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HECKE OPERATORS AND DRINFELD CUSP FORMS OF LEVEL t

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Abstract

We use a linear algebra interpretation of the action of Hecke operators on Drinfeld cusp forms to prove that when the dimension of the $\mathbb{C}_\infty$-vector space $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$ is one, the Hecke operator $\mathbf{T}_t$ is injective on $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$ and $S_{k,m}(\Gamma_0(t))$ is a direct sum of oldforms and newforms.

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1. Introduction

Let $S_k(\mathrm{SL}_2(\mathbb{Z}))$ be the space of weight k cusp forms of level one. It is well known that this space admits a basis of *Hecke eigenforms*, that is, normalised eigenfunctions for all the Hecke operators T_n. The space $S_k(\Gamma_0(N))$ still admits a basis of Hecke eigenforms, but only for those T_n such that $(n, N) = 1$. To find a basis of eigenforms for all the T_n, we have to focus on forms that are genuinely of level N and also to consider the operator U_p if $p \mid N$, as Atkin and Lehner realised [1]. More precisely, we have to distinguish between *oldforms*, that is, forms coming from lower levels $M \mid N$, and *newforms*, which are defined as the orthogonal complement of oldforms with respect to the Petersson inner product (see [10, Ch. 5]).

The present paper mainly deals with a function field counterpart of the above results. It comes after a series of papers [3, 4, 6, 13], in which we investigated:

(1) diagonalisability of Hecke operators;
(2) injectivity of Hecke operators;
(3) newforms and oldforms,

for the *Drinfeld modular forms*. Let $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$, respectively $S_{k,m}(\Gamma_0(\mathfrak{p}))$, be the space of Drinfeld cusp forms of weight k, type m and level one, respectively level $\mathfrak{p}$, where $\mathfrak{p} = (P)$ is a prime ideal of $O := \mathbb{F}_q[t]$ and q a power of a fixed prime $p \in \mathbb{Z}$ (see Section 2 for details of the definitions and notation appearing in this introduction).

Denote by $\mathbf{T}_\mathfrak{p}$ and $\mathbf{U}_\mathfrak{p}$ respectively the Hecke and Atkin–Lehner operators acting on $S_{k,m}(\mathrm{GL}_2(O))$ and $S_{k,m}(\Gamma_0(\mathfrak{p}))$.

One of the challenges in the positive characteristic setting comes from the lack of a suitable analogue of the Petersson inner product. In [3, 4], we overcame this using a combination of a combinatorial argument, that is, Teitelbaum interpretation of cusp forms as harmonic cocycles (see [12]), and of *twisted trace maps* to describe what we identify as newforms. The combinatorial method allowed us to describe explicitly the matrix associated to the $\mathbf{U}_\mathfrak{p}$-operator acting on $S_{k,m}(\Gamma_0(\mathfrak{p}))$, when $\mathfrak{p}$ is prime generated by a degree-one polynomial, and to formulate a series of conjectures, supported by numerical search, on the distribution of slopes, that is, $\mathfrak{p}$-adic valuations of eigenvalues of $\mathbf{U}_\mathfrak{p}$, as the weight varies. In [4], among other things, we gave the following conjecture.

CONJECTURE 1.1 [4, Conjecture 1.1]. With the notation as above,

(1) $\mathbf{T}_\mathfrak{p}$ is injective;
(2) $S_{k,m}(\Gamma_0(\mathfrak{p}))$ is the direct sum of oldforms $S^{\mathrm{old}}_{k,m}(\Gamma_0(\mathfrak{p}))$ and newforms $S^{\mathrm{new}}_{k,m}(\Gamma_0(\mathfrak{p}))$.

We provided some evidence: in particular,

(a) for the case $\deg(P) = 1$ and $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(O))) = 0$, in [4, Section 5];
(b) for the case $\deg(P) = 1$ and $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(O))) = 1$, in the (unpublished) Section 3 of [5].

Recently, Dalal and Kumar [9, Theorem 4.6] provided a new proof for case (b): their method is based on the analysis of the Fourier coefficients of the image of a generator via the Hecke operator $\mathbf{T}_\mathfrak{p}$ and, hopefully, it is suitable for more generalisations. Since there seems to be quite some interest in results of this type, we decided to present here our original proof of this fact based on the linear algebra interpretation of the Hecke operators $\mathbf{T}_\mathfrak{p}$ and $\mathbf{U}_\mathfrak{p}$ and of the trace maps Tr and Tr' [3, 4]. The proof is via direct computation, exploiting the symmetries of the matrices representing these operators. We believe that such symmetries are the key to improve the results but, to go further, we probably need a deeper understanding of how they reflect on the action on oldforms, that is, to find the counterpart for oldforms of the antidiagonal action on newforms (see [4, Section 5.2]). The statement and an explicit example already appeared in [6, Example 2.19].

The paper is organised as follows. In Section 2, we recall the objects we shall work with: Drinfeld modular forms, Hecke operators, degeneracy and trace maps, which will enable us to define oldforms and newforms despite the absence of an appropriate inner product. In Section 3, we specialise at $\mathfrak{p} = (t)$ and, as in [3], we associate explicit matrices to all operators. In particular, we describe a matrix M (see (3.2)) that is involved in the description of $\mathbf{U}_t$ and has many peculiar symmetries. After that, we briefly deal with the diagonalisability of M and then prove our main results.

