \bigoplus_{i\in\mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}(i))$. We claim that the non-zero classes in this group cannot be killed by a finite cover of U. To see this, note that one may view $H^1(U, \mathcal{O}_U)$ as the local cohomology group $H^2_{\{0\}}(X, \mathcal{O}_U) = H^2_{\mathfrak{m}}(R)$, where $R = k[x, y]$ is the coordinate ring of X and $\mathfrak{m} = (x, y)$ is the maximal ideal corresponding to the origin. Given a finite surjective morphism $\pi : Y \to U$, we may normalise X in π to obtain a finite surjective morphism $\overline{\pi} : \overline{Y} \to X$ which realises π as the fibre over U. As before, the cohomology group $H^1(Y, \mathcal{O}_Y)$ can be viewed as $H^2_{\overline{Y}\setminus Y}(\overline{Y}, \mathcal{O}_{\overline{Y}})$, which in turn may be viewed as $H^2_{\mathfrak{m}}(S)$ where S is the coordinate ring of $\overline{Y}$ considered as an R-module in the natural way. Under these identifications, the pullback map $H^1(U, \mathcal{O}_U) \to H^1(Y, \mathcal{O}_Y)$ corresponds to the morphism $H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(S)$ induced by the inclusion $R \to S$ coming from $\overline{\pi}$. By Example 2.3, the inclusion $R \to S$ is a direct summand as an R-module map; alternatively, one can show this property directly by proving that S is a locally free R-module thanks to the Auslander–Buchsbaum formula. In particular, the map $H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(S)$ is injective, which shows that the non-zero classes in $H^1(U, \mathcal{O}_U)$ persist after passage to finite covers.

6. Application: a result in commutative algebra

We discuss some applications of Proposition 4.2 to commutative algebra, some of which are implicit in [HL07]. The first result we want to dicuss is an analogue of Proposition 4.2 for local cohomology.

Proposition 6.1. *Let $(R, \mathfrak{m})$ be an excellent local noetherian $\mathbb{F}_p$-algebra such that R is finite over some ring A. For any A-finite Frobenius-stable submodule $M \in H^i_{\mathfrak{m}}(R)$ with $i \geqslant 1$, there exists a finite surjective morphism $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that $f^*(M) = 0$.*

Proof. Since R is excellent, we may pass to the normalisation and assume that R is normal. In particular, $H^i_{\mathfrak{m}}(R) = 0$ for $i = 0, 1$. For $i > 1$, we have an Frobenius-equivariant identification $\delta : H^{i-1}(U, \mathcal{O}_U) \simeq H^i_{\mathfrak{m}}(R)$, where $U = \mathrm{Spec}(R) - \{\mathfrak{m}\}$ is the punctured spectrum of R. Since $i > 1$, Proposition 4.2 gives us a finite surjective morphism $f : V \to U$ such that $f^*(\delta^{-1}(M)) = 0$. Setting S to be the normalisation of R in V is then easily seen to do the job. □

Next, we dualise Proposition 6.1 to obtain a global result in terms of dualising sheaves.

Proposition 6.2. *Let X be an excellent noetherian local $\mathbb{F}_p$-scheme of dimension d. Assume that X admits a dualising complex. Then there exists a finite surjective morphism $\pi : Y \to X$ with $\tau_{>-d}(\mathrm{Tr}_\pi) = 0$; here $\mathrm{Tr}_\pi : \pi_*\omega^\bullet_Y \to \omega^\bullet_X$ is the trace map.*

Proof. Let $X = \mathrm{Spec}(R)$, and fix an integer $i > 0$. We will prove by induction on the dimension $d = \dim(X)$ that there exists a finite surjective morphism $\pi' : Y' \to X$ such that $\mathcal{H}^{-d+i}(\mathrm{Tr}_{\pi'}) = 0$;

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this is enough by virtue of Lemma 3.2 and the fact that the dualising complexes appearing have bounded amplitude. We may assume that $d>0$, as there is nothing to prove when the dimension is 0. By passing to irreducible components, we may even assume that X is integral. For each non-maximal $\mathfrak{p}\in\operatorname{Spec}(R)$, we can inductively find a finite morphism $\pi_{\mathfrak{p}}:Y_{\mathfrak{p}}\to\operatorname{Spec}(R_{\mathfrak{p}})$ such that $\mathcal{H}^{-d_{R_{\mathfrak{p}}}+i}(\mathrm{Tr}_{\pi_{\mathfrak{p}}})$ is the 0 map. By duality formalism, the R-module $\mathcal{H}^{-d+i}(\omega_R^\bullet)$ localises to $\mathcal{H}^{-d_{R_{\mathfrak{p}}}+i}(\omega_{R_{\mathfrak{p}}}^\bullet)$ at $\mathfrak{p}$. Hence, the normalisation $\overline{\pi_{\mathfrak{p}}}:\overline{Y_{\mathfrak{p}}}\to X$ induces the 0 map on $\mathcal{H}^{-d+i}(\mathrm{Tr}_\pi)$ when localised at $\mathfrak{p}$. Upon finding such a cover for each non-maximal prime $\mathfrak{p}$ in the finite set of associated primes of $\mathcal{H}^{-d+i}(\omega_R^\bullet)$ and normalising X in the fibre product of the resulting collection, we find a cover $\pi:Y\to X$ such that $\mathcal{H}^{-d+i}(\mathrm{Tr}_\pi)$ has an image supported only at the closed point. Setting $Y=\operatorname{Spec}(S)$, duality tells us that the image M of $H^{d-i}_{\mathfrak{m}}(R)\to H^{d-i}_{\mathfrak{m}}(S)$ is a finite-length Frobenius-stable R-submodule. Proposition 6.1 then allows us to find a finite surjective morphism $g:\operatorname{Spec}(T)\to\operatorname{Spec}(S)$ such that $g^*(M)=0$. It follows that the composite map $\pi':\operatorname{Spec}(T)\to\operatorname{Spec}(R)$ induces the 0 map on $H^{d-i}_{\mathfrak{m}}(R)$. By duality, we see that $\mathcal{H}^{-d+i}(\mathrm{Tr}_{\pi'})=0$ as desired. □

Remark 6.3. Proposition 6.2 is also true when X is a finite-type k-scheme for some field k of characteristic p; the proof given above works since vanishing of a map of sheaves is a local statement.

Using Proposition 6.2, we discover that splinters are automatically Cohen–Macaulay.

COROLLARY 6.4. *Let $(R,\mathfrak{m})$ be an excellent noetherian local $\mathbb{F}_p$-algebra that is a splinter. Assume that R admits a dualising complex. Then R is a normal Cohen–Macaulay domain.*

Proof. The normality of R follows from the argument in Example 2.1. To verify that R is Cohen–Macaulay, it suffices to show that $\omega_R^\bullet$ is concentrated in degree d where $d=\dim(R)$, i.e. that $\mathcal{H}^{-d+k}(\omega_R^\bullet)=0$ for $k>0$. By Proposition 6.2, we can find a finite surjective morphism $\pi:\operatorname{Spec}(S)\to\operatorname{Spec}(R)$ such that $\mathcal{H}^{-d+k}(\mathrm{Tr}_\pi)=0$, where $\mathrm{Tr}_\pi:\pi_*\omega_S^\bullet\to\omega_R^\bullet$. Since R is a splinter, the inclusion $R\to S$ is a direct summand. Applying $\mathrm{R}\,\mathcal{H}\mathrm{om}(-,\omega_R^\bullet)$, we see that the trace map Tr_π is the projection onto a summand. Hence, the assumption that $\mathcal{H}^{-d+k}(\mathrm{Tr}_\pi)=0$ implies that $\mathcal{H}^{-d+k}(\omega_R^\bullet)=0$, as desired. □

Remark 6.5. One key ingredient in the proof of Proposition 6.2 is the good behaviour of local cohomology and dualising sheaves with respect to localisation. This behaviour seems to have first been observed by Grothendieck in [Gro68, Exposé VIII, Théorème 2.1], where it is used to show the following: a noetherian local ring $(R,\mathfrak{m})$ of dimension d that is Cohen–Macaulay outside the closed point and admits a dualising complex has the property that $H^i_{\mathfrak{m}}(R)$ has finite length for $i<d$. This argument can also be found in the main theorem of [HL07].

