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BOUNDARY AND ANGULAR LAYER BEHAVIOR IN SINGULARLY PERTURBED SEMILINEAR SYSTEMS

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ABSTRACT. Some authors have employed the method and technique of differential inequalities to obtain fairly general results concerning the existence and asymptotic behavior, as $\epsilon \to 0^+$, of the solutions of scalar boundary value problems

$$\epsilon y'' = h(t, y), \qquad a < t < b,$$
$$y(a, \epsilon) = A,\ y(b, \epsilon) = B.$$

In this paper, we extend these results to vector boundary value problems, under analogous stability conditions on the solution $\boldsymbol{u} = \boldsymbol{u}(t)$ of the reduced equation $\mathbf{0} = \boldsymbol{h}(\boldsymbol{t}, \boldsymbol{u})$.

Two types of asymptotic behavior are studied, depending on whether the reduced solution $\boldsymbol{u}(t)$ has or does not have a continuous first derivative in (a, b), leading to the phenomena of boundary and angular layers.

1. **Introduction**. We consider in this paper semilinear boundary value problem of the form

$$\text{(1.1)} \qquad \epsilon^2 \boldsymbol{y}'' = \boldsymbol{h}(t, \boldsymbol{y}),\ \boldsymbol{y}(a, \epsilon) = \boldsymbol{A},\ \boldsymbol{y}(b, \epsilon) = \boldsymbol{B},$$

where $\boldsymbol{y}$, $\boldsymbol{h}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are n-vectors and $\epsilon > 0$ is a small real-valued parameter. The aim is to show that under appropriate conditions, there exist solutions of (1.1) which exhibit boundary layer and angular layer behavior for all sufficiently small ϵ.

We assume that the corresponding reduced system

$$\mathbf{0} = \boldsymbol{h}(t, \boldsymbol{u})$$

has at least one solution $\boldsymbol{u}(t) = (u_1(t), \ldots, u_n(t))$. We require, as in the scalar case, that the reduced solution $\boldsymbol{u}(t)$ is I_q-stable. The definition of I_q-stability will be given in section 3. This "componentwise" I_q-stability condition will allow us to obtain estimates for each component of the solution $\boldsymbol{y}(t, \epsilon)$ of (1.1).

2. **Preliminary results**. We need the following basic result on differential inequalities ([3], chap. 1):

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LEMMA 1. *Consider the boundary problem*

$$(2.1)\qquad \boldsymbol{y}'' = \boldsymbol{h}(t, \boldsymbol{y}),\ \boldsymbol{y}(a) = \boldsymbol{A},\ \boldsymbol{y}(b) = \boldsymbol{B},$$

where $\boldsymbol{y}$, $\boldsymbol{h}$, $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are in* $\mathbb{R}^n$. *Suppose that there exist n bounding pairs* $(\alpha_i(t), \beta_i(t))$ *of* $C^{(2)}$*-functions on* $[a, b]$ *which satisfy*

$$(2.1)_1\qquad \alpha_i(a) \le A_i \le \beta_i(a),\ \alpha_i(b) \le B_i \le \beta_i(b),\ i = 1, \ldots, n;$$

$$(2.1)_2\qquad \alpha_i(t) \le \beta_i(t),\ t \text{ in } (a, b),\ i = 1, \ldots, n;$$

$$(2.1)_3\qquad \begin{cases} \alpha_i'' \ge h_i(t, y_1, \ldots, \alpha_i, \ldots, y_n) \\ \beta_i'' \le h_i(t, y_1, \ldots, \beta_i, \ldots, y_n), \end{cases}$$

for t in (a, b), $\alpha_j(t) \le y_j \le \beta_j(t)$, $j \ne i$. *Also suppose that* $\boldsymbol{h}$ *is continuous in the region* $[a, b] \times \Pi_{i=1}^n [\alpha_i, \beta_i]$.

Then the problem (2.1) *has a solution* $\boldsymbol{y}(t) = (y_1(t), \ldots, y_n(t))$ *of class* $C^{(2)}[a, b]$ *satisfying*

$$\alpha_i(t) \le y_i(t) \le \beta_i(t)$$

for t in $[a, b]$ *and* $i = 1, \ldots, n$.

The following extension of Lemma 1 will also be needed [2].