THEOREM 1.2. *Assume* $\dim_{\mathbb{C}_\infty} S_{k,m}(\mathrm{GL}_2(O)) = 1$. *Then*

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(1) $\mathbf{T}_t$ *is injective (see Theorem 3.1);*
(2) $S_{k,m}(\Gamma_0(t)) = S^{\mathrm{old}}_{k,m}(\Gamma_0(t)) \oplus S^{\mathrm{new}}_{k,m}(\Gamma_0(t))$ *(see Theorem 3.2).*

2. Setting and notation

Let K be the global function field $\mathbb{F}_q(t)$, where q is a power of a fixed prime $p \in \mathbb{Z}$. Fix the prime $1/t$ at ∞ and denote by $\mathcal{O} := \mathbb{F}_q[t]$ its ring of integers (that is, the ring of functions regular outside ∞). Let $K_\infty = \mathbb{F}_q((1/t))$ be the completion of K at $1/t$ and denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞.

2.1. Drinfeld modular forms. We work on the *Drinfeld upper half-plane*, the set $\Omega := \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(K_\infty)$ together with a structure of rigid analytic space (see [11]). The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via Möbius transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = (az+b)/(cz+d)$. Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(\mathcal{O})$: Γ has finitely many cusps, represented by $\Gamma\backslash\mathbb{P}^1(K)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, $k, m \in \mathbb{Z}$ and $\varphi : \Omega \to \mathbb{C}_\infty$, we define

$$(\varphi\,|_{k,m}\gamma)(z) := \varphi(\gamma z)(\det\gamma)^m(cz+d)^{-k}.$$

DEFINITION 2.1. A rigid analytic function $\varphi : \Omega \to \mathbb{C}_\infty$ is called a *Drinfeld modular function of weight k and type $m \in \mathbb{Z}/o(\Gamma)\mathbb{Z}$ for* Γ if

$$(\varphi\,|_{k,m}\gamma)(z) = \varphi(z) \quad \text{for all } \gamma \in \Gamma,$$

where $o(\Gamma)$ is the number of scalar matrices in Γ. A Drinfeld modular function φ of weight $k \geqslant 0$ and type m for Γ is called a *Drinfeld modular form* if φ is holomorphic at all cusps and it is called a *cusp form* if it vanishes at all cusps. The space of Drinfeld modular forms of weight k and type m for Γ will be denoted by $M_{k,m}(\Gamma)$. The subspace of cuspidal modular forms is denoted by $S_{k,m}(\Gamma)$.

This definition coincides with [7, Definition 5.1]. Other authors require the function to be meromorphic (in the sense of rigid analysis, see for example, [8, Definition 1.4]) and would call our functions *weakly modular*.

Let $\mathfrak{p} = (P) \subset \mathcal{O}$ be a prime with P irreducible of degree one. We shall work only with the arithmetic subgroup $\Gamma_0(\mathfrak{p})$ of upper triangular matrices modulo $\mathfrak{p}$ and the spaces $S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$ and $S_{k,m}(\Gamma_0(\mathfrak{p}))$ which we call respectively cusp forms of level one and of level $\mathfrak{p}$. Note that in both cases, $o(\Gamma) = q-1$. To have nontrivial forms, the weight and type must satisfy $k \equiv 2m \pmod{q-1}$.

2.2. Hecke operators. We have the Hecke operators:

$$\mathbf{T}_\mathfrak{p}(\varphi)(z) := P^{k-m}\Big(\varphi\,|_{k,m}\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)\Big)(z) + P^{k-m}\sum_{Q\in\mathbb{F}_q}\Big(\varphi\,|_{k,m}\left(\begin{smallmatrix} 1 & Q \\ 0 & P \end{smallmatrix}\right)\Big)(z), \quad \text{on } S_{k,m}(\mathrm{GL}_2(\mathcal{O}));$$

$$\mathbf{U}_\mathfrak{p}(\varphi)(z) := P^{k-m}\sum_{Q\in\mathbb{F}_q}\Big(\varphi\,|_{k,m}\left(\begin{smallmatrix} 1 & Q \\ 0 & P \end{smallmatrix}\right)\Big)(z), \quad \text{on } S_{k,m}(\Gamma_0(\mathfrak{p})).$$

2.3. Newforms and oldforms. As already mentioned in the introduction, in the positive characteristic setting, we do not have an analogue of the Petersson inner product; therefore, we need a different approach. In [4, Section 3], we defined oldforms and newforms of level t. The definition has been generalised in [6, 13], but Dalal and Kumar in [9, Section 4.3] pointed out the existence of a twisted Eisenstein form both new and old (for our definition) when the level is $\mathfrak{p}\mathfrak{q}$ ($\mathfrak{q}$ another prime different from $\mathfrak{p}$). Since we shall only work with levels one and $\mathfrak{p}$, we can still rely on our original definition, which we now recall.

