Brosche's method for representing systematic differences in positions and proper motions of stars.

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Abstract

A variant of Brosche's well-known method is proposed which allows one to represent systematic differences (many times faster than with the original method) by the use spherical harmonics. The improved economy of computer memory requirements and reduction of calculating time are achieved by replacing the two-arguments approximation with a sequence of one-argument approximations, and by enforcing an equidistant distribution of the initial differences along the α and the δ directions. The proposed method was tested on models and used for representing the systematic corrections $\Delta\delta_{\alpha}$ and $\Delta\mu_{\alpha}$ to the catalogues GC and N30.

The systematic differences between the data listed in the two catalogues are used for reducing the positions and proper motions of the stars in one of the catalogues to the system of the other. These systematic differences are obtained from individual differences by some smoothing procedure which separates the systematic from the random parts. In the classical approach, the systematic differences are split into several components (for example $\Delta\alpha_{\delta}$, $\Delta\alpha_{\alpha}$ and $\Delta\alpha_{m}$ for right ascension differences $\Delta\alpha$) each of which, being dependent on one or two arguments, is tabulated at equidistant intervals of α and δ. B. Boss and later H. Morgan were the first to use an analytical representation for systematic differences. In the construction of the GC and the N 30 catalogues, respectively, they modeled the $\Delta\alpha_{\alpha}$, $\Delta\delta_{\alpha}$, $\Delta\mu_{\alpha}$ and $\Delta\mu'_{\alpha}$ components by trigonometric expressions in right ascension. As a rule their dependence on declination had been ignored.

Brosche (1966) improved substantially Boss' and Morgan's approach by modeling the systematic differences by means of spherical harmonics, which, being dependent on both coordinates, achieve a more flexible representation of the systematic differences as functions not only of right ascension but of declination as well. Brosche's method was widely used in astrometry during the past few years (Bien et al. 1978, Kurjanova 1972, Schwan 1977, Zverev et al. 1980).

In essence, Brosche models the systematic part $f(\alpha,\delta)$ by the expression

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$$f(\alpha,\delta) = \sum_{j=o}^{g} b_j \, K_j(\alpha,\delta) + \varepsilon(\alpha,\delta)$$

where Kj (α,δ) are the spherical harmonics, as follows:

$$K_j \; (\alpha,\delta) = \begin{cases} P_{n,o}(\delta) & , \; k=0, \; 1=1 \\ P_{nk} \; (\delta) \; \sin k\alpha, & k\neq 0, \; 1=0 \\ P_{nk} \; (\delta) \; \cos k\alpha, & k\neq 0, \; 1=1. \end{cases}$$

The $P_{nk}(\delta)$ are, of course, the associated Legendre polynomials.

$$|| K_j ||^2 = \begin{matrix} 2\pi \; || P_{nk} ||_\delta^2, & k = 0, \\ \pi \; || P_{nk} ||_\delta^2, & k = 0. \end{matrix}$$

The norms of Legendre polynomials are given explicitly by

$$|| P_{nk} ||_\delta^2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{nk}^2 \; (\delta) \; \cos \; \delta d\delta = \frac{2(n-k)!(n+k)!}{2n+1} \left[\frac{2^n \; n!}{(2n)!} \right]^2 .$$

Since the spherical functions are orthogonal, the coefficients b_j are given by

$$b_j = \frac{(f, K_j)}{|| K_j ||^2} \quad , \quad j = 0, 1, \ldots , g.$$

For the square of the residual term

$$S = || \varepsilon ||^2 = \int_o^{2\pi} d\alpha \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(\alpha,\delta) - \sum_{j=0}^{g} b_j \; K_j \left[(\alpha,\delta) \right]^2 \cos\delta d\delta$$

one has

$$S = || f ||^2 - \sum_{j=0}^{g} b_j^2 \; || K_j ||^2.$$

Those are, in principle, the mathematical foundations of Brosche's method. Formula (8) cannot be used directly for the computation of the b_j because the

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stars are not sufficiently uniformly distributed in the catalogues. To remedy this, Brosche proposed to orthogonalize the vectors $K_j(\alpha_i, \delta_i)$ and then to compute with respect to this new basis some auxiliary coefficients from which the b_j may be calculated. All these procedures render the basically simple mathematical idea complicated and unwieldly, requiring powerful computers with a larger storage capacity.

In this paper, we propose a version of the method which uses spherical harmonics for a fast approximation of the function $f(\alpha, \delta)$. The principal idea of this aproach is based on the reduction of the two-arguments-approximation to the standard technique of one-argument-approximation.

