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SOME RADICAL PROPERTIES OF JORDAN MATRIX RINGS

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Introduction. Let A be a ring (not necessarily associative) in which $2x = a$ has a unique solution for each $a \in A$. Then it is known that if A contains an identity element 1 and an involution $j : x \mapsto \bar{x}$ and if J_a is the canonical involution on A_n determined by

$$a = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_n \end{pmatrix}$$

(i.e., $J_a : (x_{ij}) \mapsto a^{-1}(\bar{x}_{ij})a$) where the a_i, a_i^{-1}, $1 \leqq i \leqq n$ are symmetric elements in the nucleus of A then $H(A_n, J_a)$, the set of symmetric elements of A_n, for $n \geqq 3$ is a Jordan ring if and only if either A is associative or $n = 3$ and A is an alternative ring whose symmetric elements lie in its nucleus [**2**, p. 127]. In this paper we show that for certain radicals there is a natural connection between the radical of A and that of $H(A_n, J_a)$. In particular, if R denotes the prime or Levitzki radical then $R(H(A_n, J_a)) = H(A_n, J_a) \cap R(A)_n$. Also, if A is 3-torsion free then the same result holds for the strongly semiprime radical. As usual, the associator (x, y, z) denotes $(xy)z - x(yz)$ and the commutator $[x, y]$ denotes $xy - yx$. With this notation a ring A is *alternative* if $(y, x, x) = (x, x, y) = 0$ for all x, y in A and *Jordan* if $[x, y] = (x^2, y, x) = 0$ for all x, y in A. The *nucleus*, $N(A)$, of an arbitrary ring A is defined by

$$N(A) = \{n \in A | (n, x, y) = (x, n, y) = (x, y, n) = 0 \,\forall\, x, y \in A\}.$$

Recall that if A is an alternative ring then the Moufang laws

(1) $[(ax)y]x = a(xyx)$

(2) $x[y(xa)] = (xyx)a$

(3) $(xa)(yx) = x(ay)x$

hold for all x, y, a in A.

We shall rely heavily on the fact that if $H(A_n, J_a)$, $n \geqq 3$, is Jordan then there is a one-to-one correspondance between the j-invariant ideals I of A and the ideals of $H(A_n, J_a)$ given by $I \mapsto I_n \cap H(A_n, J_a)$. Also an ideal $K = I_n \cap H(A_n, J_a)$ of $H(A_n, J_a)$ satisfies $K^{\cdot 2} = 0$ if and only if $I^2 = 0$ [**2**, p. 129] ($K^{\cdot n}$ denotes all sums of monomials of degree $\geqq n$ in the Jordan ring K). It is also clear from the argument in [**2**] that $K^{\cdot 3} = 0$ if and only if $I^3 = 0$.

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1. The prime radical P. If A is an associative ring then it is well known that $P(A_n) = P(A)_n$. In this case J_a is an involution acting on the associative ring A_n so that by [**1**] we have $P(H(A_n, J_a)) = H(A_n, J_a) \cap P(A_n)$. Thus we have $P(H(A_n, J_a)) = H(A_n, J_a) \cap P(A)_n$ if $n > 3$. In this section we prove the same result for $n = 3$; i.e., when A is an alternative ring with identity whose symmetric elements lie in its nucleus.

LEMMA 1. *If A is an alternative ring with involution j, then A is semiprime if and only if A is j-semiprime.*

Proof. Clearly, if A is semiprime then it is j-semiprime. Conversely, if A is j-semiprime then it has no nilpotent j-invariant ideals. If A is not semiprime then it contains an ideal $I \neq 0$ such that $I^2 = 0$. Then $I + I^j$ is a j invariant ideal of A and $(I + I^j)^2(I + I^j)^2 = 0$. Since squares of ideals are ideals we have $(I + I^j)^2 = 0$ which implies that $I + I^j = 0$. Thus, I is zero, a contradiction.

The following lemma follows easily from Lemma 1 and the one to one correspondence between Jordan ideals of $H(A_n, J_a)$ which cube to zero and ideals of A which cube to zero.