We have an injective map

$$(\delta_1, \delta_{\mathfrak{p}}) : S_{k,m}(\mathrm{GL}_2(\mathcal{O}))^2 \longrightarrow S_{k,m}(\Gamma_0(\mathfrak{p}))$$

$$(\delta_1, \delta_{\mathfrak{p}})(\varphi, \psi) = \varphi(z) + \left(\psi\,|_{k,m}\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)(z) = \varphi(z) + P^m\psi(Pz).$$

DEFINITION 2.2. The space of *oldforms of level* $\mathfrak{p}$, denoted by $S^{\mathrm{old}}_{k,m}(\Gamma_0(\mathfrak{p}))$, is the subspace of $S_{k,m}(\Gamma_0(\mathfrak{p}))$ generated by $\mathrm{Im}(\delta_1, \delta_{\mathfrak{p}})$.

We recall that $R = \{\mathrm{Id}, \left(\begin{smallmatrix} 0 & -1 \\ 1 & Q \end{smallmatrix}\right) : Q \in \mathbb{F}_q\}$ is a system of coset representatives for $\Gamma_0(\mathfrak{p})\backslash\mathrm{GL}_2(\mathcal{O})$.

DEFINITION 2.3. We have the following maps defined on $S_{k,m}(\Gamma_0(\mathfrak{p}))$:

- the *Fricke involution*, which preserves the space $S_{k,m}(\Gamma_0(\mathfrak{p}))$, represented by the matrix $\gamma_{\mathfrak{p}} := \left(\begin{smallmatrix} 0 & -1 \\ P & 0 \end{smallmatrix}\right)$ and defined by $\varphi^{Fr} = (\varphi\,|_{k,m}\gamma_{\mathfrak{p}})$;
- the *trace map*, defined by

$$Tr : S_{k,m}(\Gamma_0(\mathfrak{p})) \to S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$$
$$\varphi \mapsto \sum_{\gamma\in R}(\varphi\,|_{k,m}\gamma)(z);$$

- the *twisted trace map*, defined by

$$Tr' : S_{k,m}(\Gamma_0(\mathfrak{p})) \to S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$$
$$\varphi \mapsto Tr(\varphi^{Fr}).$$

DEFINITION 2.4. The space of *newforms of level* $\mathfrak{p}$ is $S^{\mathrm{new}}_{k,m}(\Gamma_0(\mathfrak{p})) = \mathrm{Ker}(Tr) \cap \mathrm{Ker}(Tr')$.

The following important criterion is [6, Theorem 2.8 and Corollary 2.10].

THEOREM 2.5. *We have a direct sum decomposition* $S_{k,m}(\Gamma_0(\mathfrak{p})) = S^{\mathrm{old}}_{k,m}(\Gamma_0(\mathfrak{p})) \oplus S^{\mathrm{new}}_{k,m}(\Gamma_0(\mathfrak{p}))$ *if and only if the map* $\mathcal{D} := Id - P^{k-2m}(Tr')^2$ *is bijective. Moreover,*

$$\mathrm{Ker}(\mathcal{D}) = \{\delta_1\varphi : \varphi \in S_{k,m}(\mathrm{GL}_2(\mathcal{O})),\ \mathbf{T}_{\mathfrak{p}}\varphi = \pm P^{k/2}\varphi\}.$$

3. Main results

For the level t (actually for any prime of degree one), we computed the matrix associated to the operator $\mathbf{U}_t$ acting on $S_{k,m}(\Gamma_0(t))$ (using Teitelbaum's interpretation

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with harmonic cocycles in [3, Section 4] and [4, Sections 3 and 4]). For the convenience of the reader, we recall the matrices involved in our computations.

To have $S_{k,m}(\Gamma_0(t)) \neq 0$, we need $k \equiv 2m \pmod{q-1}$, and hence there exists a unique $d \in \mathbb{N}$ such that $k = 2m + (d-1)(q-1)$. For notational reasons, we put $j+1 \equiv m \pmod{q-1}$ with $0 \leqslant j \leqslant q-2$: the letters j and d provide the type m and the dimension of the matrix U associated to the action of $\mathbf{U}_t$ on $S_{k,m}(\Gamma_0(t))$. The crucial ingredient is the following matrix. For even $d = 2n$, put

$$T := \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} & (-1)^{j+1}m_{1,n} & \cdots & (-1)^{j+1}m_{1,2} & (-1)^{j+1}m_{1,1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n} & (-1)^{j+1}m_{2,n} & \cdots & (-1)^{j+1}m_{2,2} & (-1)^{j+1}m_{2,1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n} & (-1)^{j+1}m_{n,n} & \cdots & (-1)^{j+1}m_{n,2} & (-1)^{j+1}m_{n,1} \\ m_{n+1,1} & m_{n+1,2} & \cdots & 0 & 0 & \cdots & (-1)^{j+1}m_{n+1,2} & (-1)^{j+1}m_{n+1,1} \\ \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{2n-1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & (-1)^{j+1}m_{2n-1,1} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

(The reason to denote it by T will become apparent shortly.) For odd $d = 2n-1$, one just needs to modify the indices and add the central nth column

$$(m_{1,n}, \ldots, m_{n-1,n}, 0, \ldots, 0).$$

The entries of T are the binomial coefficients in $\mathbb{F}_p$,

$$m_{a,b} = \begin{cases} -\left[\binom{j+(d-a)(q-1)}{j+(d-b)(q-1)} + (-1)^{j+1}\binom{j+(d-a)(q-1)}{j+(b-1)(q-1)}\right] & \text{if } a \neq b, \\ (-1)^j\binom{j+(d-a)(q-1)}{j+(a-1)(q-1)} & \text{if } a = b. \end{cases} \tag{3.1}$$

Let A be the antidiagonal matrix

$$A = \begin{pmatrix} 0 & & (-1)^{j+1} \\ & \iddots & \\ (-1)^{j+1} & & 0 \end{pmatrix}.$$

Then $A^2 = I$ (the identity matrix of dimension d) and the main symmetry of T can be expressed as $TA = T$. This is clear for even d. For odd d and even j, note that the central column is identically 0 by (3.1), while for odd j, one is simply multiplying the central column by one. Finally, let

$$M := T - A. \tag{3.2}$$

We list here the matrices associated to the maps involved in our computations.