LEMMA 2. *Consider the problem (2.1) and suppose that there exist n bounding pairs which are piecewise —* $C^{(2)}$ *on* $[a, b]$, *namely there is a partition* $\{t_i\}_{i=0}^m$ *of* $[a, b]$ *with* $a = t_0 < t_1 < \ldots < t_m = b$ *such that on each subinterval* $[t_{i-1}, t_i]$, $i = 1, \ldots, m$, *the n bounding pairs* (α_j, β_j), $j = 1, \ldots, n$, *are twice continuously differentiable; at the partition points,* t_{i-1} *and* t_i, *the derivatives are righthand and lefthand derivatives respectively. Suppose also that* $(2.1)_1$, $(2.1)_2$, $(2.1)_3$ *hold on each subinterval* $[t_{i-1}, t_i]$. *Lastly, suppose that for each t in* $[a, b]$, $D_L\alpha_i(t) \le D_R\alpha_i(t)$ *and* $D_L\beta_i(t) \ge D_R\beta_i(t)$, *where* D_L, D_R *denote, respectively, lefthand and righthand differentation.*

Then (2.1) *has a solution* $\boldsymbol{y}(t) = (y_1t), \ldots, y_n(t))$ *of class* $C^{(2)}[a, b]$ *satisfying*

$$\alpha_i(t) \le y_i(t) \le \beta_i(t)$$

for t in $[a, b]$ *and* $i = 1, \ldots, n$.

3. **Boundary layer phenomenon**. Let q be a non-negative integer. In the following definition of I_q-stability for the reduced solution $\boldsymbol{u}(t)$, we assume that the function $\boldsymbol{h}(t, \boldsymbol{y})$ has the stated number of continuous partial derivatives with respect to y_i in $\Pi_{i=1}^n \mathcal{D}_i$, $i = 1, \ldots, n$, where

$$\mathcal{D}_i = \{(t, y_i) \colon t \in [a, b],\ |y_i - u_i(t)| \le d_i(t)\}.$$

Here $d_i(t)$ is a smooth positive function such that

$$|A_i - u_i(a)| \le d_i(t) \le |A_i - u_i(a)| + \delta, \text{ on } [a, a + \delta]$$
$$|B_i - u_i(b)| \le d_i(t) \le |B_i - u_i(b)| + \delta, \text{ on } [b - \delta, b]$$

and

$$d_i(t) \leq \delta \text{ on } [a + \delta, b - \delta].$$

where A_i, B_i are components of A, B respectively and $\delta > 0$ is a small constant.

DEFINITION. *The vector function $\boldsymbol{u} = \boldsymbol{u}(t) = (u_1(t), \ldots, u_n(t))$ is said to be I_q-stable in $[a, b]$, if there exist n positive constants $m_1, \ldots, m_n$ such that*

$$\frac{\partial^k h_i}{\partial y_i^k}(t, y_1, \ldots, u_i, \ldots, y_n) = 0 \tag{3.1}$$

for $0 \leq k \leq 2q$, $i = 1, \ldots, n$, $(t, y_j) \in \mathcal{D}_j$, $j \neq i$,
and

$$\frac{1}{(2q+1)!}\frac{\partial^{2q+1} h_i}{\partial y_i^{2q+1}}(t, y_1, \ldots, y_n) \geq m_i^2 > 0 \tag{3.2}$$

for $i = 1, \ldots, n$, $(t, \boldsymbol{y}) \in \Pi_{i=1}^n \mathcal{D}_i$.

We note that the definition of I_q-stability for a scalar function was first given by Boglaev [4], and has been employed and extended by other authors [2].

We have the following result.

