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OPTIMAL COUPLING OF JUMPY BROWNIAN MOTION ON THE CIRCLE

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Abstract

Consider a Brownian motion on the circumference of the unit circle, which jumps to the opposite point of the circumference at incident times of an independent Poisson process of rate λ. We examine the problem of coupling two copies of this 'jumpy Brownian motion' started from different locations, so as to optimise certain functions of the coupling time. We describe two intuitive co-adapted couplings ('Mirror' and 'Synchronous') which differ only when the two processes are directly opposite one another, and show that the question of which strategy is best depends upon the jump rate λ in a non-trivial way. We also provide an explicit description of a (non-co-adapted) maximal coupling for any jump rate in the case that the two jumpy Brownian motions begin at antipodal points of the circle.

Keywords: Co-adapted coupling; mirror and synchronous coupling; maximal coupling; stochastic control; Hamilton–Jacobi–Bellman equation

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1. Introduction

Consider a continuous-time stochastic process $X = (X_t)_{t\geq 0}$ on $(-\pi, \pi]$ given by

$$X_t = X_0 + \frac{1}{2}B_t + \pi N_t^{\lambda} \pmod{2\pi}, \tag{1.1}$$

where B_t is a standard $\mathbb{R}$-valued Brownian motion and N^{λ} is an independent Poisson process of rate $\lambda \geq 0$. (The factor of $\frac{1}{2}$ is introduced purely for algebraic convenience in what follows.) X is a Lévy process which can be viewed as a Brownian motion on the circumference of the unit circle which jumps to the antipodal point of the circle at incident times of N^{λ}. For this reason, we will refer to the process X as a *jumpy Brownian motion of rate* λ (JBM(λ)).

In this paper we are interested in couplings of two JBM(λ) processes started from different points of the circle. That is, we are interested in processes $(X, \hat{X})$ on $(-\pi, \pi]^2$ such that, viewed marginally, X and $\hat{X}$ each behave as a copy of JBM(λ). Given a coupling $(X, \hat{X})$, we define the *coupling time* by $T = \inf\{t \geq 0 : X_s = \hat{X}_s \text{ for all } s \geq t\}$. Recall that the tail distribution of any

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coupling time provides an upper bound on the total variation distance between the laws of X_t and $\hat{X}_t$ via the coupling inequality [1]

$$\|\mathcal{L}(X_t) - \mathcal{L}(\hat{X}_t)\|_{\mathrm{TV}} \leq \mathbb{P}(T > t), \tag{1.2}$$

where

$$\|\mathcal{L}(X_t) - \mathcal{L}(\hat{X}_t)\|_{\mathrm{TV}} = \sup_A \{\mathbb{P}(X_t \in A) - \mathbb{P}(\hat{X}_t \in A)\} \tag{1.3}$$

and where the supremum is taken over all Borel subsets of $(-\pi, \pi]$. A coupling is called *successful* if $\mathbb{P}(T > t) \to 0$ as $t \to \infty$; it is called *maximal* if it achieves equality in (1.2) for all times t.

It is well known that a maximal coupling of two random processes exists under extremely mild conditions: see [6,7,16] for discrete-time processes, and [18] for the case of càdlàg processes with Polish state-space. However, in most cases, explicit construction of a maximal coupling is extremely difficult, and it is natural for attention to focus on classes of couplings which are more readily realisable. One such class is that of *co-adapted* couplings.

Definition 1.1. A coupling $(X, \hat{X})$ is called *co-adapted* if there exists a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that X and $\hat{X}$ are both adapted to $(\mathcal{F}_t)_{t\geq 0}$ and, for any $0 \leq s \leq t$, $\mathcal{L}(X_t \mid \mathcal{F}_s) = \mathcal{L}(X_t \mid X_s)$ and $\mathcal{L}(\hat{X}_t \mid \mathcal{F}_s) = \mathcal{L}(\hat{X}_t \mid \hat{X}_s)$.

In other words, both X and $\hat{X}$ are Markov with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Kendall [11] refers to co-adapted couplings as *immersed*, since the condition of Definition 1.1 is equivalent to demanding that the natural filtrations of X and $\hat{X}$ are both immersed in a common filtration.

Even though maximal couplings certainly need not be co-adapted, there are a few processes for which maximal co-adapted couplings have been shown to exist. Probably the simplest of these is the reflection/mirror coupling of Euclidean Brownian motions, in which the path of one process, until the coupling time, is obtained by reflecting the other in the hyperplane bisecting the line joining their starting points [13]; indeed, this is the unique maximal co-adapted coupling [8], a result which holds more generally for Brownian motion on a Riemannian manifold [12]. More recently, [2] showed that a maximal co-adapted coupling for smooth elliptic diffusions on a complete Riemannian manifold can only possibly exist if the underlying space is a sphere, Euclidean space, or hyperbolic space.

Of particular relevance to the work of the current paper is the analysis of a symmetric random walk on the hypercube in [4], subsequently generalised in [3]. They considered the class of co-adapted couplings for two such random walks, and showed that there exists a *stochastically optimal* coupling within this class. In other words, they exhibited a coupling whose coupling time T^* satisfies, simultaneously for all $t \geq 0$, $\mathbb{P}(T^* > t) = \min\{\mathbb{P}(T > t) : \text{co-adapted coupling times } T\}$. Furthermore, they showed that this optimal co-adapted coupling does not achieve equality in (1.2), thus demonstrating that there does not exist a maximal co-adapted coupling for this random walk. A result in the same vein was proved in [11], which showed that there exists a stochastically optimal co-adapted coupling for the two-dimensional process consisting of Brownian motion together with its local time at 0; numerical evidence indicates that this coupling is once again not maximal. We also highlight here [9], which investigated (amongst other things) whether the reflection coupling for Brownian motions optimises various functions of the coupling time of the corresponding geometric Brownian motions. It showed that the Laplace transform of the coupling time is maximised by the reflection coupling, but that whether this coupling also solves the finite time horizon problem (to minimise the coupling time's tail distribution) depends upon an underlying drift parameter.

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1.1. Motivation and main results

By way of motivation for the study of JBM(λ), consider the following problem. Suppose that X and $\hat{X}$ start from opposite points on the circle (i.e. $|X_0 - \hat{X}_0| = \pi$, where $|\cdot|$ denotes the shortest distance between two points on the circumference of the unit circle) and that we wish to couple their evolution so as to minimise (some non-decreasing function of) the corresponding coupling time. Write $D_t = |X_t - \hat{X}_t| \in [0, \pi]$. Consider the following two cases:

1. If $\lambda = 0$ then there are no jumps; in this case the fastest coupling is achieved by setting $\hat{X}_t = \pi - X_t \pmod{2\pi}$, i.e. having the two processes *reflect* about either of the two points at distance $\pi/2$ from both X_0 and $\hat{X}_0$. The resulting process D_t is a reflected Brownian motion on $[0, \pi]$, and the corresponding coupling time is given by $T = \inf\{t : D_t = 0\} = \tau$, where τ is the first hitting time of the set $\{-\pi/2, \pi/2\}$ by the Brownian motion $\frac{1}{2}B_t$ making up the diffusion component of X.
2. Alternatively, if λ is large then we expect jumps to be occurring quickly; in this case it may be better to *synchronise* the driving Brownian motions, and to let the Poisson processes N and $\hat{N}$ be independent until the first time that we see an incident on either one. In other words, we maintain $D_t = \pi$ for all $t < J$, where $J \sim \mathrm{Exp}(2\lambda)$ is the first time that either X or $\hat{X}$ jumps, and then the coupling time is given by $T = J$.