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- The action of $\mathbf{U}_t$ on $S_{k,m}(\Gamma_0(t))$ is described by

$$U = MD := M \begin{pmatrix} t^{s_1} & & 0 \\ & \ddots & \\ 0 & & t^{s_d} \end{pmatrix},$$

where for $1 \leqslant i \leqslant d$, we put $s_i = j + 1 + (i-1)(q-1)$ (so that $s_i + s_{d+1-i} = k$ for $1 \leqslant i \leqslant d/2$ or $1 \leqslant i \leqslant (d+1)/2$ accordingly as d is even or odd).

- The matrix for the Fricke involution $Fr(t)$ is

$$t^{m-k}F = t^{m-k} \begin{pmatrix} 0 & & (-t)^{s_d} \\ & \iddots & \\ (-t)^{s_1} & & 0 \end{pmatrix} = t^{m-k} \begin{pmatrix} 0 & & (-1)^{j+1}t^{s_d} \\ & \iddots & \\ (-1)^{j+1}t^{s_1} & & 0 \end{pmatrix}.$$

Note that $F^2 = t^k I$ and $AF = D$.

- Direct computation (see [3, Section 3.3]) provides the equation

$$Tr(\varphi) = \varphi + t^{-m}\mathbf{U}_t(\varphi^{Fr}). \tag{3.3}$$

Its immediate translation in matrix form is

$$\begin{aligned} I + t^{-m}MD(t^{m-k}F) &= I + t^{-k}MAF^2 = I + MA \\ &= A^2 + (T - A)A = (A + T - A)A = TA = T. \end{aligned}$$

- The matrix for the twisted trace follows directly:

$$T' = t^{m-k}TF = \begin{cases} t^{m-k}(M + A)F = t^{m-k}(MF + D) \\ t^{m-k}TAF = t^{m-k}TD \\ t^{m-k}TAF = t^{m-k}(I + MA)F = t^{m-k}(F + MD). \end{cases}$$

- Finally, since the trace acts trivially on $\mathrm{Im}(\delta_1)$, it is easy to see that $\mathrm{Ker}(Tr - Id) = \mathrm{Im}(\delta_1)$, that is, in terms of matrices, $\mathrm{Im}(\delta_1) = \mathrm{Ker}(T - I) = \mathrm{Ker}(MA)$.

3.1. Diagonalisability of M. As seen above, the matrices M and T satisfy a number of equations. We mention a few more, leading to the diagonalisability of M (unfortunately not equivalent to the diagonalisability of $U = MD$, which is included in [4, Conjecture 1.1] and is related to the conjectures treated in this paper), but we shall not pursue this topic further here.

(i) Like all trace maps $T^2 = T$ and T is diagonalisable. This obviously leads to d^2 equations in the entries $m_{i,j}$ which are still difficult to handle for a generic d.

(ii) Let $\underline{v} = M\underline{w} \in \mathrm{Im}(M)$. Then $T\underline{v} = TM\underline{w} = T(T - A)\underline{w} = (T^2 - TA)\underline{w} = 0$, that is, $\mathrm{Im}(M) \subseteq \mathrm{Ker}(T)$. Conversely, let $\underline{v} \in \mathrm{Ker}(T)$. Then, writing $\underline{v} = -A\underline{w}$, we get $0 = T\underline{v} = -TA\underline{w} = -T\underline{w}$, that is, $\underline{w} \in \mathrm{Ker}(T)$ as well. Therefore, $M\underline{w} = (T - A)\underline{w} = -A\underline{w} = \underline{v} \in \mathrm{Im}(M)$. Hence, $\mathrm{Im}(M) = \mathrm{Ker}(T)$.

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(iii) Finally,

$$\begin{aligned} M^3 &= (T-A)^3 = (T^2 - AT - TA + I)(T-A) \\ &= (-AT+I)(T-A) = -AT^2 + T + ATA - A \\ &= T - A = M. \end{aligned}$$

Therefore, for any $p \neq 2$, the matrix M is diagonalisable and we can write

$$\underline{v} = \tfrac{1}{2}(M^2\underline{v} + \underline{v}) + \tfrac{1}{2}(M^2\underline{v} - \underline{v}) + (\underline{v} - M^2\underline{v}) := \underline{v}_1 + \underline{v}_{-1} + \underline{v}_0,$$

where each $\underline{v}_\alpha$ is in the M-eigenspace of the eigenvalue $\alpha \in \{0, 1, -1\}$. This reflects the results of [2], where we found examples of nondiagonalisability of $\mathbf{U}_t$ in characteristic two, due to the presence of inseparable eigenvalues associated to newforms.