7. Application: a question of Karen Smith

The main result of Hochster and Huneke's work [HH92] is a result in commutative algebra. While geometrising it in [Smi97c], Smith arrived at the following question (see [Smi97a]).

Question 7.1. Let X be a projective variety over a field k of characteristic p, and let $\mathcal{L}$ be a 'weakly positive' line bundle on X. For any $n\in\mathbb{Z}$ and any $0<i<\dim(X)$, does there exist a finite surjective morphism $\pi:Y\to X$ such that $H^i(X,\mathcal{L}^{\otimes n})\to H^i(Y,\pi^*\mathcal{L}^{\otimes n})$ is 0?

Using the algebraic result of Hochster and Huneke [HH92], one can show that if we take 'weakly positive' to mean ample, then Question 7.1 has an affirmative answer (see Remark 7.4).

Smith had originally hoped that 'weakly positive' could be taken to mean nef. We give some examples to show that this cannot be the case. However, first, we prove some positive results using the theorems above.

Positive results

We first examine Question 7.1 in the case of positive twists. It is clear that being ample is a sufficiently positive condition for the required vanishing: Frobenius twisting can be realised by pulling back along a finite morphism and has the effect of changing $\mathcal{L}$ by $\mathcal{L}^{\otimes p}$, and then Serre vanishing does the job. One naturally wonders whether the result passes to the closure of the ample cone, i.e. the nef cone. We show in Example 7.11 that this is not the case: there exist non-torsion degree 0 line bundles on surfaces whose middle cohomology cannot be killed by finite covers. On the other hand, Corollary 4.5 coupled with the fact that torsion line bundles can be replaced with $\mathcal{O}$ on passage to a finite cover ensures that Question 7.1 has a positive answer for torsion line bundles. The necessity of the non-torsion requirement and the observation that torsion line bundles are semiample suggests the following proposition.

Proposition 7.2. *Let X be a proper variety over a field of characteristic p, and let $\mathcal{L}$ be a semiample line bundle on X. For any $i>0$, there exists a finite surjective morphism $\pi: Y \to X$ such that the induced map $H^i(X,\mathcal{L}) \to H^i(Y,\pi^*\mathcal{L})$ is 0.*

Proof. As $\mathcal{L}$ is a semiample bundle, there exists some positive integer m such that $\mathcal{L}^{\otimes m}$ is globally generated. If we fix a basis $s_1,\ldots,s_k$ for $H^0(X,\mathcal{L}^{\otimes m})$, then the cyclic covering trick (see [Laz04a, Proposition 4.1.3]) ensures that there is a finite flat cover $\pi:\tilde{X}\to X$ such that $\pi^*(s_i)$ admits an mth root in $H^0(\tilde{X},\pi^*\mathcal{L})$ and, consequently, $\pi^*\mathcal{L}$ is globally generated. In particular, as semiamplitude is preserved under pullbacks, we may replace X with $\tilde{X}$ and assume that $\mathcal{L}$ arises as the pullback of an ample bundle $\mathcal{M}$ under a proper surjective morphism $f: X\to S$. Furthermore, once $f: X\to S$ is fixed, to show the required vanishing statement, we may always replace $\mathcal{L}$ by $\mathcal{L}^{\otimes p^j}$ for $j \gg 0$ because the Frobenius morphism $\mathrm{Frob}_X: X\to X$ is finite surjective with $\mathrm{Frob}_X^*\mathcal{L}=\mathcal{L}^{\otimes p}$. Now the projection formula for f implies that $\mathrm{R} f_*(\mathcal{L}^{\otimes p^j}) = \mathrm{R} f_*\mathcal{O}_X \otimes^{\mathrm{L}}_S \mathcal{M}^{\otimes p^j}$. Using Theorem 1.5, we may find a finite surjective morphism $\pi: Y\to X$ such that, with $g=f\circ\pi$, we have a factorisation $\mathrm{R} f_*(\mathcal{L}^{\otimes p^j}) \to g_* f^*(\mathcal{L}^{\otimes p^j}) \to \mathrm{R} g_*\pi^*(\mathcal{L}^{\otimes p^j})$ of the natural map $\pi^*: \mathrm{R} f_*(\mathcal{L}^{\otimes p^j}) \to \mathrm{R} g_*\pi^*(\mathcal{L}^{\otimes p^j})$. Applying $H^i(S,-)$ to the composite morphism gives us the desired morphism. Thus, to show the required statement, it suffices to show that $H^i(S, g_*\pi^*(\mathcal{L}^{\otimes p^j}))=0$ for $j\gg 0$. By the projection formula, we have

$$H^i(S, g_*\pi^*(\mathcal{L}^{\otimes p^j})) = H^i(S, g_*g^*(\mathcal{M}^{\otimes p^j})) = H^i(S, g_*\mathcal{O}_Y \otimes \mathcal{M}^{\otimes p^j}).$$

As $\mathcal{M}$ is ample, this group vanishes by Serre vanishing for $j\gg 0$, as required. □

Based on Proposition 7.2, one might expect that semiamplitude is a positive enough property for Question 7.1 to have an affirmative answer in the case of negative twists as well. We show in Example 7.12 that this is not true; the key feature of that example is that the semiample line bundle defines a map that is not generically finite. In fact, this feature is essentially the only obstruction: if $\mathcal{L}$ is both semiample and big, then Question 7.1 has an affirmative answer even for negative twists of $\mathcal{L}$.

Proposition 7.3. *Let X be a proper variety over a field of characteristic p, and let $\mathcal{L}$ be a semiample and big line bundle on X. For any $i<\dim(X)$, we can find a finite surjective morphism $\pi: Y\to X$ such that the induced map $H^i(X,\mathcal{L}^{-1}) \to H^i(Y,\pi^*\mathcal{L}^{-1})$ is 0.*

Proof. We first describe the idea informally. Using Proposition 5.1 and arguments similar to those in the proof of Proposition 7.2, we will reduce to the case that $\mathcal{L}$ is actually *ample* on X. In this case, we give a direct proof using Proposition 6.2; the details follow.

Fix an integer $i < \dim(X)$. As $\mathcal{L}$ is big, there is nothing to show for $i = 0$, and thus we may assume $i > 0$. As in the proof of Proposition 7.2, at the expense of replacing X by a finite flat cover, we may assume that $\mathcal{L}$ arises as the pullback of an ample line bundle $\mathcal{M}$ under a proper surjective morphism $f : X \to S$. As bigness is preserved under passage to finite flat covers, we may continue to assume that $\mathcal{L}$ is big. In particular, the map f is forced to be an alteration. By Proposition 5.1, we can find a diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "a"] \arrow[dr, "g"] & X \arrow[d, "f"] \\ S' \arrow[r, "h"] & S \end{tikzcd}$$

with π and h finite surjective such that we have a factorisation $H^i(X, \mathcal{L}^{-1}) \xrightarrow{s} H^i(S', h^*\mathcal{M}^{-1}) \xrightarrow{a^*} H^i(Y, \pi^*\mathcal{L}^{-1})$ of π^* for some map s. Moreover, given a finite cover $b : S'' \to S$, we can form the following diagram.

$$\begin{tikzcd} Y \times_{S'} S'' \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & Y \arrow[r, "\pi"] \arrow[d, "a"] \arrow[dr, "g"] & X \arrow[d, "f"] \\ S'' \arrow[r, "b"] & S' \arrow[r, "h"] & S \end{tikzcd}$$

This means that, at the level of cohomology, we have a commutative diagram as follows.

$$\begin{tikzcd} H^i(X, \mathcal{L}^{-1}) \arrow[r, "(\pi\circ\mathrm{pr}_1)^*"] \arrow[d, "s"] & H^i(Y \times_{S'} S'', (\pi \circ \mathrm{pr}_1)^*\mathcal{L}^{-1}) \\ H^i(S', h^*\mathcal{M}^{-1}) \arrow[r, "b^*"] & H^i(S'', b^*h^*\mathcal{M}^{-1}) \arrow[u, "\mathrm{pr}_2^*"'] \end{tikzcd}$$

Thus, it suffices to show that $H^i(S', h^*\mathcal{M}^{-1})$ can be killed by finite covers of S'. As h is a finite morphism, the bundle $h^*\mathcal{M}$ is ample. That f was an alteration forces $\dim(S') = \dim(X)$ and, therefore, $0 < i < \dim(S')$. In other words, we are reduced to verifying the claim in the theorem under the additional assumption that $\mathcal{L}$ is ample.