Note that the ways in which the driving Brownian motions are coupled in these two scenarios are as different as can be; the first coupling proceeds by maximising the volatility of $B_t - \hat{B}_t$ for all $t < T$, whilst the second minimises it (setting it to zero, in fact). Taking this intuition further, it is not unreasonable to suppose that there might be a critical value of λ with the property that the second of the two couplings sketched above is better than the other if and only if the jump rate exceeds this critical value.

Furthermore, it is clear that when $\lambda = 0$ the mirror coupling described in case 1 above will be *maximal*, but that this will not be true of the coupling strategy outlined in case 2: if we choose to synchronise the driving Brownian motions and couple at time J, then there is a positive probability that coupling would have occurred faster if the Brownian motions had instead been reflected. Indeed, we show in Section 2 that when $|X_0 - \hat{X}_0| = \pi$ the total variation distance between the distributions of X_t and $\hat{X}_t$ satisfies

$$\|\mathcal{L}(X_t \mid X_0 = 0) - \mathcal{L}(\hat{X}_t \mid \hat{X}_0 = \pi)\|_{\mathrm{TV}} = \mathbb{P}(\min\{J, \tau\} > t),$$

where J and τ are independent. We shall use this observation to explicitly describe a maximal, but non-co-adapted, coupling of two JBM(λ) processes started from antipodal points of the circle.

In Section 3 we restrict our attention to the class of co-adapted couplings of two JBM(λ) processes with arbitrary starting points, and consider two couplings motivated by the cases outlined above. We refer to these as *Mirror* and *Synchronous* couplings, but these names relate *solely* to the way in which the driving Brownian motions are coupled when $D_t = \pi$. A formal definition will be given later (Definition 3.1), but in order to state our main results we provide an informal description here.

Definition 1.2. *(Informal description of Mirror and Synchronous couplings.)* The Mirror and Synchronous couplings treat the jump and diffusion components of X and $\hat{X}$ as follows, until the coupling time T.

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Jumps:

- if $D_t \in (0, \pi/2]$, *synchronise* the driving Poisson processes (i.e. make $\hat{X}$ jump at time t if and only if X sees a jump at time t);
- if $D_t \in (\pi/2, \pi]$, make the driving Poisson processes *independent*.

Diffusion:

- if $D_t \in (0, \pi)$, *reflect* the driving Brownian motions (i.e. set $\hat{B}_t = -B_t$);
- if $D_t = \pi$, then:
  - for the Mirror coupling, *reflect* the driving Brownian motions (i.e. continue to set $\hat{B}_t = -B_t$);
  - for the Synchronous coupling, *synchronise* the driving Brownian motions (i.e. set $\hat{B}_t = B_t$).

Trivially, the Mirror coupling is successful for all jump rates $\lambda \geq 0$, while the Synchronous coupling is successful for any $\lambda > 0$. We note that our Synchronous coupling has a similar two-stage approach to the 'reflection/synchronised coupling' of Brownian motion together with local time described in [11]. In both cases the Brownian motions are reflected until some stopping time T_1 (in our case, the first hitting time of D_t on the set $\{0, \pi\}$); if coupling has not occurred at time T_1 then the Brownian motions are subsequently synchronised until the coupling time, T.

In Section 3 we calculate the Laplace transform of the associated coupling times, T_{M} (Mirror) and T_{S} (Synchronous), and then use ideas from stochastic control to prove that the Synchronous coupling is faster than the Mirror coupling if and only if $\lambda \geq \lambda^\star$, where $\lambda^\star = 0.083\,37\ldots$ is the unique solution to the equation

$$\operatorname{cosech}(\pi\sqrt{\lambda}) + 2\pi\sqrt{\lambda} = 2\coth(\pi\sqrt{\lambda}). \tag{1.4}$$

More precisely, for this range of λ the Synchronous coupling turns out to uniquely maximise the Laplace transform of the coupling time within the class of all co-adapted couplings.

Theorem 1.1. *For any $\lambda \geq \lambda^\star$ the Synchronous coupling uniquely maximises, within the class of co-adapted couplings, the Laplace transform of the corresponding coupling time. That is, for any $\lambda \geq \lambda^\star$ and any $x \in (0, \pi]$, the coupling time T_{S} maximises $\mathbb{E}\left[\mathrm{e}^{-\gamma T} \mid D_0 = x\right]$ simultaneously for all $\gamma \geq 0$.*

It does not appear that there is *any* co-adapted coupling which simultaneously maximises the Laplace transform for all values of γ when $\lambda < \lambda^\star$ (see Remark 3.2). However, the Mirror coupling does turn out to minimise the *mean* coupling time for this set of jump rates.

Theorem 1.2. *For any $\lambda \in [0, \lambda^\star)$, the Mirror coupling minimises the mean coupling time within the class of co-adapted couplings. That is, for any $\lambda \in [0, \lambda^\star)$ and any $x \in (0, \pi]$, the coupling time T_{M} minimises $\mathbb{E}[T \mid D_0 = x]$ within the class of co-adapted couplings.*

In the following we use the terms 'LT-optimal' and 'mean-optimal', and usually refrain from writing 'within the class of co-adapted couplings'. Note that when $\lambda = 0$ (no jumps) the Mirror coupling reduces to the reflection coupling for Brownian motion and is therefore *maximal*. Similarly, when $\lambda \to \infty$ the Synchronous coupling makes jumps from $x \mapsto \pi - x$ occur

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TABLE 1. Summary of optimal coupling properties for different values of the jump rate λ.

Jump rate	Coupling properties
$\lambda = 0$	Mirror is maximal
$0 < \lambda < \lambda^\star$	Mirror is mean-optimal; no LT-optimal coupling exists
$\lambda^\star \leq \lambda$	Synchronous is LT-optimal (and hence also mean-optimal)
$\lambda \to \infty$	T_S converges to the maximal coupling time

immediately for any $x > \pi/2$, and so in the limit the difference process D behaves like reflected Brownian motion on $[0, \pi/2]$; it is simple to see that this will once again be maximal. Table 1 summarises the non-trivial way in which the various optimality properties of our coupling strategies depend upon the jump rate.

JBM(λ) therefore makes an interesting addition to the relatively small number of examples in the literature of processes for which precise results about optimal co-adapted couplings have been established. In particular, to the best of the authors' knowledge, our results are the first of their kind for a Lévy process with both continuous and jump components.

Remark 1.1. Our original proofs of Theorems 1.1 and 1.2 were based on excursion theory of Brownian motion, using calculations similar to those in [17, Section VI.56]. We are grateful to an anonymous referee who suggested the more direct method used in Section 3. However, we still find the excursion approach very appealing, and the corresponding calculations can be found in the original arXiv version of this paper [5]. Further details can also be found in the PhD thesis of the second author [14], alongside supporting evidence for the correctness of our results obtained via direct simulation of the two couplings.