3.2. Injectivity of $\mathbf{T}_t$. In [4], we proved some special cases of Conjecture 1.1 building on the analogue of Theorem 3.2 and on the above matrices/formulae (which are not available for $\deg P \geqslant 2$). In particular, in [4, Theorem 5.5], we proved that when $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O})) = 0$, that is, there are no oldforms, the matrix M is antidiagonal and the conjectures hold. We shall now approach the case $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O})) = 1$. This will include many more cases since, for example, $\dim_{\mathbb{C}_\infty}(S_{k,0}(\mathrm{GL}_2(\mathcal{O})) = 1$ if and only if $q \leqslant d < 2q-1$, by [9, Proposition 4.3] (compare with the bounds of [4, Theorems 5.8, 5.9, 5.12 and 5.14]).

THEOREM 3.1. *Assume* $\dim_{\mathbb{C}_\infty} \mathrm{Im}(\delta_1) = 1$. *Then* $\mathbf{T}_t$ *is injective.*

PROOF. Observe that, by [6, Proposition 2.5], $\mathrm{Ker}(\mathbf{T}_t) = \mathrm{Ker}(MA) \cap \mathrm{Ker}(MDMD)$. Thanks to our assumption on the dimension of $\mathrm{Im}(\delta_1) = \mathrm{Ker}(MA)$ and because the entries of MA are in $\mathbb{F}_p$, we have $\dim_{\mathbb{C}_\infty}(\mathrm{Ker}(MA) \cap \mathrm{Ker}(MDMD)) \leqslant 1$ and we can fix a generator $\underline{a} = (a_1, \ldots, a_d) \in \mathbb{F}_p^d$. To avoid adding the transpose symbol to the several indices we shall need in the computations, with a little abuse of notation, we shall write $\underline{a}$ both for the row vector and for its transpose, the context will clarify which one we are using. Our goal is to prove $\underline{a} = 0$.

We prove the case of even dimension $d = 2n$; for odd d, the argument is exactly the same. The vector $\underline{a}$ satisfies the following equations coming from $MA\underline{a} = 0$:

$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,n}a_n + (-1)^{j+1}m_{1,n}a_{n+1} + \cdots + (-1)^{j+1}m_{1,1}a_{2n} = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,n}a_n + (-1)^{j+1}m_{2,n}a_{n+1} + \cdots + (-1)^{j+1}m_{2,1}a_{2n} = 0 \\ \vdots \\ m_{n,1}a_1 + m_{n,2}a_2 + \cdots + (m_{n,n}-1)a_n + (-1)^{j+1}m_{n,n}a_{n+1} + \cdots + (-1)^{j+1}m_{n,1}a_{2n} = 0 \\ m_{n+1,1}a_1 + m_{n+1,2}a_2 + \cdots + m_{n+1,n-1}a_{n-1} - a_{n+1} + \cdots + (-1)^{j+1}m_{n+1,1}a_{2n} = 0 \\ \vdots \\ m_{2n-1,1}a_1 - a_{2n-1} + (-1)^{j+1}m_{2n-1,1}a_{2n} = 0 \\ -a_{2n} = 0. \end{cases} \tag{3.4}$$

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Now put $\underline{p(t)} := MD\underline{a} \in \mathbb{F}_p[t]^{2n}$, with coordinates $p_i(t)$. Then (with $a_{2n} = 0$), $\underline{p}(t)$ is equal to

$$\begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,n}a_nt^{s_n} + (-1)^{j+1}m_{1,n}a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{1,2}a_{2n-1}t^{s_{2n-1}} \\ m_{2,1}a_1t^{s_1} + \cdots + m_{2,n}a_nt^{s_n} + (-1)^{j+1}m_{2,n}a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}(m_{2,2}-1)a_{2n-1}t^{s_{2n-1}} \\ \vdots \\ m_{n,1}a_1t^{s_1} + \cdots + +(-1)^{j+1}(m_{n,n}-1)a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{n,2}a_{2n-1}t^{s_{2n-1}} \\ m_{n+1,1}a_1t^{s_1} + \cdots + (-1)^{j}a_nt^{s_n} + m_{n+1,n-1}a_{n+2}t^{s_{n+2}} + \cdots + (-1)^{j+1}m_{n+1,2}a_{2n-1}t^{s_{2n-1}} \\ \vdots \\ m_{2n-1,1}a_1t^{s_1} + (-1)^{j}a_2t^{s_2} \\ (-1)^{j}a_1t^{s_1} \end{pmatrix}. \tag{3.5}$$