THEOREM 1. *Assume that the reduced system $\boldsymbol{h}(t, \boldsymbol{u}) = \boldsymbol{0}$ has an I_q-stable solution $\boldsymbol{u}(t) = (u_l(t), \ldots, u_n(t))$ of class $C^{(2)}[a, b]$. Then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the boundary value problem* (1.1) *has a solution*

$$\boldsymbol{y}(t) = \boldsymbol{y}(t, \epsilon) = (y_1(t, \epsilon), \ldots, y_n(t, \epsilon)),$$

for t in $[a, b]$, satisfying

$$|y_i(t, \epsilon) - u_i(t)| \leq L_i(t, \epsilon) + R_i(t, \epsilon) + O(\epsilon),$$

where $i = 1, \ldots, n$. Here

$$L_i(t, \epsilon) = \begin{cases} |A_i - u_i(a)| \exp[-m_i\epsilon^{-1}(t - a)], & \text{if } q = 0, \\ |A_i - u_i(a)|[1 + \sigma_{1i}\epsilon^{-1}(t - a)]^{-q^{-1}}, & \text{if } q \geq 1; \end{cases} \tag{3.3}$$

$$R_i(t, \epsilon) = \begin{cases} |B_i - u_i(b)| \exp[-m_i\epsilon^{-1}(b - t)], & \text{if } q = 0 \\ |B_i - u_i(b)|[1 + \sigma_{2i}\epsilon^{-1}(b - t)]^{-q^{-1}}, & \text{if } q \geq 1. \end{cases} \tag{3.4}$$

where

$$\sigma_{1i} = m_i \frac{q}{\sqrt{q+1}}|A_i - u_i(a)|^q, \quad \sigma_{2i} = m_i \frac{q}{\sqrt{q+1}}|B_i - u_i(b)|^q,$$

for $i = 1, \ldots, n$.

REMARK. For $q = 0$, the boundary layers at both end-points are of exponential type, while for $q \geq 1$, the boundary layers are of algebraic type.

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PROOF. Theorem 1 will follow from Lemma 1, if we can exhibit, by construction, the existence of lower and upper pairs of bounding functions $(\alpha_i(t, \epsilon), \beta_i(t, \epsilon))$ which possess the required properties $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$, with $\alpha_i(t)$, $\beta_i(t)$ replaced by $\alpha_i(t, \epsilon)$, $\beta_i(t, \epsilon)$ respectively and with h_i replaced by $h_i\epsilon^{-2}$.

By assumption (3.2), we must have $h_i(t, \mathbf{y}) \sim m_i^2 y_i^{2q+1}$, and we are led to consider the differential equation

$$\epsilon^2 z_i'' = m_i^2 z_i^{2q+1}. \tag{3.5}$$

Indeed, the function $L_i(t, \epsilon)$ is non-negative and is the solution of (3.5) such that

$$L_i(a, \epsilon) = |A_i - u_i(a)|,$$

and

$$L_i'(a, \epsilon) = -\frac{m_i}{\epsilon\sqrt{q+1}}|A_i - u_i(a)|^{q+1}.$$

This solution decreases to the right. Similarly, the function $R_i(t, \epsilon) \geq 0$ is the solution of (3.5) such that

$$R_i(b, \epsilon) = |B_i - u_i(b)|,$$

and

$$R_i'(b, \epsilon) = \frac{m_i}{\epsilon\sqrt{q+1}}|B_i - u_i(b)|^{q+1},$$

and decreases to the left.

We now define, for t in $[a, b]$ and $\epsilon > 0$, the required lower and upper functions

$$\alpha_i(t, \epsilon) = u_i(t) - L_i(t, \epsilon) - R_i(t, \epsilon) - \Gamma_i(\epsilon),$$
$$\beta_i(t, \epsilon) = u_i(t) + L_i(t, \epsilon) + R_i(t, \epsilon) + \Gamma_i(\epsilon),$$

where

$$\Gamma_i(\epsilon) = [\epsilon^2\gamma_i/m_i^2(2q+1)!]^{1/(2q+1)}.$$

Here, each γ_i is a positive constant chosen so large that

$$\gamma_i \geq M_i(2q+1)! \tag{3.6}$$

where $M_i = \max_{[a,b]}[|u_i''(t)|]$. Clearly we have $\Gamma_i(\epsilon) > 0$.

Observe that the region between α_i and β_i, that is, the set $\{(t, y_i), t \in [a, b], \alpha_i(t, \epsilon) \leq y_i \leq \beta_i(t, \epsilon)\}$ is contained in the region $\mathfrak{D}_i$ when ϵ is sufficiently small.

