Appendix I
Light-cone variables

In this appendix we introduce light-cone variables and discuss the response function in deep-inelastic electron scattering (DIS) when analyzed in terms of these quantities. The discussion follows closely that in [De73], which provides a much more extensive introduction to this topic.

Suppose that in coordinate space one has a four-vector $x_\mu = (x_1, x_2, x_3, ix_0) = (x, y, z, ict)$.[1] The light-cone variables are defined by

$$\begin{aligned} x_\pm &\equiv \frac{1}{\sqrt{2}}(z \pm ct) \\ \mathbf{x}_\perp &\equiv (x, y) \end{aligned} \tag{I.1}$$

The situation is illustrated in Fig. I.1, where the new axes are defined by the lines $x_\mp = 0$. The square of the four-vector x_μ is evidently

$$x^2 = x_\mu x_\mu = 2x_+x_- + \mathbf{x}_\perp^2 \tag{I.2}$$

In inclusive DIS we have two kinematic four-vectors $q_\mu = (k_2 - k_1)_\mu = (q_x, q_y, q_z, iq_0)$ and $p_\mu = (p_x, p_y, p_z, ip_0)$. We similarly define light-cone combinations

$$\begin{aligned} p_\pm &\equiv \frac{1}{\sqrt{2}}(p_z \pm p_0) \qquad ; \ \mathbf{p}_\perp = (p_x, p_y) \\ q_\pm &\equiv \frac{1}{\sqrt{2}}(q_z \pm q_0) \qquad ; \ \mathbf{q}_\perp = (q_x, q_y) \end{aligned} \tag{I.3}$$

The scalar products are given by

$$\begin{aligned} m\nu = p \cdot q &= p_+q_- + p_-q_+ + \mathbf{p}_\perp \cdot \mathbf{q}_\perp \\ q^2 &= 2q_+q_- + \mathbf{q}_\perp^2 \end{aligned} \tag{I.4}$$

[1] We restore $\hbar$ and c in this appendix for clarity.

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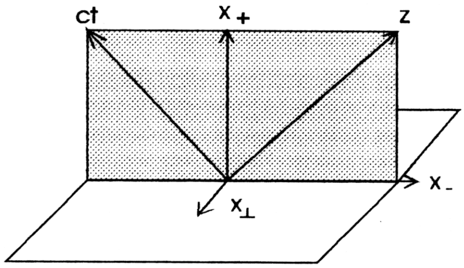

Fig. I.1. Transformation to light-cone variables.

Assume the momentum transfer $\mathbf{q}$ defines the z-axis so that $\mathbf{q}_\perp = 0$. Further, assume for simplicity that $\mathbf{p}_\perp = 0$ (as is true, for example, in the lab). From the electron scattering kinematics, one has

$$\begin{aligned} q_z &= |\mathbf{k}_2 - \mathbf{k}_1| = (k_1^2 + k_2^2 - 2k_1k_2\cos\theta)^{1/2} \\ q_0 &= k_2 - k_1 \\ q_+ &= \frac{1}{\sqrt{2}}[(k_1^2 + k_2^2 - 2k_1k_2\cos\theta)^{1/2} - (k_1 - k_2)] \\ q_- &= \frac{1}{\sqrt{2}}[(k_1^2 + k_2^2 - 2k_1k_2\cos\theta)^{1/2} + (k_1 - k_2)] \end{aligned} \tag{I.5}$$

Here we have written $|\mathbf{k}| \equiv k$.

The DIS limit is defined by $\nu \to \infty$, $q^2 \to \infty$, with constant $q^2/2m\nu = x_B$; it is evidently achieved by the following:

$$\text{Fix } (q_+, p_\mu);\ \text{ and let } q_- \to \infty \tag{I.6}$$

In this case

$$m\nu \to p_+q_- \qquad ; q^2 \to 2q_+q_- \qquad ; \frac{q^2}{2m\nu} \to \frac{q_+}{p_+} \tag{I.7}$$

To illustrate the arguments, we consider a very simplified, heuristic version of Eq. (14.18) where all indices and sums are suppressed

$$w(p,q) \equiv \frac{1}{4\pi}\int e^{iq\cdot x}\langle p|[j(z), j(0)]|p\rangle\, d^4x \tag{I.8}$$

Now

$$\begin{aligned} d^4x &= d^2x_\perp dx_- dx_+ \\ q\cdot x &= q_+x_- + q_-x_+ \end{aligned} \tag{I.9}$$