2. Maximal coupling

In this section we briefly describe the construction of a maximal, but non-co-adapted, coupling in the case that $D_0 = \pi$ (i.e. the two coupled JBM(λ) processes begin at opposite points of the circle). Let $(X_0, \hat{X}_0) = (0, \pi)$. For any set $A \subset (-\pi, \pi]$,

$$\mathbb{P}(X_t \in A) = \sum_{k=-\infty}^{\infty} \mathbb{P}(\tfrac{1}{2}B_t \in A + 2k\pi,\ N^\lambda(t) \text{ even}) + \mathbb{P}(\tfrac{1}{2}B_t \in A + (2k+1)\pi,\ N^\lambda(t) \text{ odd}).$$

Since $\mathbb{P}(N^\lambda(t) \text{ even}) = \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) \geq \mathbb{P}(N^\lambda(t) \text{ odd})$, it is clear that the set on which X_t has greater density than $\hat{X}_t$ is, for all $t \geq 0$, the half-circle $A^\star = (-\pi/2, \pi/2)$ centred at X_0. As above, let τ denote the first hitting time of the set $\{-\pi/2, \pi/2\}$ by the Brownian motion $\frac{1}{2}B_t$. By the definition of total variation distance (1.3), we see that

$$\begin{aligned} \|\mathcal{L}(X_t \mid X_0 = 0) - \mathcal{L}(\hat{X}_t \mid \hat{X}_0 = \pi)\|_{\mathrm{TV}} &= \mathbb{P}(X_t \in A^\star) - \mathbb{P}(\hat{X}_t \in A^\star) \\ &= 2\mathbb{P}(X_t \in A^\star) - 1 \\ &= \mathrm{e}^{-2\lambda t}\big(2\mathbb{P}(\tfrac{1}{2}B_t \in A^\star + 2\pi\mathbb{Z}) - 1\big) \\ &= \mathrm{e}^{-2\lambda t}\mathbb{P}(\tau > t), \end{aligned} \tag{2.1}$$

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where the final equality follows by the reflection principle. Thus we see that when X and $\hat{X}$ begin from antipodal points, the total variation distance between their laws is given by the tail distribution of the random variable $\min\{J, \tau\}$, where $J \sim \text{Exp}(2\lambda)$.

This observation leads to an explicit construction of a maximal coupling, as follows. Let $N^{2\lambda}$ be a marked Poisson process of rate 2λ, whose incident times are denoted by J_k, and with marks $Y_k \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(1/2)$. Let B be a standard Brownian motion started at 0, with hitting time τ defined as above. We first of all use these to define the JBM(λ) process X: the diffusion component of X is given by $\frac{1}{2}B$, and X makes a jump of size π at incident times J_k if and only if the corresponding mark Y_k equals 1.

We now define $\hat{X}$, started at π, in such a way that $\hat{X}$ and X couple almost surely at time $T^\star = \min\{J_1, \tau\}$:

- On the event $\{\tau < J_1\}$, we let the diffusion component of $\hat{X}$ equal $-\frac{1}{2}B$, so that $\hat{X}$ is the reflection of X about the points $\pm\pi/2$. In this case $\hat{X}$ and X will meet at time τ.
- On the event $\{J_1 < \tau\}$, we let the diffusion component of $\hat{X}$ equal $\frac{1}{2}B$ (i.e. $\hat{X}$ and X move synchronously, remaining at distance π from each other) until time J_1. We then make $\hat{X}$ jump by π at time J_1 if and only if the mark Y_1 equals 0. That is, $\hat{X}$ jumps at time J_1 if and only if X does *not* jump at that time, and since their diffusion components were synchronised, $\hat{X}$ and X will meet exactly when one of them jumps at time J_1.
- From time $\min\{J_1, \tau\}$ onwards, the diffusion and jump components of $\hat{X}$ equal those of X.

This is a valid coupling (the process $\hat{X}$, viewed marginally, really is a JBM(λ) process started from π). Furthermore, it is clearly maximal but not co-adapted: the evolution of X is adapted to the natural filtration generated by $\{N^{2\lambda}, B\}$, whereas the evolution of $\hat{X}$ until the coupling time depends on knowledge of which of the times J_1 and τ occurs first.

Remark 2.1. For general starting states ($D_0 = x < \pi$) such an explicit description of a maximal coupling is significantly more challenging. The main complication is that it is no longer true that the set on which X_t has greater density than $\hat{X}_t$ is independent of t. Rather, with $(X_0, \hat{X}_0) = (-x/2, x/2)$, the set $A_t^\star$ on which X_t has greater density takes the form $A_t^\star = (-r_t, 0) \cup (r_t, \pi)$, where $r_t = r_t(x)$ is the unique point in $(0, \pi)$ at which the densities of X_t and $\hat{X}_t$ agree. Depending upon the value of λ, r_t tends to either $\pi/2$ or π as $t \to \infty$, but not necessarily monotonically. The only exception to this rule is in the limit as $\lambda \to \infty$, when $r_t \to \pi/2$ for all values of t (and for all $x < \pi$); in this case $A_t^\star$ does not depend upon t, and an explicit maximal coupling is once again straightforward to describe. (See the comment before Table 1.)

3. Co-adapted couplings

We begin the search for optimal co-adapted couplings by adopting the perspective of stochastic control. We wish to work with a pair of JBM(λ) processes which are adapted to a common filtration of σ-algebras $(\mathcal{F}_t)$; it is more convenient to work initially with jumpy Brownian motions $(X, \hat{X})$ on the real line (i.e. processes satisfying (1.1) without the (mod 2π)) and then project to the circle. Let $(\mathcal{F}_t)_{t \geq 0}$ be any filtration to which the following independent random processes are all adapted: two standard Brownian motions, B and $\tilde{B}$; two marked Poisson processes N^λ and $\tilde{N}^\lambda$ of rate $\lambda \geq 0$, whose marks (U_t and $\tilde{U}_t$, respectively) are independent and identically distributed as Uniform [0, 1].

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First, note that any co-adapted coupling of two Brownian motions B and $\hat{B}$ can be represented by the stochastic differential equation $\mathrm{d}\hat{B}_t = \theta_t\,\mathrm{d}B_t + \sqrt{1-\theta_t^2}\,\mathrm{d}\tilde{B}_t$, where θ_t is a predictable random process taking values in $[-1, 1]$ [10, Lemma 6]. Similarly, as explained in [4], any co-adapted coupling of two Poisson processes $(N^\lambda, \hat{N}^\lambda)$ can be written as

$$\hat{N}^\lambda(\mathrm{d}t) = \mathbf{1}_{[U_t \le p_{t-}]} N^\lambda(\mathrm{d}t) + \mathbf{1}_{[\tilde{U}_t > p_{t-}]} \tilde{N}^\lambda(\mathrm{d}t), \tag{3.1}$$

where p_t is a càdlàg control process adapted to $(\mathcal{F}_t)$, taking values in [0,1]. (More accurately, [4] describes any co-adapted coupling of two unit-rate random walks on the hypercube, $\mathbb{Z}_2^n$, in terms of $(n+1)^2$ independent marked Poisson processes which are controlled by a doubly stochastic matrix-valued process; (3.1) is just a simplified parametrisation of their result when $n = 1$.)