As we are free to replace X by a Frobenius twist (which increases the positivity of $\mathcal{L}$), we may assume that $\mathcal{L}$ has the property that $H^j(X, \mathcal{L} \otimes \omega_X) = 0$ for all $j > 0$, where ω_X is the dualising *sheaf* on X. Now choose a finite surjective morphism $\pi : Y \to X$ satisfying the conclusion of Proposition 6.2. With $d = \dim(X)$, the trace map induces the following morphism of triangles in $\mathrm{D}^b(\mathrm{Coh}(X))$.

$$\begin{tikzcd} \pi_*\omega_Y[d] \arrow[r] \arrow[d, "a"] & \pi_*\omega_Y^\bullet \arrow[r] \arrow[d, "b"] \arrow[dl, dotted, "s"] & \tau_{>-d}\pi_*\omega_Y^\bullet \arrow[r] \arrow[d, "c=0"] & \pi_*\omega_Y[d+1] \arrow[d] \\ \omega_X[d] \arrow[r] & \omega_X^\bullet \arrow[r] & \tau_{>-d}\omega_X^\bullet \arrow[r] & \omega_X[d+1] \end{tikzcd}$$

Here s is a map whose existence is ensured by the equation $c=0$ (but s is not necessarily unique). Tensoring this diagram with $\mathcal{L}$, using the flatness of $\mathcal{L}$, and using the projection formula gives us the following morphism of triangles.

$$\begin{tikzcd}
\pi_*(\omega_Y \otimes \pi^*\mathcal{L})[d] \arrow[r] \arrow[d, "a_{\mathcal{L}}"] & \pi_*(\omega_Y^\bullet \otimes \pi^*\mathcal{L}) \arrow[r] \arrow[d, "b_{\mathcal{L}}"] \arrow[dl, dotted, "s_{\mathcal{L}}"'] & \tau_{>-d}\pi_*(\omega_Y^\bullet \otimes \pi^*\mathcal{L}) \arrow[r] \arrow[d, "c_{\mathcal{L}}=0"] & \pi_*(\omega_Y \otimes \pi^*\mathcal{L})[d+1] \arrow[d] \\
\omega_X \otimes \mathcal{L}[d] \arrow[r] & \omega_X^\bullet \otimes \mathcal{L} \arrow[r] & \tau_{>-d}\omega_X^\bullet \otimes \mathcal{L} \arrow[r] & \omega_X \otimes \mathcal{L}[d+1]
\end{tikzcd}$$

The commutativity of the above diagram and existence of $s_{\mathcal{L}}$ shows that for any integer i, the image of the natural trace map

$$H^{-i}(b_{\mathcal{L}}) : H^{-i}(Y, \omega_Y^\bullet \otimes \pi^*\mathcal{L}) \to H^{-i}(X, \omega_X^\bullet \otimes \mathcal{L})$$

lies in the image of the natural map

$$H^{d-i}(X, \omega_X \otimes \mathcal{L}) = H^{-i}(X, \omega_X \otimes \mathcal{L}[d]) \to H^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}).$$

Now choose i such that $0 < i < d$. By assumption, the source of the preceding map is then trivial. Hence, we find that the map $H^{-i}(b_{\mathcal{L}})$ is also 0. Dualising, it follows that

$$\pi^* : H^i(X, \mathcal{L}^{-1}) \to H^i(Y, \pi^*\mathcal{L}^{-1})$$

is trivial, as desired. $\square$

Remark 7.4. Consider the special case of Proposition 7.3 when $\mathcal{L}$ is ample. We treated this case directly in the second half of the proof above using Proposition 6.2. It is possible to replace this part of the proof by a reference to [HH92, Theorem 1.2], the main geometric theorem of that paper. We have not adopted this approach as we feel that the proof given above using Proposition 6.2 is cleaner than the algebraic approach of [HH92], which involves developing a theory of graded integral closures (see [HH92, § 4]) and reducing to a local algebra theorem.

Remark 7.5. Proposition 7.3 can be viewed as a version of Kodaira vanishing in characteristic p up to finite covers. A more natural proof would involve using the results of Deligne and Illusie [DI87]. To implement such a proof, one needs to know that any variety over a field k of characteristic p admits a proper surjective map from one that lifts to $\mathrm{W}_2(k)$. Unfortunately, we do not know whether this is true.

Remark 7.6. The results proven in this subsection were used in [BST11] to obtain a unified description of test and multiplier ideals via alterations.

Counterexamples

This subsection is dedicated to providing the examples promised earlier. The most important examples here are Examples 7.12 and 7.13. The former shows that the middle cohomology of the inverse of a semiample line bundle on a smooth projective variety cannot always be killed by finite covers, while the latter shows that the middle cohomology of a nef and big line bundle on a smooth projective variety cannot be killed by finite covers; the latter answers negatively a question asked in [Smi97a].

We start off by constructing a degree 0 line bundle on a *singular* stable curve whose cohomology cannot be killed by Frobenius; we will later use this to construct the smooth examples promised above.

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Example 7.7. Let k be a perfect field of characteristic p. Let E be an elliptic curve over k with identity $z \in E(k)$, and let $\mathcal{L}$ be a degree 0 line bundle on E that is not p-power torsion (assumed to exist). Let $C = E \sqcup_z E$ be the stable genus 2 curve obtained by glueing E to itself at z, and let $\mathcal{M} \in \mathrm{Pic}(C)$ be the line bundle obtained by glueing $\mathcal{L}$ over each copy of E to itself using the identity isomorphism $\mathcal{L}|_z \simeq \mathcal{L}|_z$. Then we claim that $H^1(C, \mathcal{M})$ is a one-dimensional k-vector space, and that $\mathrm{Frob}_C^e : C \to C$ induces an isomorphism $\mathrm{Frob}_C^{e,*} : H^1(C, \mathcal{M}) \to H^1(C, \mathcal{M}^{p^e})$ for each $e > 0$. To verify these claims, note that we have an exact triangle

$$\mathrm{R}\,\Gamma(C, \mathcal{M}) \to \mathrm{R}\,\Gamma(E, \mathcal{L}) \oplus \mathrm{R}\,\Gamma(E, \mathcal{L}) \to \mathrm{R}\,\Gamma(z, \mathcal{L}|_z)$$

where, with an abuse of notation, z denotes the reduced subscheme structure on the point z. Since $\mathcal{L}$ is chosen to be not p-power torsion, we have $\mathrm{R}\,\Gamma(E, \mathcal{L}) \simeq \mathrm{R}\,\Gamma(E, \mathcal{L}^{p^e}) = 0$ for each $e > 0$. Hence, the triangle above (and a similar one for the Frobenius pullback) degenerates to give the following diagram.

$$\begin{tikzcd} H^0(z, \mathcal{L}|_z) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Frob}_z^{e,*}"] & H^1(C, \mathcal{M}) \arrow[d, "\mathrm{Frob}_C^{e,*}"] \\ H^0(z, \mathcal{L}^{p^e}|_z) \arrow[r, "\simeq"] & H^1(C, \mathcal{M}^{p^e}) \end{tikzcd}$$

The left vertical map is an isomorphism (it is $s \mapsto s^{\otimes p^e}$ on sections), and thus so is the right vertical one. Note that this construction can be adapted to work for arbitrary genera by glueing in more copies of E.

We need a lemma on the existence of certain curves and line bundles in positive characteristic.