LEMMA 2. *If $H(A_n, J_a)$ is a Jordan ring, then $H(A_n, J_a)$ is semiprime if and only if A is semiprime.*

THEOREM 1. *If A is a ring with identity and J_a a canonical involution on A_n for $n \geqq 3$ such that $H(A_n, J_a)$ is a Jordan ring, then*

$$P(H(A_n, J_a)) = H(A_n, J_a) \cap P(A)_n.$$

Proof. As mentioned earlier we need only concern ourselves with the case $n = 3$ for which A is an alternative ring with involution j whose symmetric elements lie in its nucleus. Now, for any ideal K of A, $(A/K)_3 \cong A_3/K_3$. Therefore, if K is a j-invariant ideal of A then the involution j determines a natural involution on A/K and since A is 2-torsion free we have

$$(4) \quad H((A/K)_3, J_a) \cong H(A_3/K_3, J_a) \cong H/(H \cap K_3)$$

where H denotes $H(A_3, J_a)$ for convenience. Let $K = P(A)$. Since $A/P(A)$ is semiprime we conclude from Lemma 2 that $H(A_3/P(A)_3, J_a)$ and hence $H/(H \cap P(A)_3)$ is a semiprime Jordan ring. But this implies that $P(H) \subseteq H \cap P(A)_3$.

Conversely, by the $1 - 1$ correspondence between ideals of H and j-invariant ideals of A we may assume that $P(H) = H \cap B_3$ for some j-invariant ideal B of A. Then by (4) we have

$$0 = P(H/(H \cap B_3)) = P(H((A/B)_3, J_a)).$$

Therefore, by Lemma 2 $P(A/B) = 0$ from which it follows that $P(A) \subseteq B$. Thus $P(A_3) \cap H \subseteq B_3 \cap H = P(H)$ to complete the proof.

2. The strongly semiprime radical SP. An element x of an alternative ring A is called an *absolute zero divisor* if $xAx = 0$. Similarly an element x of a Jordan ring J is called an *absolute zero divisor* if $JU_x = 0$ where $U_x = 2R_x^2 - R_{x^2}$ for R_x the multiplication operator in J, $aR_x = ax$. An ideal B of a ring R (alternative, Jordan) is called *strongly semiprime* if R/B contains no absolute zero divisors. The strongly semiprime radical, $SP(R)$, of R is the intersection of all the strongly semiprime ideals of R. Clearly $R/(SP(R))$ is strongly semiprime. If R is associative then $P(R) = SP(R)$. If R is Jordan then $P(R) \subseteq SP(R)$ [**1**] and if R is 3-torsion free alternative then $P(R) = SP(R)$[**4**]. Finally it is shown in [**4**] that if R is a 2 and 3-torsion free alternative ring with involution and S is the Jordan ring of symmetric elements, then $P(S) = S \cap P(R)$.

If A is an associative ring then by [**1**] we have $SP(H(A_n, J_a)) = H(A_n, J_a) \cap SP(A_n)$. Thus $SP(H(A_n, J_a)) = H(A_n, J_a) \cap SP(A)_n$. Hence, if $n > 3$ and $H = H(A_n, J_a)$ is Jordan then $SP(H) = H \cap SP(A)_n$. We shall extend this to the case in which A is a 3-torsion free alternative ring with identity. Thus, throughout this section we assume that A is 3-torsion free and that $H(A_3, J_a)$ is a Jordan ring. Hence A is alternative with 1 with symmetric elements in the nucleus.

LEMMA 3. *If A is a 3-torsion free alternative ring then* $SP(H(A_3, J_a)) \supseteq H(A_3, J_a) \cap SP(A)_3$.

Proof. By our earlier remarks $P(H) \subseteq SP(H)$ and since A is 3-torsion free, $P(A) = SP(A)$. By theorem 1, $P(H) = H \cap P(A)_3$. Putting these facts together we have

$$SP(H) \supseteq P(H) = H \cap P(A)_3 = H \cap SP(A)_3.$$

We shall prove the inverse inclusion to Lemma 3 by a series of lemmas.