Combining these two results, it is clear that the joint process $(X_t, \hat{X}_t)$ satisfies

$$\begin{pmatrix} \mathrm{d}X_t \\ \mathrm{d}\hat{X}_t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ \theta_t\sqrt{1-\theta_t^2} \end{pmatrix}\begin{pmatrix} \mathrm{d}B_t \\ \mathrm{d}\tilde{B}_t \end{pmatrix} + \pi\begin{pmatrix} 1 & 0 \\ \mathbf{1}_{[U_t \le p_{t-}]}\mathbf{1}_{[\tilde{U}_t > p_{t-}]} \end{pmatrix}\begin{pmatrix} N^\lambda(\mathrm{d}t) \\ \tilde{N}^\lambda(\mathrm{d}t) \end{pmatrix}, \tag{3.2}$$

where the control process $c_t = (\theta_t, p_t)$ takes values in $[-1, 1] \times [0, 1]$ and is adapted to the filtration $(\mathcal{F}_t)$. The control process explicitly determines the dependence between X and $\hat{X}$. In particular, if $\theta_t = 0$ then their continuous components are independent; if $\theta_t = 1$ then they are *synchronised*; and if $\theta_t = -1$ then they are *mirror* or *reflection* coupled. Similarly, if $p_t = 1$ then the jump components of X and $\hat{X}$ are synchronised, whereas if $p_t = 0$ then they are independent.

In what follows, we will be interested in the difference process $Z_t = X_t - \hat{X}_t$, and in particular the time it takes for this to hit a multiple of 2π (at which time the projections of the two processes onto the unit circle will meet). Using (3.2), we see that

$$\mathrm{d}Z_t = \frac{1}{2}\big((1-\theta_t)\,\mathrm{d}B_t - \sqrt{1-\theta_t^2}\,\mathrm{d}\tilde{B}_t\big) + \pi\big((1 - \mathbf{1}_{[U_t \le p_{t-}]})N^\lambda(\mathrm{d}t) - \mathbf{1}_{[\tilde{U}_t > p_{t-}]}\tilde{N}^\lambda(\mathrm{d}t)\big).$$

Thus, Z has the same dynamics as a Lévy process on $\mathbb{R}$ whose continuous component has volatility $(1-\theta)/2$ and which makes jumps of size $+\pi$ and $-\pi$ each at rate $\lambda(1-p)$. For any $z \in \mathbb{R}$ the distances of $z+\pi$ and $z-\pi$ from the set $2\pi\mathbb{Z}$ are equal, and so the distribution of the time taken for Z to hit the set $2\pi\mathbb{Z}$ is unchanged if we alter the dynamics to make all jumps of size $+\pi$. Furthermore, the independence of N and $\tilde{N}$ means that

$$(1 - \mathbf{1}_{[U_t \le p_{t-}]})N^\lambda(\mathrm{d}t) + \mathbf{1}_{[\tilde{U}_t > p_{t-}]}\tilde{N}^\lambda(\mathrm{d}t) \overset{\mathrm{D}}{=} \mathbf{1}_{[U'_t > p_{t-}]}N^{2\lambda}(\mathrm{d}t),$$

where $\overset{\mathrm{D}}{=}$ denotes equality in distribution, and where $N^{2\lambda}$ is a marked Poisson process of rate 2λ with marks $U' \sim \mathrm{Uniform}[0, 1]$. Thus, it suffices, for any given adapted control process $c_t = (\theta_t, p_t)$, to consider the hitting time on the set $2\pi\mathbb{Z}$ of the process given by

$$\mathrm{d}Z_t = \sqrt{\frac{1-\theta_t}{2}}\,\mathrm{d}B_t + \pi\mathbf{1}_{[U'_t > p_{t-}]}N^{2\lambda}(\mathrm{d}t); \qquad Z_0 = X_0 - \hat{X}_0.$$

As in Section 1, we shall work in the following with the process $D_t = |Z_t - 2\pi\mathbb{Z}| \in [0, \pi]$, which measures the distance between the projections of X and $\hat{X}$ on the circumference of the unit circle. Until the coupling time $T = \inf\{t : D_t = 0\}$, D behaves like a *reflected* Brownian motion (with volatility controlled by θ), and with a jump (controlled by p) at time t having

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the effect of making $D_t = \pi - D_{t-}$. (That is, a jump of size π in Z results in D being reflected about the point $\pi/2$.)

This allows us to view the search for an optimal co-adapted coupling as a stochastic control problem. For a given value function, we seek a control process $c_t = (\theta_t, p_t) \in [-1, 1] \times [0, 1]$ such that the time taken for the corresponding difference process D to hit zero minimises/maximises the value function, as appropriate. As noted in [15], it suffices to restrict attention to Markov controls of the form $c_t = c(D_{t-})$.

Using this setup we can now give a precise definition of the Mirror and Synchronous couplings for two JBM(λ) processes (cf. Definition 1.2).

Definition 3.1. Suppose $D_{t-} = x \in [0, \pi]$. The Mirror coupling is the co-adapted coupling with control $c_{\mathrm{M}} = (\theta_{\mathrm{M}}, p_{\mathrm{M}})$ at time t given by

$$\text{Diffusion:} \quad \theta_{\mathrm{M}}(x) = 2\mathbf{1}_{[x=0]} - 1; \qquad \text{Jumps:} \quad p_{\mathrm{M}}(x) = \mathbf{1}_{[x \le \pi/2]}.$$

Similarly, the Synchronous coupling uses control $c_{\mathrm{S}} = (\theta_{\mathrm{S}}, p_{\mathrm{S}})$, where:

$$\text{Diffusion:} \quad \theta_{\mathrm{S}}(x) = 2\mathbf{1}_{[x=0 \text{ or } x=\pi]} - 1; \qquad \text{Jumps:} \quad p_{\mathrm{S}}(x) = \mathbf{1}_{[x \le \pi/2]}.$$

As explained above, D behaves like a Brownian motion on $(0, \pi)$, for which we can control the speed using θ, and if D sees a jump then this simply reflects it around the point $\pi/2$. Since we wish to minimise the time taken for D to hit 0, it seems intuitively sensible to maximise the speed of the diffusion when $D_t \in (0, \pi)$, and to maximise/minimise the jump rate according to whether or not a jump would reduce the value of D_t; this is exactly what both Mirror and Synchronous couplings achieve. The only difference between the two couplings is in the choice of $\theta(\pi)$: $\theta_{\mathrm{M}}(\pi) = -1$ forces the Brownian component to reflect downwards from the barrier at π, whereas $\theta_{\mathrm{S}}(\pi) = 1$ means that D waits an Exp(2λ) amount of time at level π before jumping directly to 0.

3.1. LT-optimality

In this section we prove the LT-optimality of the Synchronous coupling when $\lambda \ge \lambda^\star$, as claimed in Theorem 1.1. We begin by proving explicit formulas for the Laplace transforms of our two co-adapted coupling strategies.