Lemma 7.8. *Fix a prime p and a number $g \geqslant 2$. There exists a field k of characteristic p, a smooth projective geometrically connected curve C/k of genus g, and a degree* 0 *line bundle $\mathcal{M}$ on C such that:*

(i) *for each integer $e > 0$, the Frobenius map $\mathrm{Frob}_C^e : C \to C$ induces an injective map $\mathrm{Frob}_C^{e,*} : H^1(C, \mathcal{M}) \to H^1(C, \mathcal{M}^{p^e})$;*

(ii) *for each integer $e > 0$, the line bundle $\mathcal{M}^{-p^e}$ does not occur as a subsheaf of any quotient of $h_* \mathcal{O}_{C'}$ for any finite étale cover $h : C' \to C$.*

Proof. Our proof is indirect; we use the existence of a smooth moduli space of stable curves (see [DM69]) coupled with Example 7.7. Let $\overline{\mathcal{M}_g}(B\mathbb{G}_m)$ denote the stack over $\mathbb{F}_p$ parametrising pairs $(C, \mathcal{M})$ where C is a *stable* genus g curve and $\mathcal{M}$ is a degree 0 line bundle on C. This stack is smooth (of dimension $4g - 4$) by deformation theory: stable curves have unobstructed deformations by [Ill05, Corollary 5.32], and line bundles on any proper curve are unobstructed as $H^2(C, \mathcal{O}_C) = 0$. Let $\pi : \mathcal{X} \to \overline{\mathcal{M}_g}(B\mathbb{G}_m)$ be a smooth cover by a smooth $\mathbb{F}_p$-scheme. We will show that each of the conditions required in the lemma is (separately) satisfied by the image in $\overline{\mathcal{M}_g}(B\mathbb{G}_m)$ of a very general point of $\mathcal{X}$; this is enough to prove the lemma, as a very general subset of $\mathcal{X}$ contains points that map to $(C, \mathcal{M})$ with C smooth.

For the first condition, let $(\pi : \mathcal{C} \to \mathcal{X}, \mathcal{N})$ be the data representing π, and consider the usual diagram below.

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "\mathrm{Frob}_\pi"] \arrow[drr, bend left, "\mathrm{Frob}_{\mathcal{C}}"] \arrow[ddr, bend right, "\pi"'] & & \\
& \mathcal{C}^{(1)} \arrow[r, "\mathrm{Frob}_{\mathcal{X}}"] \arrow[d, "\pi^{(1)}"] & \mathcal{C} \arrow[d, "\pi"] \\
& \mathcal{X} \arrow[r, "\mathrm{Frob}_{\mathcal{X}}"] & \mathcal{X}
\end{tikzcd}$$

The relative Frobenius map Frob_π induces a morphism $a : \mathrm{R}^1\, \pi^{(1)}_* \,\mathrm{Frob}^*_{\mathcal{X}}\, \mathcal{N} \to \mathrm{R}^1\, \pi_*(\mathcal{N})^{\otimes p}$. Let $U \subset \mathcal{X}$ be the open subscheme where the formation of $\mathrm{R}^i\, \pi_*\mathcal{N}$ and $\mathrm{R}^i\, \pi_*(\mathcal{N}^{\otimes p})$ commutes with base change. The fibre of a over a point $x \in U(K)$ mapping to $\pi(x) = [(C, \mathcal{M})] \in \overline{\mathcal{M}_g}(K)$ (for any ring K) is given by the Frobenius pullback map $\mathrm{Frob}^*_K\, H^i(C, \mathcal{M}) \to H^i(C, \mathcal{M}^p)$. By surjectivity of π, there is a point $x_0 \in U(k_0)$ such that $\pi(x_0) \in \overline{\mathcal{M}_g}(X)$ corresponds to a pair $(C_0, \mathcal{M}_0)$ constructed as in Example 7.7. In particular, U is non-empty. Moreover, since a_{x_0} is injective (by Example 7.7), semicontinuity ensures that $\ker(a)|_V = 0$ for some non-empty open $V \subset U$. Doing the same for higher Frobenius twists and then intersecting, we see that images under π of very general points on $\mathcal{X}$ satisfy the first condition above.

For the second condition, note that for each finite étale cover $h : C' \to C$, the bundle $h_*\mathcal{O}_{C'}$ is a semistable degree 0 vector bundle: this can be checked after finite étale base change on C, but then it is clear as C' splits completely over some finite étale cover. As the category of semistable vector bundles of degree 0 is an artinian and noetherian k-linear abelian category, only finitely many degree 0 line bundles occur as subsheaves of quotients of $h_*\mathcal{O}_{C'}$ in a given finite étale cover $h : C' \to C$. There are only countably many possibilities for such h by the finite generation of $\pi_{1,\text{ét}}(C)$, so there are only countably many possibilities for degree 0 line bundles that occur as subsheaves of quotients of $h_*\mathcal{O}_{C'}$ as h varies over all finite étale covers $h : C' \to C$. Thus, for any smooth projective curve C, a very general degree 0 line bundle $\mathcal{M}$ will satisfy the second condition. In particular, a very general point of $\mathcal{X}$ will satisfy the second condition. □

We now arrive at the key example of this section.

Example 7.9. We will show that for a pair $(C, \mathcal{M})$ satisfying the conditions of Lemma 7.8, the group $H^1(C, \mathcal{M})$ cannot be killed by finite covers; the proof uses the Harder–Narasimhan filtration for vector bundles on a curve (see [Laz04b, § 6.4.A]).

Assume, towards a contradiction, that there exists a finite surjective map $f : C' \to C$ such that $f^* : H^1(C, \mathcal{M}) \to H^1(C', f^*\mathcal{M})$ has a non-zero kernel. By replacing C' with a cover if necessary, we may assume that C' is normal and that the extension of function fields induced by f is normal. By taking invariants at the level of function fields, we can factor f as $C' \xrightarrow{a} C \xrightarrow{b} C$ with a generically étale and b a power of Frobenius (see [Har77, Proposition IV.2.5]). Our assumptions on $\mathcal{M}$ imply that $b^* : H^1(C, \mathcal{M}) \to H^1(C, b^*\mathcal{M})$ is injective, and hence $a^* : H^1(C, b^*\mathcal{M}) \to H^1(C', f^*\mathcal{M})$ must have a kernel. Now consider the exact sequence

$$0 \to \mathcal{O}_C \to a_*\mathcal{O}_{C'} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined to be the quotient. Tensoring with $b^*\mathcal{M}$ and taking cohomology, we see that if $H^1(C, b^*\mathcal{M}) \to H^1(C', f^*\mathcal{M})$ has a kernel, then $H^0(b^*\mathcal{M} \otimes \mathcal{Q}) \neq 0$ or, equivalently,

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that $\mathcal{M}^{-p^e} = b^*\mathcal{M}^{-1}$ occurs as a subsheaf of $\mathcal{Q}$. We will show that this contradicts the second property of Lemma 7.8.