LEMMA 4. *If A strongly semiprime and $X = (x_{ij})$; $i, j = 1, 2, 3$ is an element of $H(A_3, J_a)$ such that $H(A_3, J_a)U_X = 0$ then $x_{ii} = 0$ for $i = 1, 2, 3$.*

Proof. Let $s \in S$, the set of symmetric elements of A. Then

$$\widehat{sa}_1 = \begin{pmatrix} sa_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in H(A_3, J_a)$$

and by hypothesis $\widehat{sa}_1 U_X = 0$. A direct calculation shows that the (1, 1) component of $\widehat{sa}_1 U_X$ is

$$(sa_1)U_{x_{11}} + \tfrac{1}{2}[(sa_1, x_{12}, x_{21}) - (x_{12}, x_{21}, sa_1)] + \tfrac{1}{2}[(sa_1, x_{13}, x_{31}) - (x_{13}, x_{31}, sa_1)].$$

Since A is alternative (or since $sa_1 \in N(A)$) each of the last two terms is zero. Therefore $(sa_1)U_{x_{11}} = 0$. Also, since $X \in H(A_3, J_a)$ it follows that $X = Ya$ where $Y = (y_{ij}) \in H(A_3)$, the set of symmetric elements of A_3 under

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the standard involution (i.e., $a = I$) [**2**, p. 60]. Therefore $x_{11} = y_{11}a_1$ and $(sa_1)U_{y_{11}a_1} = 0$. Consider the involution I_{a_1} on A where $I_{a_1} : x \mapsto a_1^{-1}\bar{x}a_1$. Then the set T of symmetric elements of A under the involution I_{a_1} is given by $T = \{sa_1 | s \in S\}$. Thus $TU_{y_{11}a_1} = 0$. Since $y_{11} \in S$, $y_{11}a_1 \in T$. Therefore, $y_{11}a_1$ is an absolute zero divisor of the Jordan ring T. But by [**4**]

$$SP(T) = T \cap SP(A).$$

Since A is strongly semiprime, it follows that $SP(T) = 0$. Therefore $y_{11}a_1 = x_{11} = 0$. In similar fashion by letting U_X act on $\hat{\hat{sa}}_2 = (sa_2)e_{22}$ and on $\hat{\hat{sa}}_3 = (sa_3)e_{33}$ and by considering the involutions I_{a_2} and I_{a_3} we may conclude that $x_{22} = x_{33} = 0$.

It follows from our proof that any element X of $H(A_3, J_a)$ which is an absolute zero divisor is of the form

$$X = \begin{pmatrix} 0 & y_{12}a_2 & y_{13}a_3 \\ \bar{y}_{12}a_1 & 0 & y_{23}a_3 \\ \bar{y}_{13}a_1 & \bar{y}_{23}a_2 & 0 \end{pmatrix}$$

LEMMA 5. *If A is strongly semiprime and*

$$X = \begin{pmatrix} 0 & y_{12}a_2 & 0 \\ \bar{y}_{12}a_1 & 0 & y_{23}a_3 \\ \bar{y}_{13}a_1 & \bar{y}_{23}a_2 & 0 \end{pmatrix}$$

is an absolute zero divisor of $H(A_3, J_a)$ then $y_{ij}S\bar{y}_{ij} = 0$ for $i, j = 1, 2, 3$.