Given the jump rate λ and a constant $\gamma \ge 0$, we write $\alpha = \sqrt{2(2\lambda + \gamma)}$ and $\beta = \sqrt{2\gamma}$, and use these to define the following four non-negative constants (depending implicitly upon λ and γ):

$$\kappa_1 = \frac{2\cosh\left(\alpha\frac{\pi}{2}\right)\sinh\left(\beta\frac{\pi}{2}\right)}{2\cosh\left(\alpha\frac{\pi}{2}\right)\cosh\left(\beta\frac{\pi}{2}\right) - 1}; \qquad \kappa_2 = \frac{\beta}{\alpha}\kappa_1 \operatorname{sech}\left(\alpha\frac{\pi}{2}\right);$$

$$\kappa_3 = \operatorname{sech}\left(\beta\frac{\pi}{2}\right)\left(\frac{\beta}{2\alpha}\operatorname{cosech}\left(\alpha\frac{\pi}{2}\right) + \sinh\left(\beta\frac{\pi}{2}\right)\right); \qquad \kappa_4 = \left(\frac{\beta}{\alpha}\right)^2 \operatorname{cosech}\left(\alpha\frac{\pi}{2}\right).$$

Lemma 3.1. *The Laplace transforms for the coupling times T_{M} and T_{S} under the Mirror and Synchronous couplings started from distance $x \in [0, \pi]$ are given by the following formulas.*

Mirror coupling:

$$\mathbb{E}_x\left[\mathrm{e}^{-\gamma T_{\mathrm{M}}}\right] = w_{\mathrm{M}}(x) := \begin{cases} \cosh(\beta x) - \kappa_1 \sinh(\beta x), & 0 \le x \le \frac{\pi}{2}; \\ w_{\mathrm{M}}(\pi - x) - \kappa_2 \sinh\left(\alpha\left(x - \frac{\pi}{2}\right)\right), & \frac{\pi}{2} < x \le \pi. \end{cases}$$

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Synchronous coupling:

$$\mathbb{E}_x\left[\mathrm{e}^{-\gamma T_{\mathrm{S}}}\right] = w_{\mathrm{S}}(x) := \begin{cases} \cosh(\beta x) - \kappa_3 \sinh(\beta x), & 0 \le x \le \frac{\pi}{2}; \\ w_{\mathrm{S}}(\pi - x) - \kappa_4 \sinh\left(\alpha\left(x - \frac{\pi}{2}\right)\right) & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Remark 3.1. Note that when $x = \pi$, the final formula in Lemma 3.1 simplifies to

$$\mathbb{E}_\pi\left[\mathrm{e}^{-\gamma T_{\mathrm{S}}}\right] = 1 - \left(\frac{\beta}{\alpha}\right)^2 = \frac{2\lambda}{2\lambda + \gamma};$$

this is just the Laplace transform of an Exp(2λ) distribution, which is consistent with the observation that under the Synchronous coupling the two processes will meet precisely when one of them jumps to the other side of the circle. Similarly, the formula for the Mirror coupling when $\lambda = 0$ and $x = \pi$ reduces to $\mathbb{E}_\pi[\mathrm{e}^{-\gamma T_{\mathrm{M}}}] = \mathrm{sech}(\pi\sqrt{2\gamma})$, which is the Laplace transform of the time taken for a standard Brownian motion to hit $\pm\pi$ (and hence of the time taken for a Brownian motion started from, and reflected at, π to hit 0).

Before proceeding any further, it will be helpful to quickly establish some basic properties of the function w_{S} defined in Lemma 3.1.

Lemma 3.2. *For any fixed $\gamma \ge 0$, $w_{\mathrm{S}}(x)$ is a convex function of $x \in [0, \pi]$, and $w''_{\mathrm{S}}(\pi) = 0$.*

Proof. It is clear from the definition of w_{S} that this function is continuous on $[0, \pi]$ and C^2 on $(0, \pi)$, with second derivative given by

$$w''_{\mathrm{S}}(x) = \begin{cases} \beta^2 w_{\mathrm{S}}(x), & 0 \le x \le \pi/2; \\ \beta^2 w_{\mathrm{S}}(\pi - x) - \alpha^2\kappa_4 \sinh\left(\alpha\left(x - \frac{\pi}{2}\right)\right), & \pi/2 < x \le \pi. \end{cases}$$

This is clearly non-negative for $x \le \pi/2$, and zero for $x = \pi$. For $x > \pi/2$, inserting the formula for $w_{\mathrm{S}}(\pi - x)$ and expanding shows that

$$w''_{\mathrm{S}}(x) \propto s_\beta c_\beta \cosh(\beta(\pi - x)) - \left(\frac{\beta s_\beta}{2\alpha s_\alpha} + s_\beta^2\right)\sinh(\beta(\pi - x)) - \frac{s_\beta c_\beta}{s_\alpha}\sinh\left(\alpha\left(x - \frac{\pi}{2}\right)\right), \tag{3.3}$$

where we have written $c_u = \cosh\left(u\frac{\pi}{2}\right)$ and $s_u = \sinh\left(u\frac{\pi}{2}\right)$. Now note that $u \mapsto us_u$ is an increasing function, and that $u \mapsto \sinh\left(u\left(x - \frac{\pi}{2}\right)\right)/s_u$ is decreasing for any fixed $x \in [\pi/2, \pi]$. Thus (recalling that $\beta \le \alpha$) the right-hand side of (3.3) can be bounded below by

$$\begin{aligned} &s_\beta c_\beta \cosh(\beta(\pi - x)) - \left(\frac{1}{2} + s_\beta^2\right)\sinh(\beta(\pi - x)) - c_\beta \sinh\left(\beta\left(x - \frac{\pi}{2}\right)\right) \\ &\qquad = \frac{1}{2}\sinh(\beta(\pi - x)) \ge 0. \end{aligned}$$

□

Proof of Lemma 3.1.We deal first with the Synchronous coupling. It is clear that $w_{\mathrm{S}}(0) = 1$, and we have already observed that $w_{\mathrm{S}}(\pi) = 2\lambda/(2\lambda + \gamma)$, so w_{S} certainly takes the correct values at the two boundary points. Given $D_0 = x \in [0, \pi]$, consider the (bounded) process $W_t(x) = \mathrm{e}^{-\gamma t} w_{\mathrm{S}}(D_t)$. We shall show that W is a martingale until time T_{S}, and then the optional stopping theorem will quickly yield that $\mathbb{E}_x\left[\mathrm{e}^{-\gamma T_{\mathrm{S}}}\right] = \mathbb{E}\left[W_{T_{\mathrm{S}}}(x)\right] = W_0(x) = w_{\mathrm{S}}(x)$, as required.