Since a is generically étale, a theorem of Lazarsfeld [PS00, Appendix, Proposition A] implies that $\mathcal{Q}^\vee$ is a nef vector bundle; the result in [PS00] is stated in characteristic 0, but this assumption is only used to ensure generic separability. The bundle $a_*\mathcal{O}_{C'}$ therefore has non-positive degree (as it is an extension of the antinef vector bundle $\mathcal{Q}$ by $\mathcal{O}_C$). This implies that the maximal slope occurring in the Harder–Narasimhan filtration for $a_*\mathcal{O}_{C'}$ is 0. Since $\mathcal{O}_C$ is a subbundle of $a_*\mathcal{O}_{C'}$ of maximal degree, we have an induced exact sequence of semistable degree 0 vector bundles

$$0 \to \mathcal{O}_C \to \mathrm{Fil}^0(a_*\mathcal{O}_{C'}) \to \mathrm{Fil}^0(\mathcal{Q}) \to 0$$

where $\mathrm{Fil}^0(\mathcal{E})$ denotes the piece of the Harder–Narasimhan filtration with slope greater than or equal to 0. We showed above that $\mathcal{M}^{-p^e}$ occurs as a subsheaf of $\mathcal{Q}$. Since $\mathcal{M}^{-p^e}$ has slope 0, it also occurs as a subsheaf of $\mathrm{Fil}^0(\mathcal{Q})$. Hence, $\mathcal{M}^{-p^e}$ occurs as a subsheaf of a quotient of $\mathrm{Fil}^0(a_*\mathcal{O}_{C'})$. On the other hand, the algebra structure on $a_*\mathcal{O}_{C'}$ descends to give an algebra structure on $\mathrm{Fil}^0(a_*\mathcal{O}_{C'})$, so the latter corresponds to the structure sheaf of some finite cover $h : D \to C''$ factoring the map a. Since a is generically étale, the same is true of h. By construction, we also have $\deg(h_*\mathcal{O}_D) = 0$. The Reimann–Hurwitz and Riemann–Roch theorems then show that h is finite étale. Thus, the line bundle $\mathcal{M}^{-p^e}$ occurs as a subsheaf of a quotient of $h_*\mathcal{O}_D$ for $h : D \to C$ finite étale, which contradicts the assumptions on the pair $(C, \mathcal{M})$, finishing the proof.

Remark 7.10. Example 7.9 requires one to work with very general line bundles and hence does not answer the following question: does the conclusion of Proposition 7.2 hold for nef line bundles if the base field is $\overline{\mathbb{F}_p}$? The strategy of Example 7.9 cannot work: any degree 0 line bundle $\mathcal{M}$ on a curve C over $\overline{\mathbb{F}_p}$ is torsion since $\mathrm{Pic}^0(C)$ is torsion, and hence $H^1(C, \mathcal{M})$ can be killed by Proposition 7.2. This question has now been answered negatively by Adrian Langer in [Lan11] using the examples in [Tot09].

Using Example 7.9, we can easily produce an example of a nef line bundle $\mathcal{L}$ on a surface X whose middle cohomology cannot be killed by passage to finite covers. In fact, the bundle constructed has degree 0 and can thus be viewed as the inverse of a nef bundle as well; this dual perspective negatively answers Question 7.1 for the case of positive or negative twists when 'weakly positive' is taken to mean nef.

Example 7.11. Let $(C, \mathcal{M})$ be as in Example 7.9. Then $\mathcal{L} = \mathcal{M} \boxtimes \mathcal{O}_C = \mathrm{pr}_1^* \mathcal{M}$ is a nef line bundle on $X = C \times C$ with $\mathrm{pr}_1^* : H^1(C, \mathcal{M}) \xrightarrow{\simeq} H^1(X, \mathrm{pr}_1^* \mathcal{M}) = H^1(X, \mathcal{L})$. We claim that there does not exist a finite surjective morphism $\pi : Y \to X$ inducing the 0 map $\pi^* : H^1(X, \mathcal{L}) \to H^1(Y, \pi^*\mathcal{L})$. If π were such a map, then choosing a multisection of $\mathrm{pr}_1 \circ \pi$ and normalising it gives a finite flat morphism $f : C' \to C$ inducing the 0 map on $H^1(C, \mathcal{M})$. However, as shown in Example 7.9, this cannot happen.

Our next example is of a semiample line bundle $\mathcal{L}$ on a surface X such that the middle cohomology of $\mathcal{L}^{-1}$ cannot be killed by finite covers. Thus, it negatively answers Question 7.1 in the case of negative twists when 'weakly positive' is taken to mean even semiample, not just nef.

Example 7.12. Consider the bundle $\mathcal{L} = \mathcal{O}(2) \boxtimes \mathcal{O} = \mathrm{pr}_1^* \mathcal{O}(2)$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$ over a field k. This is a semiample bundle with $H^1(X, \mathcal{L}^{-1}) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. We claim

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that there is no finite surjective morphism $g: Y \to X$ inducing the 0 map $H^1(X, \mathcal{L}^{-1}) \to H^1(Y, \pi^*\mathcal{L}^{-1})$. If there were such a map g, then $\mathrm{pr}_1 \circ g : Y \to \mathbb{P}^1$ is an alteration inducing the 0 map on $H^1(\mathbb{P}^1, \mathcal{O}(-2))$. Choosing a multisection of $\mathrm{pr}_1 \circ g$ and normalising it gives a finite flat morphism $f : C \to \mathbb{P}^1$ inducing the 0 map on $H^1(\mathbb{P}^1, \mathcal{O}(-2))$. However, this cannot happen: the morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2) \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}_0 \oplus \mathcal{O}_\infty \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*f^*\mathcal{O}(-2) \arrow[r] & f_*\mathcal{O} \arrow[r] & f_*\mathcal{O}_{f^{-1}(0)} \oplus f_*\mathcal{O}_{f^{-1}(\infty)} \arrow[r] & 0
\end{tikzcd}$$

gives us the following morphism of exact sequences.

$$\begin{tikzcd}
k = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \arrow[r, hook] \arrow[d, "\simeq"] & H^0(\mathbb{P}^1, \mathcal{O}_0 \oplus \mathcal{O}_\infty) \arrow[r, two heads, "a"] \arrow[d, "b"] & H^1(\mathbb{P}^1, \mathcal{O}(-2)) = k \arrow[d, "d"] \\
k = H^0(C, \mathcal{O}_C) \arrow[r, hook, "c"] & H^0(C, \mathcal{O}_{f^{-1}(0)} \oplus \mathcal{O}_{f^{-1}(\infty)}) \arrow[r] & H^1(C, f^*\mathcal{O}(-2))
\end{tikzcd}$$

The surjectivity of a gives $\dim(H^0(\mathbb{P}^1, \mathcal{O}_0 \oplus \mathcal{O}_\infty)) = 2$, while the injectivity of b ensures that $\dim(\mathrm{im}(b)) = 2$. As $\dim(\mathrm{im}(c)) = 1$, it follows that $\mathrm{im}(b)$ strictly contains $\mathrm{im}(c)$; therefore $\dim(\mathrm{im}(d)) = 1$, which is what we wanted.

Finally, we conclude by giving an example showing that the conclusion of Proposition 7.2 fails for nef and big line bundles.

Example 7.13. Let $(C, \mathcal{M})$ be as in Example 7.9, and let $\mathcal{L}$ be an ample line bundle on C. Let $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$, let $X = \mathbb{P}(\mathcal{E})$, and let $\pi : X \to C$ be the natural projection. With $\mathcal{O}_\pi(1)$ denoting the Serre line bundle on X, we will show that:

- the line bundle $\mathcal{O}_\pi(1)$ is nef;
- the line bundle $\mathcal{O}_\pi(1)$ is big;
- the group $H^1(X, \mathcal{O}_\pi(1))$ is non-zero and cannot be annihilated by finite covers of X.

We will first verify that $\mathcal{O}_\pi(1)$ is nef. Using the Barton–Kleiman criterion (see [Laz04b, Proposition 6.1.18]), it suffices to show that for any quotient $\mathcal{E} \twoheadrightarrow \mathcal{N}$ with $\mathcal{N}$ invertible, we must have $\deg(\mathcal{N}) \geqslant 0$. This claim follows from the formula

$$\mathrm{Hom}(\mathcal{E}, \mathcal{N}) = \mathrm{Hom}(\mathcal{L}, \mathcal{N}) \oplus \mathrm{Hom}(\mathcal{M}, \mathcal{N})$$

and the fact that neither $\mathcal{L}$ nor $\mathcal{M}$ admits a map to a line bundle with negative degree.