Proof. Since $N(A)$ is a subring of A, $sa_i \in N(A)$ for every $s \in S$. Thus, it follows that the (2, 2) component of $\hat{sa}_1 \; U_X$ can be written as $2\bar{y}_{12}a_1sa_1y_{12}a_2$ and its (3, 3) component as $2\bar{y}_{13}a_1sa_1y_{13}a_3$. Also the (3, 3) component of $\hat{\hat{sa}}_2 \; U_X$ is $2\bar{y}_{23}a_2sa_2y_{23}a_3$. Therefore, since A is 2-torsion free we have $\bar{y}_{ij}a_isa_iy_{ij}a_j = 0$ for all $i \leqq j$. Since a_j is invertible this reduces to $\bar{y}_{ij}a_isa_iy_{ij} = 0$. If we multiply on the left by $a_iy_{ij}s'$ and on the right by $s'\bar{y}_{ij}a_i$ for any $s' \in S$ we obtain $(a_iy_{ij}s'\bar{y}_{ij}a_i)s(a_iy_{ij}s'\bar{y}_{ij}a_i) = 0$. Now $a_iy_{ij}s'\bar{y}_{ij}a_i \in S$. Therefore it is an absolute zero divisor of S. But S is strongly semiprime since A is. Therefore we have $a_iy_{ij}S\bar{y}_{ij}a_i = 0$ for all $i \leqq j$. Since a_i is invertible it follows that $y_{ij}S\bar{y}_{ij} = 0$ for all $i \leqq j$. Continuing, we obtain $\bar{y}_{ij}sy_{ij}S\bar{y}_{ij}sy_{ij} = 0$ for any $s \in S$. Thus since $\bar{y}_{ij}sy_{ij} \in S$ it follows that $\bar{y}_{ij}Sy_{ij} = 0$ for all $i \leqq j$. But since $\bar{y}_{ij} = y_{ji}$ we have $y_{ji}S\bar{y}_{ji} = 0$ for all $i \leqq j$. Thus, in all cases $y_{ij}S\bar{y}_{ij} = 0$.

LEMMA 6. *Under the hypothesis of Lemma 5, $y_{ij}Sy_{ij} = y_{ij} + \bar{y}_{ij} = 0$ for $i, j = 1, 2, 3$.*

Proof. We first show that $y_{ij}Sy_{ij} = 0$ for all i, j. By hypothesis

$$[(xa_2)e_{12} + (\bar{x}a_1)e_{21}]U_X = 0$$

since for any x in A $(xa_2)e_{12} + (\bar{x}a_1)e_{21} \in H(A_3, J_1)$. Thus the (2, 1) component of $[(xa_2)e_{12} + (\bar{x}a_1)e_{21}]U_X$ is zero for each x in A. Now, since $y_{ij}S\bar{y}_{ij} = 0$

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for all i, j and since $a_i \in S$ a straightforward computation gives

$$(5)\quad [(\bar{x}a_1)(y_{12}a_2) + (\bar{y}_{12}a_1)(xa_2)](\bar{y}_{12}a_1) + (\bar{y}_{12}a_1)[(xa_2)(\bar{y}_{12}a_1) + (y_{12}a_2)(\bar{x}a_1)] + [(\bar{x}a_1)(y_{13}a_3)](\bar{y}_{13}a_1) + (y_{23}a_3)[(\bar{y}_{23}a_2)(\bar{x}a_1)] = 0.$$

Now, let $x = a_1^{-1}sa_2^{-1}$ for $s \in S$. Then $x \in N(A)$ and since $y_{ij}S\bar{y}_{ij} = 0$ the (2, 1) component reduces to $2\bar{y}_{12}s\bar{y}_{12}a_1 = 0$. Thus, $\bar{y}_{12}s\bar{y}_{12} = 0$ for each $s \in S$ so that $\bar{y}_{12}S\bar{y}_{12} = 0 = y_{12}Sy_{12}$. In similar fashion by considering

$$[(xa_3)e_{13} + (\bar{x}a_1)e_{31}]U_X$$

and $[(xa_3)e_{23} + (\bar{x}a_2)e_{32}]U_X$ we get $y_{ij}Sy_{ij} = 0$ for $i, j = 1, 2, 3$.

For the second part of the lemma consider

$$SU_{y_{ij}+\bar{y}_{ij}} = SU_{y_{ij},y_{ij}} + SU_{y_{ij}} + SU_{\bar{y}_{ij}} = y_{ij}S\bar{y}_{ij} + \bar{y}_{ij}Sy_{ij} + \bar{y}_{ij}S\bar{y}_{ij} + y_{ij}Sy_{ij} = 0$$

by our previous result and Lemma 5. Since $y_{ij} + \bar{y}_{ij} \in S$, if $y_{ij} + \bar{y}_{ij} \neq 0$ we would have a contradiction to the fact that A, and consequently S, is strongly semiprime. Therefore $y_{ij} + \bar{y}_{ij} = 0$ for all i, j.