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The martingale property is trivial when $x = \pi$. For $x \in (0, \pi)$, we apply Itô's formula to $W_t(x)$ to obtain

$$\begin{aligned} \mathrm{e}^{\gamma t}\,\mathrm{d}W_t(x) &= \left(-\gamma w_{\mathrm{S}}(D_t) + \frac{1}{2}w''_{\mathrm{S}}(D_t)\right)\mathrm{d}t + w'_{\mathrm{S}}(D_t)\,\mathrm{d}D_t \\ &= \left(-\gamma w_{\mathrm{S}}(D_t) + \frac{1}{2}w''_{\mathrm{S}}(D_t)\right)\mathrm{d}t + \mathrm{d}Q_t \\ &\quad + \mathbf{1}_{[D_t > \pi/2]}\{w_{\mathrm{S}}(\pi - D_t) - w_{\mathrm{S}}(D_t)\}N^{2\lambda}(\mathrm{d}t), \end{aligned} \tag{3.4}$$

where Q_t is a martingale. Using the formula for w''_{S} established in the proof of Lemma 3.2, and recalling that $\alpha^2 = 2(2\lambda + \gamma)$ and $\beta^2 = 2\gamma$, (3.4) simplifies to become

$$\mathrm{e}^{\gamma t}\,\mathrm{d}W_t(x) = \mathrm{d}Q_t + \mathbf{1}_{[D_t > \pi/2]}\kappa_4 \sinh\left(\alpha\left(D_t - \frac{\pi}{2}\right)\right)(N^{2\lambda}(\mathrm{d}t) - 2\lambda\,\mathrm{d}t). \tag{3.5}$$

The final bracketed term on the right-hand side is of course a compensated Poisson process, and we conclude that W is indeed a martingale.

The Mirror coupling case is almost identical: repeating the argument above using w_{M} in place of w_{S} shows that, when $x \in (0, \pi)$, (3.5) holds with κ_4 replaced by κ_2. It only remains to check that the appropriate boundary conditions are satisfied for w_{M}: here we require $w_{\mathrm{M}}(0) = 1$ and $w'_{\mathrm{M}}(\pi) = 0$, and these follow trivially from the definition of w_{M}. □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\mathcal{C}$ denote the set of all successful co-adapted couplings. For any coupling $c \in \mathcal{C}$, and any fixed $\gamma \geq 0$, write $w_c(x) = \mathbb{E}_x\left[\mathrm{e}^{-\gamma T_c}\right]$ for the Laplace transform of the associated coupling time when starting from $D_0 = x \in [0, \pi]$. The value function $\hat{w}(x) = \sup_{c \in \mathcal{C}} w_c(x)$ solves the Hamilton–Jacobi–Bellman (HJB) equation [15, Chapter 3], which may be derived here using Itô's formula in a similar manner to (3.4). We first of all deal with the case when $x \in [0, \pi)$, for which the HJB equation is as follows:

$$\sup_{\theta,\, p}\left(-\gamma\hat{w}(x) + \frac{(1 - \theta(x))}{4}\hat{w}''(x) + 2\lambda(1 - p(x))\{\hat{w}(\pi - x) - \hat{w}(x)\}\right) = 0. \tag{3.6}$$

Consider the Laplace transform for the Synchronous coupling, $w_{\mathrm{S}}(x)$. We saw in Lemma 3.2 that this is convex on $[0, \pi]$; moreover, using the formula in Lemma 3.1 it may quickly be checked that $w_{\mathrm{S}}(\pi - x) - w_{\mathrm{S}}(x) > 0$ if and only if $x \in (\pi/2, \pi]$. Thus, if we replace $\hat{w}(x)$ with $w_{\mathrm{S}}(x)$ on the left-hand side of (3.6), the corresponding supremum for $x \in [0, \pi)$ is obtained by taking $\theta(x) = -1$ and $p(x) = \mathbf{1}_{[x \leq \pi/2]}$; since these values agree with the control $(\theta_{\mathrm{S}}, p_{\mathrm{S}})$ in Definition 3.1, it also follows that this supremum is indeed zero. Therefore, the function w_{S} satisfies (3.6) for $x \in [0, \pi)$; if we can also show that $w_c(\pi) \leq w_{\mathrm{S}}(\pi)$ for all couplings $c \in \mathcal{C}$, then it will follow that $\hat{w} = w_{\mathrm{S}}$, i.e. that the Synchronous coupling is LT-optimal (see [15, Theorem 3.1]).

Starting from $D_0 = \pi$, if $\theta_c(\pi) \neq 1$ then the diffusion component of D reflects downwards off the barrier at π. This, along with the fact that $w''_{\mathrm{S}}(\pi) = 0$ (Lemma 3.2), implies that the relevant quantity to be maximised over admissible controls (θ_c, p_c) becomes

$$-\gamma w_{\mathrm{S}}(\pi) + 2\lambda(1 - p_c(\pi))\{1 - w_{\mathrm{S}}(\pi)\} - \sqrt{\frac{1 - \theta_c(\pi)}{2}}w'_{\mathrm{S}}(\pi). \tag{3.7}$$

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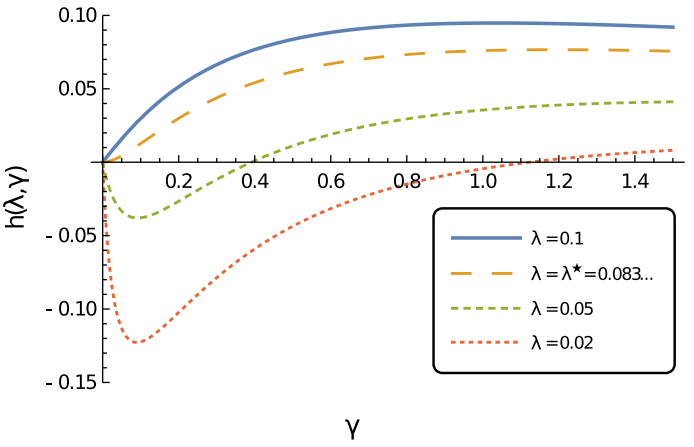

FIGURE 1. The function $h(\lambda, \gamma)$ for four representative values of λ. For $\lambda < \lambda^\star$ the function takes first negative then positive values as γ increases; for $\lambda \geq \lambda^\star$ the function is non-negative for all γ.

We therefore need to show that this expression is maximised by taking $p_c(\pi) = p_S(\pi) = 0$ and $\theta_c(\pi) = \theta_S(\pi) = 1$. The first of these is trivial, since $w_S(\pi) \in (0, 1]$ for all values of λ and γ; the second will follow if and only if $w_S'(\pi) \geq 0$ for all values of $\gamma \geq 0$. To make explicit the dependence of $w_S'(\pi)$ on the underlying parameters λ and γ, let us define $h(\lambda, \gamma) := w_S'(\pi) = \beta\kappa_3 - \alpha\kappa_4 \cosh\left(\alpha\frac{\pi}{2}\right)$. The function h is continuous in both arguments, with $h(\lambda, 0) = 0$ and

$$\left.\frac{\mathrm{d}h(\lambda, \gamma)}{\mathrm{d}\gamma}\right|_{\gamma=0} = \pi + \frac{\operatorname{cosech}(\pi\sqrt{\lambda})}{2\sqrt{\lambda}} - \frac{\coth(\pi\sqrt{\lambda})}{\sqrt{\lambda}}.$$