In the DIS limit of Eq. (I.6), the integrand in Eq. (I.8) oscillates very rapidly, and the resulting integral goes to zero, unless there is a finite

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contribution from the region where $x_+ \to 0$. If $x_+ \to 0$, then Eq. (I.2) implies that $x^2 \to \mathbf{x}_\perp^2$. This now represents a space-like separation of two points. The principle of *microscopic causality* states that the commutator of two hermitian observables (here the currents) must vanish for space-like separations since their measurements cannot interfere outside of the light cone. Hence the only contribution to the integral in Eq. (I.8) will come from the region where $(x_+, \mathbf{x}_\perp) \to 0$, which implies (for any finite x_-) that $x^2 \to 0$; this defines the light cone and illustrates the utility of the new variables. To obtain the asymptotic form of the response function in the DIS region, one is led to an investigation of the structure of the commutator of the two currents on the light cone.

Geometrically, the forward light cone is a cone around the ct axis that lies in the second quadrant in Fig. I.1. Both the x_+ and x_- axes lie in the surface of the cone. In the DIS limit, one is forced to the $x_+ = 0$ plane, which is tangent to the light cone along the negative x_- axis. Since by causality the commutator of the currents vanishes outside the light cone, the only contribution to the integral in Eq. (I.8) comes from the negative x_- axis in the DIS limit.

What kind of singularities exist on the light cone for the commutator of two hermitian operators in field theory? To get some insight, consider the very simple example of a free, massless, real (neutral), scalar field

$$\phi(x_\mu) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \left(c_{\mathbf{k}} e^{ik\cdot x} + c_{\mathbf{k}}^\dagger e^{-ik\cdot x}\right) \tag{I.10}$$

It is one of the standard introductory exercises in field theory to show that the commutator of this field taken at two different space-time points is given by

$$\begin{aligned}
[\phi(x_\mu), \phi(y_\mu)] &= \frac{\hbar}{ic}\Delta(x_\mu - y_\mu) \\
\Delta(x_\mu) &= \frac{i}{(2\pi)^3} \int d^4k\, \varepsilon(k_0)\, \delta(k^2)\, e^{ik\cdot x} \\
&= \frac{1}{2\pi}\varepsilon(x_0)\, \delta(x^2)
\end{aligned} \tag{I.11}$$

Here $d^4k = d^3k dk_0$. The invariant commutator has a delta-function singularity on the light cone.

One can quite generally define

$$w(p, q) \equiv \frac{1}{i\pi} \int e^{iq\cdot x} G(p, x) D(x^2)\, d^4x \tag{I.12}$$

Assume now that the free-field singularities of the commutator have been isolated in $D(x^2)$ and that the function $G(p, x)$ contains the details of

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the currents and the states. From Lorentz invariance one must have $G(p, x) = G(p \cdot x, x^2)$. In the DIS limit one requires the singularities of the commutator on the light cone. In extracting the asymptotic limit, one can then replace the regular coefficient G by its value on the light cone

$$G(p \cdot x, x^2) \approx G(p \cdot x, 0) \equiv g(p \cdot x) \qquad ; \text{DIS} \tag{I.13}$$

Introduce the Fourier transform of this function

$$g(\sigma) \;=\; \int e^{-i\alpha\sigma} F(\alpha)\, d\alpha \tag{I.14}$$

Substitution into Eq. (I.12) then gives

$$w(p, q) \;\approx\; \frac{1}{i\pi} \int F(\alpha)\, d\alpha \int e^{-i(\alpha p - q)\cdot x} D(x^2)\, d^4x \tag{I.15}$$

Again, for simplicity and illustration, suppose the light-cone singularity structure is that of Eq. (I.11). A four-dimensional Fourier transform then leads to

$$w(p, q) \;\approx\; 2 \int \varepsilon(\alpha p_0 - q_0)\, \delta[(\alpha p - q)^2]\, F(\alpha) d\alpha \tag{I.16}$$

In the DIS limit $q_0 \to -\infty$ and with $p^2/q^2 \ll 1$,

$$\begin{aligned} w(p, q) \;&\approx\; 2 \int \delta(2\alpha p \cdot q - q^2) F(\alpha)\, d\alpha \\ &\approx\; \frac{1}{m\nu} F(x_B) \end{aligned} \tag{I.17}$$

One thus derives the scaling relation of the quark–parton model from the free-field singularities, and details of the structure, of the commutator of the currents on the light cone.