We now verify bigness of $\mathcal{O}_\pi(1)$. By definition, this amounts to showing that $h^0(X, \mathcal{O}_\pi(n))$ grows quadratically in n (we follow the usual convention that $h^0(X, \mathcal{F}) = \dim(H^0(X, \mathcal{F}))$ for a coherent sheaf $\mathcal{F}$ on X). Standard calculations about projective space bundles show that

$$\pi_*\mathcal{O}_\pi(n) \simeq \mathrm{R}\,\pi_*\mathcal{O}_\pi(n) \simeq \mathrm{Sym}^n(\mathcal{E})$$

for $n > 0$. The Leray spectral sequence for π then gives us that

$$H^0(X, \mathcal{O}_\pi(n)) = H^0(C, \mathrm{Sym}^n(E)) = H^0\Big(C, \bigoplus_{i+j=n} \mathcal{L}^i \otimes \mathcal{M}^j\Big).$$

Since $\mathcal{L}$ is ample and $\mathcal{M}$ has degree 0, the Riemann–Roch estimate tells us that $H^0(C, \mathcal{L}^i \otimes \mathcal{M}^j)$ grows like i (for big enough i). Hence we find

$$h^0(X, \mathcal{O}_\pi(n)) = \sum_{i+j=n} h^0(C, \mathcal{L}^i \otimes \mathcal{M}^j) \sim 1 + 2 + \cdots + n = \frac{n(n-1)}{2},$$

thereby verifying the bigness of $\mathcal{O}(1)$.

It remains to check the cohomology-annihilation claim. The Leray spectral sequence shows that

$$H^1(X, \mathcal{O}_\pi(1)) = H^1(C, \mathcal{E}) = H^1(C, \mathcal{L}) \oplus H^1(C, \mathcal{M}).$$

In particular, this group is non-zero since the second factor is non-zero. Moreover, the natural projection $\mathcal{E} \twoheadrightarrow \mathcal{M}$ defines a section $s : C \to X$ of π such that $s^*\mathcal{O}_\pi(1) \simeq \mathcal{M}$. Hence, we find that s^* induces a map $H^1(X, \mathcal{O}_\pi(1)) \to H^1(C, \mathcal{M})$ which is simply the projection on the second factor under the preceding isomorphism. In particular, if there were a finite cover $\pi : Y \to X$ such that $\pi^*(H^1(X, \mathcal{O}_\pi(1)) = 0$, then upon restricting Y to $s : C \to X$ we would obtain a finite cover of C annihilating $H^1(C, \mathcal{M})$, contradicting what we proved in Example 7.9.

Question 7.14. The examples given above do not answer the following question: given a smooth projective variety X and a nef and big line bundle $\mathcal{L}$, can the group $H^i(X, \mathcal{L}^{-1})$ be killed by finite covers of X for $0 < i < \dim(X)$? If $\mathcal{L}$ is assumed to be semiample, then Proposition 7.3 provides a positive answer. If the bigness condition is dropped, then Example 7.11 provides a negative answer. A positive answer to this question would give an 'up to finite covers' analogue of Kawamata–Viehweg vanishing, and would be quite useful for applications.

8. Application: some more global examples of D-splinters

The goal of this section is to show that the complete flag variety for an algebraic group group G is a D-splinter. When $G = \mathrm{GL}_n$, we give a direct proof in § 8.1 relying on the results of § 5; this provides the first non-toric example of a projective variety that is a D-splinter in this paper. For general G, we give a proof in § 8.2 using the results of [SS10, LRT06]; the latter was suggested to us by the referee.

8.1 The general linear group

Our goal is to show that the variety of complete flags in a vector space is a splinter. We begin by recording an elementary criterion to test when a finite morphism is 'split'.

Lemma 8.1. *Let X be a Gorenstein projective scheme of equidimension n over a field k, and let $\pi : Y \to X$ be a proper morphism. Then the existence of a section of $\mathcal{O}_X \to \mathrm{R}\,\pi_*\mathcal{O}_Y$ is equivalent to the injectivity of $H^n(X, \omega_X) \to H^n(Y, \pi^*\omega_X)$.*

Proof. By the projection formula and the flatness of ω_X, we have $H^n(Y, \pi^*\omega_X) = H^n(X, \omega_X \otimes \mathrm{R}\,\pi_*\mathcal{O}_Y)$. Thus, the injectivity of $H^n(X, \omega_X) \to H^n(Y, \pi^*\omega_X)$ is equivalent to the injectivity of

$$H^n(X, \omega_X) \to H^n(X, \omega_X \otimes \mathrm{R}\,\pi_*\mathcal{O}_Y).$$

This map is the map on H^n induced by the natural map $\omega_X \to \omega_X \otimes \mathrm{R}\,\pi_*\mathcal{O}_Y$. Serre duality tells us that this injectivity is equivalent to the surjectivity of

$$\mathrm{Hom}(\mathrm{R}\,\pi_*\mathcal{O}_Y \otimes \omega_X, \omega_X) \to \mathrm{Hom}(\omega_X, \omega_X).$$

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Since ω_X is invertible, the preceding surjectivity is equivalent to the surjectivity of

$$(\pi^*)^\vee = \mathrm{ev}_1 : \mathrm{Hom}(\mathrm{R}\,\pi_*\mathcal{O}_Y, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$$

induced by the natural map $\mathcal{O}_X \to \mathrm{R}\,\pi_*\mathcal{O}_Y$. On the other hand, the surjectivity of this map is also clearly equivalent to $\mathcal{O}_X \to \mathrm{R}\,\pi_*\mathcal{O}_Y$ admitting a section; the claim follows. □

Next, we discuss a criterion that allows us to propagate the splinter condition from a subvariety to the entire variety. The criterion is formulated in terms of the existence of nice resolutions of dualising sheaves.

Proposition 8.2. *Let X be a Gorenstein projective variety of equidimension n over a field k of positive characteristic p. Let $i : Z \hookrightarrow X$ be a closed equidimensional subvariety that is itself Gorenstein, and let c be the codimension $\dim(X) - \dim(Z)$. Assume that there exists a resolution of ω_Z of the form*

$$[\omega_X = \mathcal{E}_c \to \mathcal{E}_{c-1} \to \cdots \to \mathcal{E}_0] \simeq \omega_Z$$

where, for each $0 \leqslant i < c$, the sheaf $\mathcal{E}_i$ is an iterated extension of inverses of semiample and big line bundles. If Z is a splinter, so is X.

The assumption on $\mathcal{E}_i$ (for $0 \leqslant i < c$) means that $\mathcal{E}_i$ admits a finite filtration with associated graded pieces given by inverses of semiample and big line bundles $\mathcal{L}_{i,j}^{-1}$. This implies that $H^k(X, \mathcal{E}_i)$ admits a finite filtration with associated graded pieces contained in $H^k(\mathcal{L}_{i,j})$ for any k.

Proof. Let $\alpha \in H^{n-c}(X, \omega_Z) \simeq H^{n-c}(Z, \omega_Z)$ be a generator (under Serre duality). Let $f : Y \to X$ be an alteration. By the assumption on Z, we know that $f^*(\alpha)$ is not zero in $H^{n-c}(Y, f^*\omega_Z)$. The pullback $\mathrm{L}f^*\alpha \in H^{n-c}(Y, \mathrm{L}f^*\omega_Z)$ also has to be non-zero by the natural map $\mathrm{L}f^*\omega_Z \to f^*\omega_Z$. Note that this holds for *any* alteration $f : Y \to X$; this observation will be applied later in the proof to a different map.

Pulling back the given resolution for ω_Z to Y, we obtain a resolution

$$[f^*\omega_X = f^*\mathcal{E}_c \to f^*\mathcal{E}_{c-1} \to \cdots \to f^*\mathcal{E}_0] \simeq \mathrm{L}f^*\omega_Z.$$

The hypercohomology spectral sequence associated to the stupid filtration of this complex takes the form

$$E_p^{1,q}(Y \to X) : H^q(Y, f^*\mathcal{E}_p) \Rightarrow H^{q-p}(Y, \mathrm{L}f^*\omega_Z).$$

We will trace the behaviour of the class $\mathrm{L}f^*\alpha \in H^{n-c}(Y, \mathrm{L}f^*\omega_Z)$ through the spectral sequence. The terms contributing to this group in the spectral sequence are $H^q(Y, f^*\mathcal{E}_p)$ with $q - p = n - c$. Since $\dim(Y) = n$, the contributing terms $H^q(Y, f^*\mathcal{E}_p)$ have $q < n$ whenever $p < c$. We will first show, by applying Proposition 7.3, that these numerics imply that $\mathrm{L}f^*\alpha$ has to be non-zero in $H^n(Y, f^*\mathcal{E}_0)$, and then we will explain why this is enough to prove the claim.