This derivative at $\gamma = 0$ is a strictly increasing function of λ, equalling zero if and only if $\lambda = \lambda^\star$ (recall the defining equation for $\lambda^\star$, (1.4)). Furthermore, $h(\lambda, \gamma)$ is non-negative for all γ if and only if $\lambda \geq \lambda^\star$; if $\lambda < \lambda^\star$ then there exist values of $\gamma > 0$ for which h is negative, and others for which it is positive: see Figure 1. It follows that $w_S'(\pi)$ is non-negative for all $\gamma \geq 0$ exactly when $\lambda \geq \lambda^\star$, and we conclude that for this range of jump rates the Synchronous coupling is LT-optimal, as required. □

Remark 3.2. For $\lambda < \lambda^\star$, the proof given above fails only when $x = \pi$, when it is no longer true that $h(\lambda, \gamma)$ is non-negative for all γ. For such a λ, if γ is such that $h(\lambda, \gamma) < 0$, then the expression in (3.7) is clearly maximised by setting $\theta_c(\pi) = \theta_M(\pi) = -1$, i.e. by using the Mirror coupling. However, any attempt to prove LT-optimality of the Mirror coupling when $\lambda < \lambda^\star$ is thwarted by the function $w_M''(\pi)$ taking both positive and negative values as γ varies: as above, we arrive at the situation where for some values of γ the optimal control is to set $\theta(\pi) = 1$, and for others it is to set $\theta(\pi) = -1$. Figure 2 shows that when $\lambda < \lambda^\star$, $w_M(\pi) \geq w_S(\pi)$ when γ is small, but that this relationship is reversed for larger γ, and so neither the Mirror nor Synchronous strategy is LT-optimal for these jump rates. This implies that when $\lambda < \lambda^\star$ there is no co-adapted coupling which maximises the Laplace transform $\mathbb{E}\left[\mathrm{e}^{-\gamma T}\right]$ simultaneously for all values of γ.

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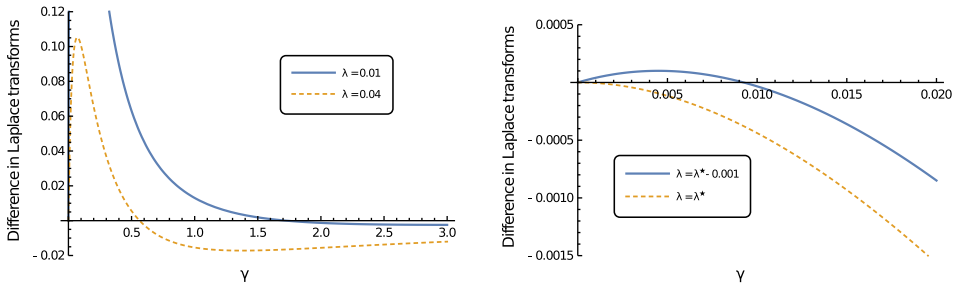

FIGURE 2. Difference between Laplace transforms corresponding to Mirror and Synchronous strategies: both graphs show $\mathbb{E}_\pi[\mathrm{e}^{-\gamma T_\mathrm{M}} - \mathrm{e}^{-\gamma T_\mathrm{S}}]$ as a function of γ, for different jump rates $\lambda \le \lambda^\star$. Neither coupling is LT-optimal when $\lambda < \lambda^\star$.

3.2. Mean-optimality

In this final section we compare the mean coupling times under the Mirror and Synchronous couplings.

Lemma 3.3. *The expectations of the coupling times under the Mirror and Synchronous couplings started from distance $x \in [0, \pi]$ are given by*

$$\mathbb{E}_x[T] = \begin{cases} x(\pi - x) + C(\lambda)x & \text{if } 0 \le x \le \frac{\pi}{2}, \\ x(\pi - x) + C(\lambda)(\pi - x) + C(\lambda)\frac{\sinh(\sqrt{\lambda}(2x-\pi))}{\sqrt{\lambda}} & \text{if } \frac{\pi}{2} < x \le \pi, \end{cases} \tag{3.8}$$

where

$$C(\lambda) = \begin{cases} C_\mathrm{M}(\lambda) = \frac{\pi}{2\cosh(\pi\sqrt{\lambda})-1} & \text{if } T = T_\mathrm{M} \text{ (Mirror coupling)}, \\ C_\mathrm{S}(\lambda) = \frac{\operatorname{cosech}(\pi\sqrt{\lambda})}{2\sqrt{\lambda}} & \text{if } T = T_\mathrm{S} \text{ (Synchronous coupling)}. \end{cases}$$

Proof. This can be proved in a couple of ways. Probably the simplest, but most tedious, is to start from Lemma 3.1 and calculate $\mathbb{E}_x[T] = \lim_{\gamma\to 0}\left(-(\mathrm{d}/\mathrm{d}\gamma)\mathbb{E}_x\left[\mathrm{e}^{-\gamma T}\right]\right)$. Alternatively, the line of argument used in Lemma 3.1 can be followed: letting $v(x)$ be given by the function in (3.8), it is straightforward to check that this has appropriate boundary conditions and that the process $V_t(x) = v(D_t) + t$ is a martingale until the time that D first hits 0. □

Note that $C(\lambda) = \min\{C_\mathrm{M}(\lambda), C_\mathrm{S}(\lambda)\}$, and that $C_\mathrm{M}(\lambda)$ and $C_\mathrm{S}(\lambda)$ are both positive, decreasing functions of λ, which agree precisely when $\lambda = \lambda^\star$ (recall the definition of $\lambda^\star$ in (1.4)). Thus, from any starting state $x \in (0, \pi]$, $\mathbb{E}_x[T_\mathrm{M}] < \mathbb{E}_x[T_\mathrm{S}]$ if and only if $\lambda < \lambda^\star$.

The proof of Theorem 1.2 (mean-optimality of the Mirror coupling for $\lambda < \lambda^\star$) is very similar to that given above for the LT-optimality of the Synchronous coupling. In this case it suffices to show that the function $v_\mathrm{M}(x) = \mathbb{E}_x[T_\mathrm{M}]$ satisfies the HJB equation

$$\inf_{\theta,\, p}\left(1 + \tfrac{1}{4}(1-\theta(x))v_\mathrm{M}''(x) + 2\lambda(1-p(x))\left\{v_\mathrm{M}(\pi - x) - v_\mathrm{M}(x)\right\}\right) = 0;$$

this is straightforward, and the details are omitted for the sake of brevity.