With local currents constructed out of bilinear combinations of quark fields, one first separates the points in the quark fields and introduces the notion of bilocal operators when evaluating the required current commutators [De73]. To quote from [De73], … "The important lesson we learn … is that the behavior of the structure function in the inelastic region is strictly related to the light-cone behavior of the commutator of the currents. The nature of the commutator singularity at $x_+ = 0$ determines the precise nature of the scaling, while the scaling function can be expressed as the Fourier transform of $g(\sigma)$, which in turn is related to the matrix element of a bilocal operator."

The idea of using the commutation relations of free-quark currents on the light cone to derive the DIS quark–parton results is due to Fritzsch and Gell-Mann [Fr71, De73]. It is Wilson's operator product expansion

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that provides a systematic way of looking at the short-distance behavior of a field theory [Wi69].

With the asymptotically-free theory QCD, one can justify the use of the free-field results at very short distances.[2] One can then proceed to calculate *corrections* to these free-field results. A useful way to proceed is to make use of the analysis in chapter 14 to rewrite the expression in Eq. (I.8). First, introduce a scattering amplitude analogous to that for forward virtual Compton scattering

$$a(p,q) \equiv \frac{i}{2\pi}\int e^{iq\cdot z}(p|P[j(z),j(0)]|p)\,d^4z \tag{I.18}$$

Here P denotes the time-ordered product

$$P[j(z),j(0)] \equiv j(z)j(0)\theta(z_0) + j(0)j(z)\theta(-z_0) \tag{I.19}$$

This expression is immediately analyzed in terms of Feynman diagrams [Fe71]; the necessary Feynman rules for QCD are given in chapter 25. Insertion of a complete set of states and explicit evaluation of the integrals in Eq. (I.18), with the inclusion of an adiabatic damping factor for convergence in the time integrals, leads to the Low equation for the scattering amplitude

$$\begin{aligned} a(p,q) &= \frac{1}{\pi}\sum_f (2\pi)^3\left[\frac{\delta^{(3)}(\mathbf{q}+\mathbf{p}'-\mathbf{p})}{q_0+p_0'-p_0-i\eta} - \frac{\delta^{(3)}(\mathbf{q}-\mathbf{p}'+\mathbf{p})}{q_0-p_0'+p_0+i\eta}\right] \\ &\quad\times\langle p|j(0)|p'\rangle\langle p'|j(0)|p\rangle(\Omega E) \end{aligned} \tag{I.20}$$

Now take the imaginary part of this expression.[3] As in chapter 14, the second term does not contribute by the stability of the target, and

$$\operatorname{Im} a(p,q) = \sum_f (2\pi)^3\delta^{(4)}(q+p'-p)\langle p|j(0)|p'\rangle\langle p'|j(0)|p\rangle(\Omega E) \tag{I.21}$$

The right side is recognized as the analog of Eq. (14.8) for the simplified response function in Eq. (I.8), and therefore

$$\operatorname{Im} a(p,q) = w(p,q) \tag{I.22}$$

Thus by taking the imaginary part of the scattering amplitude written in terms of Feynman diagrams, one can evaluate the response function in DIS.

The quark–parton result for the DIS response function in the impulse approximation in the $\mathbf{p}\to\infty$ frame is derived in chapter 12; it is evidently

[2] Indeed, the non-abelian gauge theory QCD was originally developed to do just that!

[3] More generally, take the *absorptive* part.

obtained by considering the imaginary part of the scattering diagram where the scattering takes place from a single non-interacting quark in the target (the so-called *handbag* diagram). The probability of finding such a quark in the target, $F(x_B)$, still depends on the strong-coupling aspects of the theory. From the above analysis, this result is equivalent to keeping the contribution of the singularities of the free-quark commutator on the light cone, with an amplitude $g(\sigma)$ again determined by the dynamics.

By considering additional Feynman diagrams, with radiative corrections, one can obtain perturbation-theory corrections to the response function of DIS. The evolution equations then allow one to obtain renormalization-group-improved results [Al77, Ch84, Ro90, Wa95].

The topics of operator product expansion, QCD radiative corrections, and evolution equations are explored in many texts (e.g. [Ch84]). In particular, the reader is referred to [Ro90] for an extensive discussion of the current theory of DIS scattering from the proton (with a summary of experimental results). Hopefully, the present text and this appendix will make that discussion more meaningful.