Clearly, α_i and β_i satisfy the required properties $(2.1)_1$ and $(2.1)_2$. It remains to show that the property $(2.1)_3$, with $h_i\epsilon^{-2}$ in place of h_i, also holds. Applying Taylor's Theorem and the hypothesis that $\mathbf{u}(t)$ is I_q-stable, we have

$$\begin{aligned}\epsilon^2\alpha_i'' - h_i(t, y_1, \dots, \alpha_i, \dots, y_n) &= \epsilon^2 u_i'' - \epsilon^2 L_i'' - \epsilon^2 R_i'' + \frac{1}{(2q+1)!}\frac{\partial^{2q+1}h_i}{\partial y_i^{2q+1}} \\ &\quad \times (t, y_1, \dots, \theta_i, \dots, y_n)[\alpha_i(t,\epsilon) - u_i(t)]^{2q+1} \\ &= \epsilon^2 u_i'' - \epsilon^2 L_i'' - \epsilon^2 R_i'' + \frac{1}{(2q+1)!}\frac{\partial^{2q+1}h_i}{\partial y_i^{2q+1}} \\ &\quad \times (t, y_1, \dots, \theta_i, \dots, y_n)(L_i + R_i + \Gamma_i)^{2q+1},\end{aligned}$$

where θ_i is some intermediate point between $\alpha_i(t,\epsilon)$ and $u_i(t)$. The point (t,θ_i) is therefore in $\mathfrak{D}_i$ if ϵ is sufficiently small, say, $\epsilon \leq \epsilon_0$. Since L_i, R_i, Γ_i are all positive and since both L_i and R_i satisfy (3.5), it follows by virtue of (3.2) and (3.6) that

$$\begin{aligned}\epsilon^2\alpha_i'' - h_i(t, y_1, \dots, \alpha_i, \dots, y_n) &\geq -\epsilon^2|u_i''| + m_i^2\Gamma_i^{2q+1} \\ &\geq -\epsilon^2 M_i + \frac{\epsilon^2\gamma_i}{(2q+1)!} = \epsilon^2\left[\frac{\gamma_i}{(2q+1)!} - M_i\right] \geq 0,\end{aligned}$$

and so

$$\epsilon^2\alpha_i'' \geq h_i(t, y_1, \dots, \alpha_i, \dots, y_n).$$

The proof for β_i is similar. Therefore Theorem 1 follows from Lemma 1.

4. **Angular layer phenomenon**. We now turn to the following situation: suppose that the reduced equation $\boldsymbol{h}(t,\boldsymbol{u}) = \boldsymbol{0}$ has a pair of $C^{(2)}$-solutions $\boldsymbol{u}_1 = \boldsymbol{u}_1(t)$ and $\boldsymbol{u}_2 = \boldsymbol{u}_2(t)$ which intersect at an interior point $t = T$ in (a,b). That is to say, $\boldsymbol{u}_1(T) = \boldsymbol{u}_2(T)$, but $\boldsymbol{u}_1'(T) \neq \boldsymbol{u}_2'(T)$, or if we define the reduced solution $\boldsymbol{u}(t)$ by

$$\boldsymbol{u}(t) = \begin{cases}\boldsymbol{u}_1(t), & a \leq t \leq T,\\ \boldsymbol{u}_2(t), & T \leq t \leq b,\end{cases}$$

then $\boldsymbol{u}'(T^-) \neq \boldsymbol{u}'(T^+)$. Thus, the essential characteristic of this situation is that the reduced solution $\boldsymbol{u}(t)$ does not have a continuous first derivative in (a,b), but has a 'corner' at an interior point.

We wish to determine if results similar to Theorem 1 can be obtained under appropriate stability assumptions on this type of reduced solution $\boldsymbol{u}(t)$. In view of the corner or angular nature of the reduced solution $\boldsymbol{u}(t)$, we expect that the bounding functions will be more complex than those considered earlier in Theorem 1. Furthermore, each component of the original solution can, in general, be expected to exhibit an angular or corner layer at a different interior point and also simultaneously exhibit boundary layer behavior at the end-points. This situation is demonstrated by an example in Section 5.