Figure 3 gives an impression of how the mean coupling time formulas from Lemma 3.3 behave as functions of the jump rate and the starting distance x. The left-hand plot shows that $\mathbb{E}_x[T_\mathrm{M}] < \mathbb{E}_x[T_\mathrm{S}]$ for all $x \in (0, \pi]$ when $\lambda = 0.05 < \lambda^\star$, with this inequality being reversed

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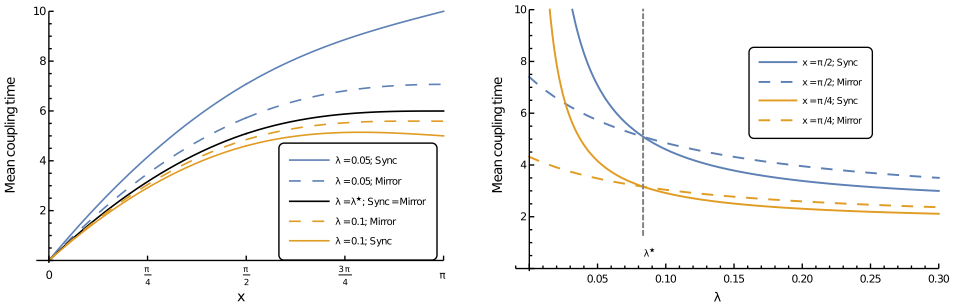

FIGURE 3. Expected coupling time under the Synchronous and Mirror couplings. The left-hand plot shows the mean coupling time as a function of $D_0 = x$ for three representative values of λ. The right-hand plot shows the mean coupling time as a function of λ for two particular values of D_0.

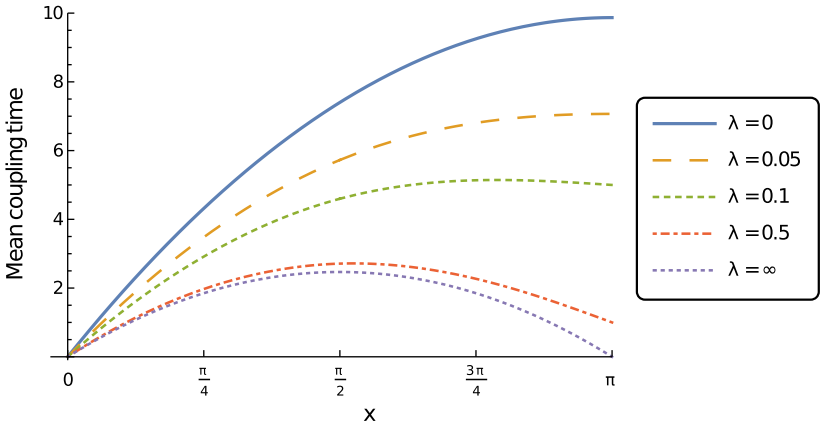

FIGURE 4. Expected coupling time from $D_0 = x$ when using the mean-optimal strategy.

for $\lambda = 0.1 > \lambda^\star$. The right-hand plot graphs $\mathbb{E}_x[T_M]$ and $\mathbb{E}_x[T_S]$ as functions of λ when $x \in \{\pi/4, \pi/2\}$: for both starting distances the intersection of the two mean coupling times at $\lambda = \lambda^\star$ is evident.

Now let $T_{\min}$ denote the coupling time of the mean-optimal co-adapted coupling (i.e. Mirror for $\lambda < \lambda^\star$, Synchronous otherwise). As expected, when $\lambda = 0$ we have $\mathbb{E}_x[T_{\min}] = x(2\pi - x)$, and when $\lambda \to \infty$ we obtain $\mathbb{E}_x[T_{\min}] = x(\pi - x)$ (i.e. the mean time for a Brownian motion started at x to hit $\{0, 2\pi\}$ or $\{0, \pi\}$, respectively; recall the comment immediately following Theorem 1.2). Figure 4 plots $\mathbb{E}_x[T_{\min}] = \mathbb{E}_x[T_M] \wedge \mathbb{E}_x[T_S]$ as a function of the initial distance $x \in [0, \pi]$, nicely showing the monotonic dependence upon the jump rate λ.

Finally, let us compare $T_{\min}$ to a *maximal* coupling. We know that the optimal co-adapted coupling is maximal when $\lambda = 0$; given the construction of a maximal coupling in Section 2 it should come as no surprise that this is not the case for any $\lambda > 0$. From (2.1) we can easily calculate the Laplace transform and subsequently the mean of the maximal coupling time $T^\star$ when starting from opposite sides of the circle ($x = \pi$):

$$\mathbb{E}_\pi[e^{-\gamma T^\star}] = \frac{1}{2\lambda + \gamma}\left(2\lambda + \frac{\gamma}{\cosh\left(\pi\sqrt{2(2\lambda + \gamma)}\right)}\right); \quad \mathbb{E}_\pi[T^\star] = \frac{1 - \operatorname{sech}(2\pi\sqrt{\lambda})}{2\lambda}. \tag{3.9}$$

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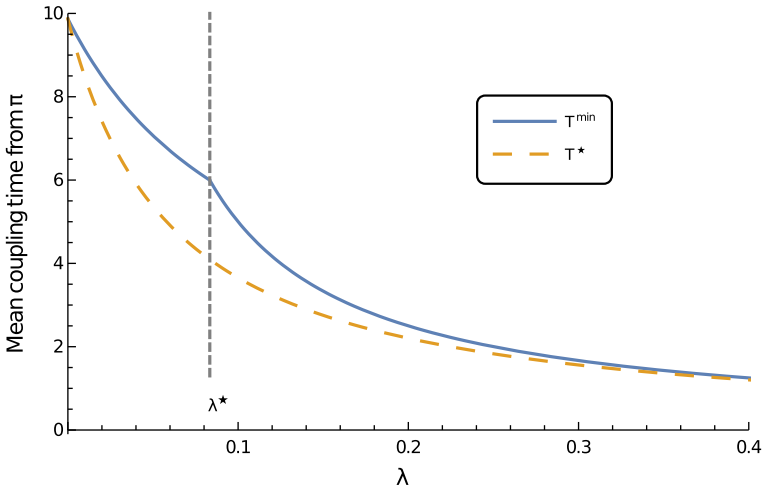

FIGURE 5. Expectation of the mean-optimal co-adapted coupling time ($T_{\min}$) and maximal coupling time ($T^\star$) when $D_0 = \pi$.

On the other hand, Lemma 3.3 tells us that $T_{\min}$ satisfies

$$\mathbb{E}_\pi[T_{\min}] = \min\{C_{\mathrm{M}}(\lambda),\, C_{\mathrm{S}}(\lambda)\}\,\frac{\sinh(\pi\sqrt{\lambda})}{\sqrt{\lambda}}. \tag{3.10}$$

Thus,

$$\frac{\mathbb{E}_\pi[T^\star]}{\mathbb{E}_\pi[T_{\min}]} = \begin{cases} (\pi\sqrt{\lambda})^{-1}(2\cosh(\pi\sqrt{\lambda})-1)\sinh(\pi\sqrt{\lambda})\operatorname{sech}(2\pi\sqrt{\lambda}), & 0 \le \lambda < \lambda^\star, \\ 1-\operatorname{sech}(2\pi\sqrt{\lambda}), & \lambda^\star \le \lambda, \end{cases}$$

and we note that for all values of $\lambda > 0$ this ratio is strictly less than one. (For the case $\lambda < \lambda^\star$, this follows from the observation that

$$\frac{(2\cosh(z)-1)\sinh(z)\operatorname{sech}(2z)}{z} = \frac{\int_z^{2z}\cosh(y)\,\mathrm{d}y}{z\cosh(2z)},$$

along with the convexity of cosh.) Graphs of the expressions in (3.9) and (3.10) are shown in Figure 5: we see that the difference between the mean coupling times is largest when $\lambda = \lambda^\star$.

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