Since the bundles $\mathcal{E}_i$ are assumed to be iterated extensions of inverses of semiample and big line bundles for $i < c$, the same is true for the pullbacks $f^*\mathcal{E}_i$. Proposition 7.3 then produces a finite surjective morphism $g : Y' \to Y$ such that $H^j(Y, f^*\mathcal{E}_i) \to H^j(Y', g^*f^*\mathcal{E}_i)$ is 0 for $j < n$ (and $i < c$ still). Since we know that $\mathrm{L}(f \circ g)^*\alpha$ is non-zero by the earlier argument, the image of $\mathrm{L}f^*\alpha$ also has to be non-zero in $H^n(Y, f^*\mathcal{E}_0) = H^n(Y, f^*\omega_X)$ under the natural coboundary map $H^{n-c}(Y, \mathrm{L}f^*\omega_Z) \to H^n(Y, f^*\omega_X)$.

Now note that we also have a analogous spectral sequence

$$E_p^{1,q}(X \to X) : H^q(X, \mathcal{E}_p) \Rightarrow H^{q-p}(X, \omega_Z)$$

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and a morphism of spectral sequences $E_p^{1,q}(X \to X) \to E_p^{1,q}(Y \to X)$ by pulling back classes. This gives rise to the following commutative square.

$$\begin{tikzcd} k \simeq H^{n-c}(X, \omega_Z) \arrow[r, "\delta_X"] \arrow[d, "a"] & H^n(X, \omega_X) \simeq k \arrow[d, "b"] \\ H^{n-c}(Y, \mathrm{L}f^*\omega_Z) \arrow[r, "\delta_Y"] & H^n(Y, f^*\omega_X) \end{tikzcd}$$

We have just verified that $\delta_Y \circ a$ is non-zero and, hence, injective. By a diagram chase, δ_X is injective and, hence, bijective. Another chase then implies that b is injective. By Proposition 8.1, we are done. $\square$

Remark 8.3. Consider the special case of Proposition 8.2 where all the line bundles occurring in the $\mathcal{E}_i$ are antiample. Since X is Gorenstein, one may be tempted to say that the given proof of Proposition 8.2 goes through without using Proposition 7.3 as we can simply use Frobenius to kill cohomology after dualising. However, this is false: we applied Proposition 7.3 to finite covers $Y \to X$ rather than to X itself, and there is no reason we can suppose that Y is Gorenstein. If we alter Y to a Gorenstein (or even regular) scheme, then we lose ampleness and are once again in a position where we need to use Proposition 7.3.

Remark 8.4. The assumptions in Proposition 8.2 are extremely strong. Consider the case where $Z \hookrightarrow X$ is a divisor. The natural resolution (and, in fact, the only possible one) to consider is

$$[\omega_X \to \omega_X(Z)] \simeq \omega_Z.$$

The assumptions of Proposition 8.2 will be satisfied precisely when $\omega_X^{-1}(-Z)$ is semiample and big. This implies that ω_X^{-1} is also big. In particular, X is birationally Fano.

Proposition 8.2 looks slightly bizarre at first glance. However, it is a useful argument in inductive proofs. Here is a typical application.

Proposition 8.5. *Let V be a vector space of dimension d over a field k, and let* $\mathrm{Flag}(V)$ *be the moduli space of complete flags* $(0 = F_0 \subset F_1 \subset \cdots F_{d-1} \subset F_d = V)$ *in V. Then* $\mathrm{Flag}(V)$ *is a D-splinter.*

Proof. We work by induction on the dimension d. The case $d = 0$ being trivial, we may assume that $\mathrm{Flag}(W)$ is a splinter for any vector space W of dimension at most $d - 1$. If we let $\mathbb{P}(V)$ denote the projective space of *hyperplanes* in V, then there is a natural morphism $\pi : \mathrm{Flag}(V) \to \mathbb{P}(V)$ given by sending a complete flag $(0 = F_0 \subset F_1 \subset \cdots F_{d-1} \subset F_d = V)$ to the hyperplane $(F_{d-1} \subset V)$. The morphism π can easily be checked to be projective and smooth. Let $W \subset V$ be a fixed hyperplane, and let $b \in \mathbb{P}(V)(k)$ be the corresponding point. The fibre $\pi^{-1}(b)$ is identified with $\mathrm{Flag}(W)$. We will apply Proposition 8.2 with $Z = \mathrm{Flag}(W)$ and $X = \mathrm{Flag}(V)$ to get the desired result.

The structure sheaf $\kappa(b)$ of the point $b : \mathrm{Spec}(k) \hookrightarrow \mathbb{P}(V)$ can be realised as the zero locus of a section of $\mathcal{O}(1)^{\oplus(d-1)}$ by thinking of b as the intersection of $(d-1)$ hyperplanes in general position. This gives us a Koszul resolution

$$[\mathcal{O}(-(d-1)) \simeq \wedge^{d-1}(\mathcal{O}(-1)^{\oplus(d-1)}) \to \cdots \to \mathcal{O}(-1)^{\oplus(d-1)} \to \mathcal{O}] \simeq \kappa(b).$$

Twisting by $\mathcal{O}(-1)$, we find a resolution

$$[\omega_{\mathbb{P}(V)} \to \mathcal{M}_{d-2} \to \cdots \to \mathcal{M}_1 \to \mathcal{M}_0] \simeq \kappa(b)$$

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with each $\mathcal{M}_i$ being a direct sum of inverses of ample line bundles with degrees between 1 and $d-2$. Pulling this data back along π, we find a resolution

$$[\pi^*\omega_{\mathbb{P}(V)} \to \pi^*\mathcal{M}_{d-2} \to \cdots \to \pi^*\mathcal{M}_1 \to \pi^*\mathcal{M}_0] \simeq \pi^*\kappa(b) = \mathcal{O}_Z.$$

Twisting by the relative dualising sheaf ω_π, we find

$$[\omega_\pi \otimes \pi^*\omega_{\mathbb{P}(V)} \to \omega_\pi \otimes \pi^*\mathcal{M}_{d-2} \to \cdots \to \omega_\pi \otimes \pi^*\mathcal{M}_1 \to \omega_\pi \otimes \pi^*\mathcal{M}_0] \simeq \omega_\pi|_Z.$$

Since π is smooth, we make the identifications $\omega_X \simeq \omega_\pi \otimes \pi^*\omega_{\mathbb{P}(V)}$ and $\omega_Z \simeq \omega_\pi|_Z$. Thus, we obtain a resolution

$$[\omega_X \to \omega_\pi \otimes \pi^*\mathcal{M}_{d-2} \to \cdots \to \omega_\pi \otimes \pi^*\mathcal{M}_1 \to \omega_\pi \otimes \pi^*\mathcal{M}_0] \simeq \omega_Z$$

with $\mathcal{M}_i$ as above. Standard calculations with flag varieties (see Lemma 8.6) now show that the terms $\omega_\pi \otimes \pi^*\mathcal{M}_i$ are direct sums of inverses of semiample and big line bundles. In particular, this resolution has the form required in Propositon 8.2. Hence, we win by induction. $\square$

We needed to calculate the positivity of certain natural line bundles on the flag variety in Proposition 8.5. Since we were unable to find a satisfactory reference, we carry out the calculation here.