THEOREM 2. *Assume that*
(1) *there exists functions* $\boldsymbol{u}_1 = (u_{11}(t), \dots, u_{1n}(t))$ *and* $\boldsymbol{u}_2 = (u_{21}(t), \dots, u_{2n}(t))$ *with* $u_{ji}(t)$ *of class* $C^{(2)}$ *on* $[a, T_i]$ *and* $[T_i, b]$ *respectively, satisfying for* $j = 1, 2,$;

$$h_i(t, y_1, \dots, u_{ji}, \dots, y_n) = 0$$

for $t \in [a, b]$ *and* y_k in D_k, $k \neq i$. *Moreover,* $u_{1i}(T_i) = u_{2i}(T_i)$ *and* $u'_{1i}(T_i) < u'_{2i}(T_i)$, T_i *in* (a, b), *where*

$$D_k = \{y_k : |y_k - u_k(t)| \leq d_k(t)\}$$

with

$$u_k(t) = \begin{cases} u_{1k}(t), & t \in [a, T_k] \\ u_{2k}(t), & t \in [T_k, b], \end{cases}$$

and d_k *is a smooth positive function such that*

$$|A_k - u_k(t)| \leq d_k(t) \leq |A_k - u_k(t)| + \delta \text{ on } \left[a, a + \frac{\delta}{2}\right]$$

$$|B_k - u_k(t)| \leq d_k(t) \leq |B_k - u_k(t)| + \delta \text{ on } \left[b - \frac{\delta}{2}, b\right]$$

for $\delta > 0$ *a small constant;*

(2) *for a nonnegative integer* q, *the function* $\boldsymbol{h}$ *is continuous in* $(t, \boldsymbol{y})$ *and* $C^{(2q+1)}$ *with respect to* y_i *in* D_i;

(3) $u_j(t)$ *is* I_q*-stable for* $j = 1, 2$ *in* $[a, T_i]$ *and* $[T_i, b]$ *respectively.*

Then there exists an $\epsilon_0 > 0$ *such that for each* ϵ, $0 < \epsilon \leq \epsilon_0$, *there exists a solution* $\boldsymbol{y} = \boldsymbol{y}(t, \epsilon) = (y_1(t, \epsilon), \ldots, y_n(t, \epsilon))$ *of* (1.1). *Moreover, for* t *in* $[a, b]$

$$|y_i(t, \epsilon) - u_i(t)| \leq L_i(t, \epsilon) + R_i(t, \epsilon) + C_q \epsilon^{1/(q+1)},$$

for $i = 1, 2, \ldots, n$, *where* C_q *is a positive, computable constant independent of* ϵ,

$$L_i(t, \epsilon) = |A_i - u_{1i}(a)| E_i(t, \epsilon), \tag{4.1}$$

$$R_i(t, \epsilon) = |B_i - u_{2i}(b)| F_i(t, \epsilon), \tag{4.2}$$

$$E_i(t, \epsilon) = \begin{cases} \exp\left[-\frac{m_i}{\epsilon}(t - a)\right], & \text{if } q = 0 \\ \left[1 + \frac{\sigma_{1i}}{\epsilon}(t - a)\right]^{-(1/q)}, & \text{if } q \geq 1, \end{cases}$$

$$F_i(t, \epsilon) = \begin{cases} \exp\left[-\frac{m_i}{\epsilon}(b - t)\right], & \text{if } q = 0, \\ \left[1 + \frac{\sigma_{2i}}{\epsilon}(b - t)\right]^{-(1/q)}, & \text{if } q \geq 1 \end{cases}$$

and

$$\sigma_{1i} = \frac{m_i q}{\sqrt{q + 1}} |A_i - u_{1i}(a)|^q, \quad \sigma_{2i} = \frac{m_i q}{\sqrt{q + 1}} |B_i - u_{2i}(b)|^q.$$

PROOF. The theorem follows from Lemma 2, if we can show the existence of functions $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ which satisfy the required differential inequalities.

For t in $[a, b]$ and $\epsilon > 0$, define

$$\alpha_i(t,\epsilon) = \begin{cases} u_{1i}(t) - |A_i - u_{1i}(a)|E_i(t,\epsilon) \\ \qquad - |B_i - u_{2i}(b)|F_i(T_i,\epsilon) - \Gamma_i(\epsilon), & t \in [a, T_i], \\ u_{2i}(t) - |B_i - u_{2i}(b)|F_i(t,\epsilon) \\ \qquad - |A_i - u_{2i}(a)|E_i(T_i,\epsilon) - \Gamma_i(\epsilon), & t \in [T_i, b] \end{cases}$$