We are now able to prove the main theorem of this section.

THEOREM 2. *If A is a 3-torsion free ring with identity and J_a a canonical involution on A_n for $n \geqq 3$ such that $H(A_n, J_a)$ is a Jordan ring, then $SP(H(A_n, J_a)) = H(A_n, J_a) \cap SP(A)_3$.*

Proof. If $n > 3$ then A is associative and we are done as mentioned earlier. Assume now that $n = 3$ so that A is an alternative ring. In view of Lemma 3 it is sufficient to prove that $SP(H(A_3, J_a)) \subseteq H(A_3, J_a) \cap SP(A)_3$. We first establish the result in the case in which A is strongly semiprime. In this case, if $H(A_3, J_a)$ is not strongly semiprime then there is an element $0 \neq X \in H(A_3, J_a)$ such that $H(A_3, J_a)U_X = 0$. Then the results of Lemmas 4, 5, and 6 apply to

$$X = \begin{pmatrix} 0 & y_{12}a_2 & y_{13}a_3 \\ \bar{y}_{12}a_1 & 0 & y_{23}a_3 \\ \bar{y}_{13}a_1 & \bar{y}_{23}a_2 & 0 \end{pmatrix}.$$

By Lemmas 5 and 6, $y_{ij} + \bar{y}_{ij} = y_{ij}\bar{y}_{ij} = 0$. Thus $y_{ij}^2 = 0$ for all i, j. Since $\bar{y}_{ij} = -y_{ij}$, (5) reduces to:

$$(6)\quad [(y_{12}a_1)(xa_2) - (\bar{x}a_1)(y_{12}a_2)](y_{12}a_1) + (y_{12}a_1)[(xa_2)(y_{12}a_1) - (y_{12}a_2)(\bar{x}a_1)] - [(\bar{x}a_1)(y_{13}a_3)](y_{13}a_1) - (y_{23}a_3)[y_{23}a_2)(\bar{x}a_1)] = 0$$

for any $x \in A$.

In (6) consider the term $[(\bar{x}a_1)(y_{12}a_2)](y_{12}a_1) = [(((\bar{x}a_1)y_{12})a_2)y_{12}]a_1$ since $a_1, a_2 \in N(A)$. But by (1), $[(((\bar{x}a_1)y_{12})a_2)y_{12}]a_1 = [(\bar{x}a_1)(y_{12}a_2y_{12})]a_1$. But by Lemma 6, $y_{12}a_2y_{12} = 0$. Therefore $[(\bar{x}a_1)(y_{12}a_2)](y_{12}a_1) = 0$. Similarly

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$[(\bar{x}a_1)(y_{13}a_3)](y_{13}a_1) = 0$. Also $(y_{12}a_1)[(y_{12}a_2)(\bar{x}a_1)] = (y_{12}a_1)[y_{12}(a_2\bar{x}a_1)] = y_{12}[a_1[y_{12}(a_2\bar{x}a_1)]] = (y_{12}a_1y_{12})(a_2\bar{x}a_1)$ by (2). Therefore, by Lemma 6, $(y_{12}a_1)[(y_{12}a_2)(\bar{x}a_1)] = 0$. Similarly, $(y_{23}a_3)[(y_{23}a_2)(\bar{x}a_1)] = 0$. Thus, (6) reduces to $2y_{12}(a_1xa_2)y_{12}a_1 = 0$. Since A is 2-torsion free and a_1 and a_2 are invertible this becomes $y_{12}Ay_{12} = 0$ so that by hypothesis $y_{12} = 0$. In similar fashion we get $y_{ij} = 0$ for all i, j. Thus $X = 0$ and we have established that A strongly semiprime implies that $H(A_3, J_a)$ is strongly semiprime.