Brosche's expression (1) is basically a trigonometric polynomial

$$f(\alpha, \delta) = \phi(\delta) + \sum_{k=1}^{M} \left[A_k(\delta) \sin K\alpha + C_k(\delta) \cos k\alpha\right] + \varepsilon(\alpha, \delta),$$

where

$$\phi(\delta) = \sum_{n=0}^{N} b_{n^2} P_{n,o}(\delta) + \varepsilon_\phi(\delta),$$

$$a_k(\delta) = \sum_{n=k}^{N_a} b_{n^2+2k-1} P_{nk}(\delta) + \varepsilon_{ak}(\delta),$$

$$c_k(\delta) = \sum_{n=k}^{N_c} b_{n^2+2k} P_{nk}(\delta) + \varepsilon_{ck}(\delta),$$

$$k = 1, 2, \ldots, M.$$

Here, $\varepsilon_\phi(\delta)$, $\varepsilon_{a_k}(\delta)$, $\varepsilon_{c_k}(\delta)$ are the residuals after representing the functions $\phi(\delta)$, $a_k(\delta)$, $c_k(\delta)$ by Legendre polynomials. From these formulas it is immediately obvious that the desired b_j can be obtained by a two-step procedure of which each step is the customary one-dimensial expansion.

Assume that the $p \times q$ values of the initial function are given at the centers of the spherical trapezia $(24^h/p) \times (180°/q)$. The Fourier coefficients $a_k(\delta)$ and $c_k(\delta)$ in (11) can then be computed by

$$\phi(\delta) = \frac{1}{p} \sum_{i=o}^{p-1} f(\alpha_i, \delta) \quad ,$$

$$a_k(\delta) = \frac{2}{p}\sum_{i=o}^{p-1} f(\alpha_i,\delta)\sin k\alpha_i,$$

$$c_k(\delta) = \frac{2}{p}\sum_{i=o}^{p-1} f(\alpha_i,\delta)\cos k\alpha_i,$$

where

$$\alpha_i = \frac{2\pi}{p}\left(i+\frac{1}{2}\right),$$
$$i = 0,\ 1,\ \ldots,\ p-1$$
$$k = 0,\ 1,\ \ldots,\ M,$$

assuming that within a narrow zone bounded by two parallels of declination, the dependence of $f(\alpha,\delta)$ on δ is not strong, the values of the computed Fourier coefficients may be regarded as valid for the centers of these zones. The square of the approximation norm

$$S(\delta) = ||\ \varepsilon\ ||^2_\alpha = \int_o^{2\pi} \varepsilon^2(\alpha,\delta)d\alpha$$

in each zone of declination is defined by the expression

$$S(\delta) = ||\ f\ ||^2_\alpha - 2\pi\ \phi^2(\delta) - \pi\sum_{k=1}^{M}\left[a_k^2(\delta) + c_k^2(\delta)\right],$$

thus we get for the root-mean-square errors of the Fourier coefficients

$$\sigma_\phi(\delta) = \frac{\sigma(\delta)}{\sqrt{2\pi}},\quad \sigma_{a_k,c_k} = \frac{\sigma(\delta)}{\sqrt{\pi}},$$

where

$$\sigma(\delta) = \sqrt{\frac{p}{2\pi}\ \frac{S(\delta)}{(P-2M-1)}}.$$

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Evaluation of Eqs. (15-21) yields estimates of the alues of the Fourier coefficients $\phi(\delta_j)$, $a_k(\delta_j)$, $c_k(\delta_j)$ and of their errors $\sigma(\delta_j)$, $\sigma_\phi(\delta_j)$, $\sigma_{a_k, c_k}(\delta_j)$ at the points

$$\delta_j = \frac{\pi}{2} - \frac{\pi}{q}\left(j+\frac{1}{2}\right) , \qquad j = 0, 1, \ldots , q - 1.$$

As the second step, we determine the coefficients b_i by developing the just obtained Fourier coefficients in Legendre polynomials as functions of declination. The associated Legendre polynomials are orthogonal in the sense of the relationship

$$(P_{nk}, P_{n'k}) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} P_{nk}(\delta)\, P_{n'k}(\delta) \cos\delta\, d\delta = \delta_{n,n'}\, ||P_{nk}||^2_\delta ,$$

and therefore, we get directly

$$b_{n^2} = \frac{\pi}{q\, ||P_{n,o}||^2_\delta} \sum_{j=o}^{q-1} \phi(\delta_j) P_{n,o}(\delta_j) \cos\delta_j ,$$

$$b_{n^2+2k-1} = \frac{\pi}{q||P_{nk}||^2_\delta} \sum_{j=o}^{q-1} a_k(\delta_j)\, P_{nk}(\delta_j) \cos\delta_j ,$$

$$b_{n^2+2k} = \frac{\pi}{q||P_{nk}||^2_\delta} \sum_{j=o}^{q-1} c_k(\delta_j) P_{nk}(\delta_j) \cos\delta_j .$$

The squares of residuals norms of each harmonic are

$$I_\phi = ||\varepsilon_\phi||^2_\delta = ||\phi||^2_\delta - \sum_{n=o}^{N} b^2_{n^2}\, ||P_{n,o}||^2_\delta ,$$

$$I_{a_k} = ||\varepsilon_{a_k}||^2_\delta = ||a_k||^2_\delta - \sum_{n-k}^{N_a} b^2_{n^2+2k-1}\, ||P_{nk}||^2_\delta ,$$

$$I_{c_k} = ||\varepsilon_{c_k}||^2_\delta = ||c_k||^2_\delta - \sum_{n=k}^{N_a} b^2_{n^2+2k} ||P_{nk}||^2_\delta .$$