Lemma 8.6. *Let V be an n-dimensional vector space over a field k, and let $\pi : \mathrm{Flag}(V) \to \mathbb{P}(V)$ be the natural morphism. For all $i > 0$ and all n, the line bundles $\omega_\pi \otimes \pi^*\mathcal{O}(-i)$ are inverses of semiample and big line bundles.*

Proof. For $n = 2$, the map π is an isomorphism, and the claim is obvious. Assume $n \geqslant 3$. Let

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = V \otimes \mathcal{O}_{\mathrm{Flag}(V)}$$

be the universal flag on $\mathrm{Flag}(V)$ with $\dim(V_i) = i$. For each $i \geqslant 1$, let $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ be the associated line bundle. The tangent bundle of $\mathrm{Flag}(V)$ admits a filtration whose pieces are of the form

$$\mathcal{H}\mathrm{om}(\mathcal{V}_i, \mathcal{L}_{i+1}) \simeq \mathcal{V}_i^\vee \otimes \mathcal{L}_{i+1}$$

for $1 \leqslant i \leqslant n-1$. This filtration gives us the formula

$$\omega_{\mathrm{Flag}(V)}^{-1} \simeq \bigotimes_{i=1}^{n-1} (\det(\mathcal{V}_i)^{-1} \otimes \det(\mathcal{L}_{i+1})^i).$$

Since each $\mathcal{V}_i$ is filtered with pieces of the form $\mathcal{L}_j$ for $1 \leqslant j \leqslant i$, we find that

$$\omega_{\mathrm{Flag}(V)}^{-1} \simeq \bigotimes_{i=1}^{n-1} (\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \otimes \cdots \otimes \mathcal{L}_i^{-1} \otimes \mathcal{L}_{i+1}^i).$$

Collecting terms, we find that

$$\omega_{\mathrm{Flag}(V)}^{-1} \simeq \left(\bigotimes_{i=1}^{n-1} \mathcal{L}_i^{2i-n}\right) \otimes \mathcal{L}_n^{n-1}.$$

The inverse $\mathcal{M}_j$ of the line bundle $\omega_\pi \otimes \pi^*\mathcal{O}(-j)$ can be written as

$$\mathcal{M}_j \simeq \omega_\pi^{-1} \otimes \pi^*\mathcal{O}(j) \simeq \omega_{\mathrm{Flag}(V)}^{-1} \otimes \pi^*(\omega_{\mathbb{P}(V)} \otimes \mathcal{O}(j)).$$

Using the formula for $\omega^{-1}_{\mathrm{Flag}(V)}$ that we arrived at earlier and the fact that π is defined by the tautological quotient $\mathcal{O}_{\mathrm{Flag}(V)} \otimes V \twoheadrightarrow \mathcal{L}_n$, we can simplify the preceding formula to get

$$\mathcal{M}_j \simeq \left(\bigotimes_{i=1}^{n-1} \mathcal{L}_i^{2i-n}\right) \otimes \mathcal{L}_n^{n-1} \otimes \mathcal{L}_n^{-n+j} \simeq \left(\bigotimes_{i=1}^{n-1} \mathcal{L}_i^{2i-n}\right) \otimes \mathcal{L}_n^{j-1}.$$

Our goal is to show that $\mathcal{M}_j$ is semiample and big for $j > 0$. Being the pullback of a very ample line bundle, the factor $\mathcal{L}_n^{j-1}$ is semiample and effective for $j > 0$. Hence, it suffices to show that

$$\mathcal{N} := \bigotimes_{i=1}^{2i-n} \mathcal{L}_i^{2i-n}$$

is semiample and big. Since we have assumed that $n \geqslant 3$, the centre $c = \lfloor (n-1)/2 \rfloor$ is strictly positive. We may then write

$$\mathcal{N} \simeq \bigotimes_{k=1}^{c} (\mathcal{L}_{n-k} \otimes \mathcal{L}_k^{-1})^{\otimes(n-2k)}.$$

Schubert calculus (see [Ful97, § 10.2, Proposition 3]) tells us that the line bundles $\mathcal{L}_a \otimes \mathcal{L}_b^{-1}$ are ample when $a > b$. In particular, all the factors in the preceding factorisation of $\mathcal{N}$ are ample. Since $c \geqslant 1$, this factorisation is also non-empty. It follows then that $\mathcal{N}$ is an ample line bundle, as desired. $\square$

Remark 8.7. Proposition 8.5 can be improved slightly to say that $\omega_\pi \otimes \pi^*\mathcal{O}(-i)$ is actually the inverse of an ample line bundle. This claim follows directly from the homogeneity of $\mathrm{Flag}(V)$. Indeed, let $\mathcal{L}$ be a semiample and big line bundle on a projective variety X that is homogeneous for a connected group G. Let $f : X \to \mathbb{P}^N$ denote the map defined by a suitably large power of $\mathcal{L}$. If $\mathcal{L}$ were not ample, then there would be a proper curve $C \subset X$ that is contracted by f. By the rigidity lemma (see [MFK94, Proposition 6.1]), the same is true for any curve algebraically equivalent to C. However, since X is homogeneous, translates of C under G actually cover X. Since G is connected, all translates of C are algebraically equivalent to C. It follows then that $\dim(\mathrm{im}(f)) < \dim(X)$, contradicting the bigness of $\mathcal{L}$.

As a corollary, we obtain a further family of examples; this will be generalised in Corollary 8.10.

Corollary 8.8. *Let V be a finite-dimensional vector space, and let X be a partial Flag variety for V. Then X is a splinter. In particular, all Grassmannians* $\mathrm{Gr}(k, n)$ *are splinters.*

Proof. There is a natural morphism $\pi : \mathrm{Flag}(V) \to X$ given by remembering the corresponding flag. It can be checked that π is a smooth projective morphism whose fibres are iterated fibrations of projective spaces. In particular, $\mathrm{R}\,\pi_*\mathcal{O}_{\mathrm{Flag}(V)} \simeq \pi_*\mathcal{O}_{\mathrm{Flag}(V)} \simeq \mathcal{O}_X$. The claim for X now follows from that for $\mathrm{Flag}(V)$ proven in Proposition 8.5. $\square$

8.2 Arbitrary groups

Following a suggestion of the referee, we now turn to the flag varieties of arbitrary groups. The results in this section subsume those of § 8.1 but are more dependent on the literature. We first show that globally F-regular projective varieties are splinters; see [SS10] for the relevant definitions.

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Proposition 8.9. *Let X be a projective variety over a field k of characteristic p. If X is globally F-regular, then X is a D-splinter.*

Proof. Let $\mathcal{L}$ denote an ample line bundle on X, and let $S = \bigoplus_{i \geqslant 0} H^0(X, \mathcal{L}^i)$ denote the corresponding section ring with $\mathfrak{m} \subset S$ the ideal of elements of strictly positive degree. Then $\mathrm{Spec}(S)$ can be viewed as an affine cone over X, and the punctured spectrum $U = \mathrm{Spec}(S) - \{\mathfrak{m}\}$ is the total space of $\mathbb{G}_m$-torsor $\pi : \mathcal{L}^{-1} - 0(X) \to X$; here $0(X) \subset \mathcal{L}^{-1}$ denotes the 0 section. By [SS10, Proposition 5.3(1)], the ring S is globally F-regular. It is well-known that this forces S to be a splinter (this can be proven by a variant of the argument in Example 2.3, for example). Since a finite cover of U extends to one of S (by normalisation, for example), U is also a splinter. The structure map $\mathcal{O}_X \to \pi_* \mathcal{O}_U$ splits (as it does for any $\mathbb{G}_m$-torsor), so it follows that X is a splinter, as required. □

We can now show the promised result on flag varieties.

Corollary 8.10. *Let G be a geometrically reductive algebraic group over a field k of characteristic p, and let $P \subset B$ be a parabolic. Then G/P is a splinter.*

Proof. We explain the idea first. As a first step, one uses that $\pi : G/B \to G/P$ satisfies $\mathrm{R}\,\pi_* \mathcal{O}_{G/B} \simeq \mathcal{O}_{G/P}$ to reduce to the case where $P = B$. Next, one reduces to showing that the Bott–Samelson variety X for G (see [BS55]) is a splinter: the reason is that X admits a proper birational map $\pi : X \to G/B$ satisfying $\mathcal{O}_{G/B} \simeq \mathrm{R}\,\pi_* \mathcal{O}_X$. The claim for X is shown by proving that X is globally F-regular and using Proposition 8.9. Instead of reproducing this work here, we refer to [LRT06, Theorem 2.2] for a proof that G/P is globally F-regular. □

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Bhargav Bhatt bhattb@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

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