Assume now that A is not strongly semiprime. Since $A/(SP(A))$ is strongly semiprime it follows from our previous remark that $H((A/(SP(A)))_3, J_a)$ is strongly semiprime. But, as in the proof of theorem 1,

$$H((A/(SP(A)))_3, J_a) \cong H(A_3/(SP(A))_3, J_a) \cong H(A_3, J_a)/(H(A_3, J_a) \cap (SP(A)_3).$$

Therefore $H(A_3, J_a)/(H(A_3, J_a) \cap (SP(A)_3)$ is strongly semiprime. It follows from the definition of the strongly semiprime radical that $SP(H(A_3, J_a)) \subseteq H(A_3, J_a) \cap SP(A)_3$, completing the proof.

It is not known in general whether the prime radical and the strongly semiprime radical coincide for Jordan rings. In the case of a Jordan matrix ring, however, we have:

COROLLARY. *If $H(A_n, J_a)$, $n \geqq 3$, is a Jordan matrix ring determined by a 2 and 3-torsion free ring A with identity then $P(H(A_n, J_a)) = SP(H(A_n, J_a))$.*

Proof. If $n = 3$ then A is alternative and since A is 3-torsion free $P(A) = SP(A)$. Thus, $SP(H(A_3, J_a)) = H(A_3, J_a) \cap SP(A)_3 = H(A_3, J_a) \cap P(A)_3 = P(H(A_3, J_a))$ by Theorems 1 and 2. In case $n > 3$ then A is associative and the same proof works without the assumption of 3-torsion freeness.

3. The Levitzki radical L. Recall that a ring is called *locally nilpotent* if every finitely generated subring is nilpotent. The *Levitzki radical*, $L(A)$, of a ring A (associative, alternative, Jordan) is the maximal locally nilpotent ideal of A. $L(A)$ contains all locally nilpotent ideals of A and $A/L(A)$ is Levitzki semisimple. It is known that if A is a 2-torsion free associative ring with involution * and S is the set of *-symmetric elements of A then $L(S) = S \cap L(A)$ [**3**]. We first treat the easy case in which A is associative. In this case we need not assume that A contains an identity element.

LEMMA 7. *If A is an associative ring then $L(A_n) = L(A)_n$ for any positive integer n.*

Proof. $L(A)$ is a locally nilpotent ideal of A. Therefore, if C is a finitely generated subring of $L(A)_n$ generated by

$$M_1 = \sum_{i,j=1}^{n} r_{1ij}e_{ij}, \ldots, M_h = \sum_{i,j=1}^{n} r_{hij}e_{ij},$$

then the ring D generated by all the r_{tij}, $t = 1, \ldots, h$; $i, j = 1, \ldots, n$ is a finitely generated subring of $L(A)$. Therefore there is a positive integer m such that $D^m = 0$. But then $C^m = 0$ and $L(A)_n$ is locally nilpotent. Therefore $L(A)_n \subseteq L(A_n)$.

For the converse assume first that A contains an identity element. Then if

$$\sum_{i,j=1}^{n} b_{ij}e_{ij} \in L(A_n)$$

it is easy to see that $b_{ij}\, e_{ij} \in L(A_n)$ for every i, j. Therefore $e_{1i}(b_{ij}\, e_{ij})e_{j1} = b_{ij}\, e_{11} \in L(A_n)$ for every i, j. Therefore any finitely generated subring of $(b_{ij}\, e_{11})$, the ideal of A_n generated by $b_{ij}\, e_{11}$, is nilpotent. In particular, if $r_1, r_2, \ldots, r_k$ are elements of (b_{ij}), the ideal of A generated by b_{ij}, then the subring of A_n generated by $r_1e_{11}, r_2e_{11}, \ldots, r_ke_{11}$ is nilpotent. Therefore, the subring of A generated by $r_1, r_2, \ldots, r_k$ is nilpotent and (b_{ij}) is locally nilpotent. Therefore, $b_{ij} \in L(A)$ for every i, j. Hence $\sum b_{ij}e_{ij} \in L(A)_n$ and $L(A_n) \subseteq L(A)_n$.