Since $MD\underline{p(t)} = 0$, we also have the equations:

$$\begin{cases} m_{1,1}t^{s_1}p_1(t) + \cdots + m_{1,n}t^{s_n}p_n(t) + (-1)^{j+1}m_{1,n}t^{s_{n+1}}p_{n+1}(t) + \cdots + (-1)^{j+1}(m_{1,1}-1)t^{s_{2n}}p_{2n}(t) = 0 \\ m_{2,1}t^{s_1}p_1(t) + \cdots + m_{2,n}t^{s_n}p_n(t) + (-1)^{j+1}m_{2,n}t^{s_{n+1}}p_{n+1}(t) + \cdots + (-1)^{j+1}m_{2,1}t^{s_{2n}}p_{2n}(t) = 0 \\ \vdots \\ m_{n,1}t^{s_1}p_1(t) + \cdots + m_{n,n}t^{s_n}p_n(t) + (-1)^{j+1}(m_{n,n}-1)t^{s_{n+1}}p_{n+1}(t) + \cdots + (-1)^{j+1}m_{n,1}t^{s_{2n}}p_{2n}(t) = 0 \\ \vdots \\ m_{2n-1,1}t^{s_1}p_1(t) + (-1)^{j}t^{s_2}p_2(t) + (-1)^{j+1}m_{2n-1,1}t^{s_{2n}}p_{2n}(t) = 0 \\ (-1)^{j}t^{s_1}p_1(t) = 0. \end{cases} \tag{3.6}$$

Note that in (3.6), we have polynomials in $\mathbb{F}_p[t]$. From now on, we shall use the identity principle for polynomials to solve the equations in the a_i. From the last row in (3.6), we get $p_1(t) = 0$, and comparing with (3.5) and recalling the s_i are distinct,

$$m_{1,1}a_1 = m_{1,2}a_2 = \cdots = m_{1,n}a_n = m_{1,n}a_{n+1} = \cdots = m_{1,2}a_{2n-1} = 0.$$

Substituting in the first and second-last equations in (3.4), we obtain

$$a_1 = a_{2n-1} = 0,$$

which also means that $p_{2n}(t) = 0$. We can rewrite (3.4), (3.5) and (3.6) as

$$\begin{cases} (m_{2,2}-1)a_2 + \cdots + m_{2,n}a_n + (-1)^{j+1}m_{2,n}a_{n+1} + \cdots + (-1)^{j+1}m_{2,3}a_{2n-2} = 0 \\ \vdots \\ m_{n,2}a_2 + \cdots + (m_{n,n}-1)a_n + (-1)^{j+1}m_{n,n}a_{n+1} + \cdots + (-1)^{j+1}m_{n,3}a_{2n-2} = 0 \\ m_{n+1,2}a_2 + \cdots + m_{n+1,n-1}a_{n-1} - a_{n+1} + \cdots + (-1)^{j+1}m_{n+1,3}a_{2n-2} = 0 \\ \vdots \\ m_{2n-2,2}a_2 - a_{2n-2} = 0 \\ a_1 = a_{2n-1} = a_{2n} = 0, \end{cases} \tag{3.7}$$

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$$\underline{p(t)} = \begin{pmatrix} 0 \\ m_{2,2}a_2t^{s_2} + \cdots + m_{2,n}a_nt^{s_n} + (-1)^{j+1}m_{2,n}a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{2,3}a_{2n-2}t^{s_{2n-2}} \\ \vdots \\ m_{n,2}a_2t^{s_2} + \cdots + m_{n,n}a_nt^{s_n} + (-1)^{j+1}(m_{n,n}-1)a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{n,3}a_{2n-2}t^{s_{2n-2}} \\ m_{n+1,2}a_2t^{s_2} + \cdots + (-1)^j a_nt^{s_n} + m_{n+1,n-1}a_{n+2}t^{s_{n+2}} + \cdots + (-1)^{j+1}m_{n+1,3}a_{2n-2}t^{s_{2n-2}} \\ \vdots \\ (-1)^j a_2t^{s_2} \\ 0 \end{pmatrix} \tag{3.8}$$

and

$$\begin{cases} m_{1,2}t^{s_2}p_2(t) + \cdots + m_{1,n}t^{s_n}p_n(t) + (-1)^{j+1}m_{1,n}t^{s_{n+1}}p_{n+1}(t) + \cdots (-1)^{j+1}m_{1,2}t^{s_{2n-1}}p_{2n-1}(t) = 0 \\ m_{2,2}t^{s_2}p_2(t) + \cdots + m_{2,n}t^{s_n}p_n(t) + (-1)^{j+1}m_{2,n}t^{s_{n+1}}p_{n+1}(t) + \cdots + (-1)^{j+1}(m_{2,2}-1)t^{s_{2n-1}}p_{2n-1}(t) = 0 \\ \vdots \\ m_{n,2}t^{s_2}p_2(t) + \cdots + m_{n,n}t^{s_n}p_n(t) + (-1)^{j+1}(m_{n,n}-1)t^{s_{n+1}}p_{n+1}(t) + \cdots + (-1)^{j+1}m_{n,2}t^{s_{2n-1}}p_{2n-1}(t) = 0 \\ \vdots \\ (-1)^j t^{s_2}p_2(t) = 0 \\ p_1(t) = p_{2n}(t) = 0. \end{cases} \tag{3.9}$$

We repeat the same argument starting now from the second-last equation in (3.9), which yields $p_2(t) = 0$. This means

$$m_{2,2}a_2 = \cdots = m_{2,n}a_n = m_{2,n}a_{n+1} = \cdots = m_{2,3}a_{2n-2} = 0,$$

which, substituted in the first equation of (3.7), gives $a_2 = 0$. From the second-last equations in (3.7) and (3.8), $a_{2n-2} = 0$ and $p_{2n-1}(t) = 0$ as well. Iterating the process, we see that the specular symmetries between MD ($(-1)^j$ on the antidiagonal) and MA (-1 on the diagonal) and the positions of the $m_{i,i} - 1$ lead to $\underline{a} = 0$. □