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For the root-mean-square errors of the b_j with respect to δ-approximation we have

$$\sigma^{*}_{b_{n^2}} = \frac{1}{||P_{n,o}||_\delta} \sqrt{\frac{q}{\pi} \frac{I_\phi}{(q - N - 1)}} ,$$

$$\sigma^{*}_{b_{n^2+2k-1}} = \frac{1}{||P_{nk}||_\delta} \sqrt{\frac{q}{\pi} \frac{I_{a_k}}{(q - N_a - K)}} ,$$

$$\sigma^{*}_{b_{n^2+2k}} = \frac{1}{||P_{nk}||_\delta} \sqrt{\frac{q}{\pi} \frac{I_{c_k}}{(q - N_c - k)}}$$

This concludes the representation of the systematic differences $f(\alpha,\delta)$.

In order to calculate root-mean-square errors of the total representation, with respect to the both coordinates we multiply Eq. (19) by $\cos\delta$ and integrate with respect to δ from $-\pi/2$ to $\pi/2$. Taking into consideration Eqs. (6) and (10) we find

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(\delta)\cos\delta\, d\delta + 2\pi\, I_\phi + \pi \sum_{k=1}^{M} (I_{a_k} + I_{c_k}) \approx$$

$$\approx \frac{\pi}{q} \sum_{j=o}^{q-1} S(\delta_j)\, \cos\delta_j + 2\pi\, I_\phi + \pi \sum_{k=1}^{M} (I_{A_k} + I_{C_k}) .$$

This expression establishes a relationship between the norm $S = ||\varepsilon||^2$ of the residual term by Eq. (1) with the squares of the norms of the residuals in Eqs. (11-14). Considering Eq. (6) for the b_j, the root-mean-square error with respect to the two-coordinates-approximation is given by

$$\sigma_{b_j} = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{\sigma}{||P_{n,o}||_\delta} , & k = 0 \\ \frac{1}{\sqrt{\pi}} \frac{\sigma}{||P_{nk}||_\delta} , & k \neq 0, \end{cases}$$

where

$$\sigma^2 = \frac{pq}{2\pi^2} \quad \frac{S}{(pq - G)} \quad ,$$

and G is the total number of terms in Eqs. (11) - (14)..

We now consider the criteria for significance and retention of the highest order terms in the expansions Eqs. (11)-(14), which formalize the split of the systematic differences into systematic and random components. $|M|$ in Eq. (11) must be below the Nyquist limit since the p - the (even) number of divisions on the equator yields $M < p/2$. After the Fourier coefficients are computed one has to retain for further analysis only those which significantly depend on declination. The terms N, N_a, N_c in Eqs. (12)-(14) can be chosen to some statistical principles. Using, for example, the χ^2-criterion, the succesive expansion following Eqs. (14)-(26) is to be continued until

$$\sum_{j=o}^{q-1} \frac{\varepsilon_\phi^2(\delta_j)}{\sigma_\phi^2(\delta_j)} \leq q - N ,$$

$$\sum_{j=o}^{q-1} \frac{\varepsilon_{a_k}^2(\delta_j)}{\sigma_{a_k}^2(\delta_j)} \leq q - (N_a - k) ,$$

$$\sum_{j=o}^{q-1} \frac{\varepsilon_{c_k}^2(\delta_j)}{\sigma_{c_k}^2(\delta_j)} \leq q - (N_c - k) ,$$

$$k = 1, 2, \ldots , M$$

Another possible way to establish the highest order terms is the evaluation of the dispersion changes according to the F-criterion after succesive determination of the coefficients b_j with the help of Eqs. (24)-(26).

The algorithm described was tested on models and used for the representation of the systematic corrections of the type $\Delta\alpha_\alpha$ and $\Delta\mu_\alpha$ between the fundamental catalogues GC and N30 (V.V.Vityazev and E.V.Vityazeva 1984). The new procedure simplifies Brosche's original method substantially. It requires only about one tenth the storage capacity as well as one tenth the calculating time.

It is easy to see that according to Eqs. (12-14) one can use not only the associated Legendre polynomials but any other set of orthogonal functions (Legendre polynomials, trigonometric functions and so on). For the catalogues distributed stars, but only to those catalogues where the distribution of stars is somehow judged to be uniform in the zone which they cover. This is, for example, indeed the case for the catalogue PFKSZ-2. If the distribution of stars is uneven along the α and the δ directions, the method may be used provided it is preceded by suitable averaging which musy pay careful attention to choosing the averaging intervals properly, to calculate the response function of the filter by means of which the Fourier coefficients are to be restored before the b_j are calculated.

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