$$\beta_i(t,\epsilon) = \begin{cases} u_{1i}(t) + |A_i - u_{1i}(a)|E_i(t,\epsilon) + H_i(t,\epsilon) + \Delta_i(\epsilon), & t \in [a, T_i] \\ u_{2i}(t) + |B_i - u_{2i}(b)|F_i(t,\epsilon) + \Omega_i(\epsilon), & t \in [T_i, b], \end{cases}$$

where

$$\Delta_i(\epsilon) = (b - t)|A_i - u_{1i}(a)|E_i'(T_i,\epsilon) + \Gamma_i(\epsilon) \\ \qquad + |B_i - u_{2i}(b)|[F_i(T_i,\epsilon) + (b - T_i)F_i'(T_i,\epsilon)],$$

$$\Omega_i(\epsilon) = H_i(T_i,\epsilon) + (b - t)|B_i - u_{2i}(b)|F_i'(T_i,\epsilon) + \Gamma_i(\epsilon) \\ \qquad + |A_i - u_{1i}(a)|[E_i(T_i,\epsilon) + (b - T_i)E_i'(T_i,\epsilon)],$$

$$H_i(t,\epsilon) = \begin{cases} \dfrac{\epsilon}{m_i}[u_{2i}'(T_i) - u_{1i}'(T_i)] \exp\left[-\dfrac{m_i}{\epsilon}(T_i - t)\right], & \text{if } q = 0, \\ \dfrac{q\epsilon^{1/(q+1)}[u_{2i}'(T_i) - u_{1i}'(T_i)]}{k_i[1 + k_i\epsilon^{-(q+1)^{-1}}(T_i - t)]^{q^{-1}}}, & \text{if } q \geq 1 \end{cases}$$

$$k_i = |m_i\sqrt{q+1}\, q^{(q+1)}[u_{2i}'(T_i) - u_{1i}'(T_i)]^q|^{(q+1)^{-1}}.$$

Here $\Gamma_i(\epsilon) = [\gamma_i\epsilon^2/m_i(2q + 1)!]^{(2q+1)^{-1}}$, and $\gamma_i > 0$ is a constant chosen so large that

$$\gamma_i \geq M_i(2q + 1)!, \quad M_i = \max\left\{\max_{[a,T_i]} |u_{1i}''(t)|, \quad \max_{[T_i,b]} |u_{2i}''(t)|\right\}.$$

We observe that $\alpha_i \leq \beta_i$, $\alpha_i(a,\epsilon) \leq A_i \leq \beta_i(a,\epsilon)$, $\alpha_i(b,\epsilon) \leq B_i \leq \beta_i(b,\epsilon)$, and that $D_R\alpha_i(T_i) \geq D_L\alpha_i(T_i)$ and $D_R\beta_i(T_i) \leq D_L\beta_i(T_i)$, for all sufficiently small values of ϵ.

It only remains to verify that the differential inequalities

$$\text{(4.3)} \qquad \begin{cases} \epsilon^2\alpha_i''(t,\epsilon) \geq h_i(t, y_1, \ldots, \alpha_i(t,\epsilon), \ldots, y_n) \\ \epsilon^2\beta_i''(t,\epsilon) \leq h_i(t, y_1, \ldots, \beta_i(t,\epsilon), \ldots, y_n) \end{cases}$$

are satisfied on $[a, T_i]$ and $[T_i, b]$. We only verify the inequality for β_i, since the verification for α_i is similar.

We can easily see that the terms $\Delta_i(\epsilon)$ and $\Omega_i(\epsilon)$ are nonnegative for ϵ sufficiently small, even though they contain the negative terms $(b - t)\,[A_i - u_{1i}(a)]E_i'(a,\epsilon)$ and