3.3. Direct sum. We use the criterion of Theorem 2.5 to show that $\mathrm{Ker}(\mathcal{D}) = 0$. Note that $\varphi \in \mathrm{Ker}(\mathcal{D})$ yields $\varphi - t^{k-2m}(Tr')^2(\varphi) = 0$, that is, $\varphi = t^{k-2m}(Tr')^2(\varphi) \in S_{k,m}(\mathrm{GL}_2(O))$; and hence φ is old and, in particular, belongs to $\mathrm{Im}(\delta_1) = \mathrm{Ker}(MA)$. So we write $\varphi = \delta_1\psi$ and, by [3, (3.2)], $\mathbf{U}_t(\delta_1\psi) = \delta_1\mathbf{T}_t\psi - t^{k-m}\delta_t\psi$ is old as well. Moreover,

$$\begin{aligned} t^{2m-k}\delta_1\psi &= (Tr')^2(\delta_1\psi) = (Tr')(Tr'(\delta_1\psi)) \\ &= Tr'((\delta_1\psi)^{Fr} + t^{m-k}\mathbf{U}_t(\delta_1\psi)) \quad \text{(use the twisted version of (3.3))} \\ &= Tr(((\delta_1\psi)^{Fr})^{Fr}) + t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi)) \\ &= t^{2m-k}Tr(\delta_1\psi) + t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi)) \\ &= t^{2m-k}\delta_1\psi + t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi)), \end{aligned}$$

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that is, $Tr'(\mathbf{U}_t(\delta_1\psi)) = 0$. We can similarly prove that $Tr(\mathbf{U}_t(\delta_1\psi)) = 0$ (that is, $\mathbf{U}_t(\delta_1\psi)$ is old and new), but the equations coming from MA and $T'U$ will be enough for our purposes.

THEOREM 3.2. *If* $\dim_{\mathbb{C}_\infty} \mathrm{Im}(\delta_1) = 1$, *then* $S_{k,m}(\Gamma_0(t)) = S^{\mathrm{old}}_{k,m}(\Gamma_0(t)) \oplus S^{\mathrm{new}}_{k,m}(\Gamma_0(t))$.

PROOF. Take $\underline{a} \in \mathbb{F}_p^d$ satisfying $MA\underline{a} = 0$ and representing an element $\eta = \delta_1\varphi \in \mathrm{Ker}(\mathcal{D})$, so that, as seen above, $TF(MD\underline{a}) = 0$. We prove that these two relations yield $\underline{a} = 0$, so that $\mathrm{Ker}(\mathcal{D}) = 0$ and $\mathcal{D}$ is invertible. As before, we only treat the case of even $d = 2n$.

The equation $MA\underline{a} = 0$ gives again the system (3.4) (in particular, $a_{2n} = 0$). Writing $\underline{p(t)} = MD\underline{a}$ as in (3.5), from $TF(MD\underline{a}) = 0$,

$$\begin{cases} m_{1,1}t^{s_1}p_1(t) + \cdots + m_{1,n}t^{s_n}p_n(t) + m_{1,n}(-t)^{s_{n+1}}p_{n+1}(t) + \cdots + m_{1,1}(-t)^{s_{2n}}p_{2n}(t) = 0 \\ m_{2,1}t^{s_1}p_1(t) + \cdots + m_{2,n}t^{s_n}p_n(t) + m_{2,n}(-t)^{s_{n+1}}p_{n+1}(t) + \cdots + m_{2,1}(-t)^{s_{2n}}p_{2n}(t) = 0 \\ \vdots \\ m_{n,1}t^{s_1}p_1(t) + \cdots + m_{n,n}t^{s_n}p_n(t) + m_{n,n}(-t)^{s_{n+1}}p_{n+1}(t) + \cdots + m_{n,1}(-t)^{s_{2n}}p_{2n}(t) = 0 \\ m_{n+1,1}t^{s_1}p_1(t) + \cdots + m_{n+1,1}(-t)^{s_{2n}}p_{2n}(t) = 0 \\ \vdots \\ m_{2n-2,1}t^{s_1}p_1(t) + m_{2n-2,2}t^{s_2}p_2(t) + m_{2n-2,2}(-t)^{s_{2n-1}}p_{2n-1}(t) + m_{2n-2,1}(-t)^{s_{2n}}p_{2n}(t) = 0 \\ m_{2n-1,1}t^{s_1}p_1(t) + m_{2n-1,1}(-t)^{s_{2n}}p_{2n}(t) = 0. \end{cases} \tag{3.10}$$

In the last equation of (3.10), the term of highest degree is $m_{2n-1,1}(-t)^{s_{2n}}(-1)^j a_1 t^{s_1} = -m_{2n-1,1}a_1t^k$ (note that $p_1(t)$ has degree at most s_{2n-1} because $a_{2n} = 0$); therefore, $m_{2n-1,1}a_1 = 0$ and the second-last equation in (3.4) tells us that $a_{2n-1} = 0$. Now, (3.4) and (3.5) turn into