If A does not contain an identity element then if we imbed A into a ring A' with 1 in the usual way then it is straightforward to see that $L(A) = A \cap L(A')$ and $L(A_n) = A_n \cap L(A'_n)$ (this is also true as a consequence of the fact that the Levitzki radical is hereditary on associative rings). Therefore $L(A_n) = A_n \cap L(A'_n) = A_n \cap L(A')_n = (A \cap L(A'))_n = L(A)_n$.

COROLLARY. *If A is an associative ring with involution j and $H = H(A_n, J_a)$ is a Jordan matrix ring determined by the canonical involution J_a, then $L(H) = H \cap L(A)_n$.*

Proof. A_n is an associative ring with involution J_a. Therefore by [**3**] $L(H) = H \cap L(A_n) = H \cap L(A)_n$.

If A is an arbitrary ring and $x_1, x_2, \ldots, x_n$ are elements of A, denote by $[x_1, x_2, \ldots, x_n]$ the subring of A generated by $x_1, x_2, \ldots, x_n$. If $H(A_3, J_a)$ is a Jordan matrix ring then denote by $J[x_1, x_2, \ldots, x_n]$ the Jordan subring of $H(A_3, J_a)$ generated by the elements $x_i[jk]$ for $i = 1, 2, \ldots, n$ and $j, k = 1, 2, 3$.

The following technical lemma will be useful in extending the previous result.

LEMMA 8. *Let A be an alternative ring and let $H(A_3, J_a)$ be a Jordan matrix ring. Then if M_k is a monomial of $[x_1, x_2, \ldots, x_n]$ of degree k it follows that $M_k[ij] \in J[x_1, \ldots, x_n]^{\cdot k}$ for $i \neq j$.*

Proof. We use the fact that if $x, y \in A$ and i, j, l are all different then $2x[ij] \cdot y[jl] = xy[il]$ and proceed by induction on k. If $k = 1$ the result is certainly true. Suppose true for any $s < k$. Now either $M_k = Mx_t$ or $M_k = x_tM$ for some x_t and a monomial M of degree $k - 1$ or $M_k = M_sM_t$ where $s < k$ and $t < k$. If $M_k = Mx_t$ then if i, j, and l are all different, $M_k[ij] = Mx_t[ij] = 2M[il] \cdot x[lj] \in J^{\cdot k-1} \cdot J = J^{\cdot k}$ by the induction hypothesis. Similarly if $M_k = x_tM$. Finally if $M_k = M_sM_t$ then $M[ij] = M_sM_t[ij] =$

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$2M_s[il] \cdot M_t[lj]$ for i, l, and j all different. By hypothesis $M_s \in J^{\cdot s}$ and $M_t \in J^{\cdot t}$. Therefore in all cases $M_k \in J[x_1, x_2, \ldots, x_n]^{\cdot k}$.

THEOREM 3. *If A is a ring with identity element and J a canonical involution on A_n, $n \geqq 3$, such that $H = H(A_n, J_a)$ is a Jordan ring, then $L(H(A_n, J_a)) = H(A_n, J_a) \cap L(A)_n$.*

Proof. If $n > 3$ then A is associative so the result is true by the corollary to Lemma 7. Suppose then that $n = 3$ so that A is alternative. It is apparent that $H \cap L(A)_n \subseteq L(H)$ as in the proof of Lemma 7. For the converse first note that $L(H) = H \cap B_n$ for some j-invariant ideal B of A. Also B is a locally nilpotent ideal of A. For if $x_1, x_2, \ldots, x_n$ are elements of B then $J[x_1, x_2, \ldots, x_n]$ is a finitely generated subring of $L(H)$. Hence

$$J[x_1, x_2, \ldots, x_n]^{\cdot k} = 0$$

for some k. Thus, by Lemma 8, if M_k is a monomial of $[x_1, x_2, \ldots, x_n]$ of degree k then $M_k = 0$. Hence $[x_1, x_2, \ldots, x_n]$ is nilpotent of degree $\leqq k$. Therefore B is locally nilpotent and $B \subseteq L(A)$. Hence, we get $L(H) \subseteq H \cap L(A)_n$ to complete the proof.

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