$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,n}a_n + (-1)^{j+1}m_{1,n}a_{n+1} + \cdots + (-1)^{j+1}m_{1,3}a_{2n-2} = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,n}a_n + (-1)^{j+1}m_{2,n}a_{n+1} + \cdots + (-1)^{j+1}m_{2,3}a_{2n-2} = 0 \\ \vdots \\ m_{n,1}a_1 + m_{n,2}a_2 + \cdots + (m_{n,n}-1)a_n + (-1)^{j+1}m_{n,n}a_{n+1} + \cdots + (-1)^{j+1}m_{n,3}a_{2n-2} = 0 \\ m_{n+1,1}a_1 + m_{n+1,2}a_2 + \cdots + m_{n+1,n-1}a_{n-1} - a_{n+1} + \cdots + (-1)^{j+1}m_{n+1,3}a_{2n-2} = 0 \\ \vdots \\ m_{2n-2,1}a_1 + m_{2n-2,2}a_2 - a_{2n-2} = 0 \\ m_{2n-1,1}a_1 = 0 \\ a_{2n-1} = a_{2n} = 0 \end{cases} \tag{3.11}$$

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and

$$\underline{p(t)} = \begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,n}a_nt^{s_n} + (-1)^{j+1}m_{1,n}a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{1,3}a_{2n-2}t^{s_{2n-2}} \\ m_{2,1}a_1t^{s_1} + \cdots + m_{2,n}a_nt^{s_n} + (-1)^{j+1}m_{2,n}a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{2,3}a_{2n-2}t^{s_{2n-2}} \\ \vdots \\ m_{n,1}a_1t^{s_1} + \cdots + m_{n,n}a_nt^{s_n} + (-1)^{j+1}(m_{n,n}-1)a_{n+1}t^{s_{n+1}} + \cdots + (-1)^{j+1}m_{n,3}a_{2n-2}t^{s_{2n-2}} \\ m_{n+1,1}a_1t^{s_1} + \cdots + (-1)^{j}a_nt^{s_n} + m_{n+1,n-1}a_{n+2}t^{s_{n+2}} + \cdots + (-1)^{j+1}m_{n+1,3}a_{2n-2}t^{s_{2n-2}} \\ \vdots \\ (-1)^{j}a_2t^{s_2} \\ (-1)^{j}a_1t^{s_1} \end{pmatrix}.$$

Consider the second-last equation in (3.10):

$$m_{2n-2,1}t^{s_1}p_1(t) + m_{2n-2,2}t^{s_2}p_2(t) + m_{2n-2,2}(-t)^{s_{2n-1}}p_{2n-1}(t) + m_{2n-2,1}(-t)^{s_{2n}}p_{2n}(t) = 0.$$

The term with the highest possible degree $s_1 + s_{2n} = s_2 + s_{2n-1} = k$ is

$$m_{2n-2,2}(-t)^{s_{2n-1}}(-1)^{j}a_2t^{s_2} + m_{2n-2,1}(-t)^{s_{2n}}(-1)^{j}a_1t^{s_1} = -(m_{2n-2,2}a_2 + m_{2n-2,1}a_1)t^k,$$

and hence $m_{2n-2,1}a_1 + m_{2n-2,2}a_2 = 0$. Looking at the system (3.11), we obtain $a_{2n-2} = 0$; and hence the degree of $p_i(t)$ is bounded by s_{2n-3} for all i.

The proof goes on in the same way. It may be less evident than the one for Theorem 3.1 (where the a_i vanished in pairs), but looking always at the terms of degree k of the $(2n-i)$th equation of (3.10) and substituting in (3.11), we are able to prove that $a_{2n-i} = 0$ and, as an immediate consequence from (3.5), that all the $p_i(t)$ have degree at most s_{2n-i-1}. For example, midway through the proof we get

$$\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{p(t)} = \begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,n}a_nt^{s_n} \\ \vdots \\ m_{n,1}a_1t^{s_1} + \cdots + m_{n,n}a_nt^{s_n} \\ m_{n+1,1}a_1t^{s_1} + \cdots + (-1)^{j}a_nt^{s_n} \\ m_{n+2,1}a_1t^{s_1} + \cdots + (-1)^{j}a_{n-1}t^{s_{n-1}} \\ \vdots \\ (-1)^{j}a_1t^{s_1} \end{pmatrix}.$$

Therefore, what remains of (3.4) is

$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,n}a_n = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,n}a_n = 0 \\ \vdots \\ m_{n,1}a_1 + m_{n,2}a_2 + \cdots + (m_{n,n}-1)a_n = 0 \\ a_{n+1} = \cdots = a_{2n} = 0. \end{cases} \tag{3.12}$$

Finally, we observe that the nth equation of (3.10) is

$$m_{n,1}t^{s_1}p_1(t) + \cdots + m_{n,n}t^{s_n}p_n(t) + m_{n,n}(-t)^{s_{n+1}}p_{n+1}(t) + \cdots + m_{n,1}(-t)^{s_{2n}}p_{2n}(t) = 0.$$

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As before, the term of degree k must have coefficient 0 and it appears only in the final terms starting from $m_{n,n}(-t)^{s_{n+1}} p_{n+1}(t)$. So we get

$$m_{n,n}a_n + m_{n,n-1}a_{n-1} + \cdots + m_{n,1}a_1 = 0$$

and, by (3.12), $a_n = 0$ as well. Iterating we get $\underline{a} = 0$ and so our claim. □

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