$[A_i - u_{1i}(a)]E_i'(a, \epsilon)$ respectively. On $[a, T_i]$, by differentiating β_i, substituting into $(2.1)_3$ and expanding by Taylor's Theorem, we have

$$h_i(t, y_1, \ldots, \beta_i, \ldots, y_n) - \epsilon^2\beta_i'' = h_i(t, y_1, \ldots, u_{1i}, \ldots, y_n)$$
$$+ \sum_{k=1}^{2q}\left\{\frac{1}{k!}\frac{\partial^k h_i}{\partial y_i^k}(t, y_1, \ldots, u_{1i}, \ldots, y_n)[(A_i - u_{1i}(a)]E_i(t, \epsilon)\right.$$
$$\left. + H_i(t, \epsilon) + \Delta_i(\epsilon)]^k\right\} + \frac{1}{(2q+1)!}\cdot\frac{\partial^{2q+1}h_i}{\partial y_i^{2q+1}}(t, y_1, \ldots, \eta_{1i}, \ldots, y_n)$$
$$\times [(A_i - u_{1i}(a))E_i(t, \epsilon) + H_i(t, \epsilon) + \Delta_i(\epsilon)]^{2q+1}$$
$$- \epsilon^2 u_{1i}''(t) - \epsilon^2[A_i - u_{1i}(a)]E_i''(t, \epsilon) - \epsilon^2 H_i''(t, \epsilon),$$

where η_{1i} is the appropriate intermediate value. In view of the I_q-stability of $\boldsymbol{u}$ and the fact that $\Delta_i(\epsilon) \geq 0$, it follows that

$$h_i(t, y_1, \ldots, \beta_i, \ldots, y_n) - \epsilon^2\beta_i'' \geq m_i[(A_i - u_{1i}(a))^{2q+1}E_i^{2q+1}(t, \epsilon)$$
$$+ H_i^{2q+1}(t, \epsilon) + \Delta_i^{2q+1}(\epsilon)] - \epsilon^2 M_i - \epsilon^2(A_i - u_{1i}(a))$$
$$\times E_i''(t, \epsilon) - \epsilon^2 H_i''(t, \epsilon).$$

By construction, the functions E_i and H_i satisfy the differential equation

$$\epsilon^2 Z_i'' = m_i^2 Z_i^{2q+1},$$

and so

$$h_i(t, y_1, \ldots, \beta_i, \ldots, y_n) - \epsilon^2\beta_i'' \geq m_i^2\Delta_i^{(2q+1)}(\epsilon) - \epsilon^2 M_i$$
$$\geq m_i^2\left[\frac{\gamma_i\epsilon^2}{m_i^2(2q+1)!}\right] - \epsilon^2 M_i = \epsilon^2\left[\frac{\gamma_i}{(2q+1)!} - M_i\right] \geq 0.$$

The verification of the differential inequality for $\beta_i(t, \epsilon)$ for t in $[T_i, b]$ is similar and so we omit details.

REMARK. If some of the derivatives of the functions u_{1i} and u_{2i} satisfy the inequality $u_{1i}'(T_i) > u_{2i}'(T_i)$, then it is possible to obtain results which are analogous to Theorem 2. We can simply make the change of dependent variable $y_i \to -y_i$ and apply Theorem 2 to the transformed problem.

5. **An example**. Consider the problem

$$\epsilon^2\boldsymbol{y}'' = \boldsymbol{h}(t, \boldsymbol{y}), \qquad -1 < t < 1,$$
$$\boldsymbol{y}(-1, \epsilon) = \boldsymbol{A}, \qquad \boldsymbol{y}(1, \epsilon) = \boldsymbol{B},$$

where $\boldsymbol{h}(t, \boldsymbol{y})$ is the column vector

$$((y_1 - |t|)^{2q+1}(1 + G(y_2)), \qquad (y_2 - 1 + |t|)^{2q+1}(1 + H(y_1))).$$

Here q is a nonnegative integer, $G(y_2) \geq 0$, $H(y_1) \geq 0$.

The reduced solution is the column vector $(|t|, 1 - |t|)$ and does not have a continuous derivative at $t = 0$. The reduced solution is stable, since

$$\frac{\partial^{2q+1} h_i}{\partial y_1^{2q+1}} = 1 + G(y_2) \geq 1 > 0,$$

$$\frac{\partial^{2q+1} h_2}{\partial y^{2q+1}} = 1 + H(y_1) \geq 1 > 0.$$

By Theorem 2 there exists a solution $\mathbf{y} = (y_1(t, \epsilon), y_2(t, \epsilon))$ for ϵ sufficiently small which satisfies the following inequalities:

$$|y_1 - |t|| \leq L_1 + R_1 + C_q \epsilon^{(q+1)^{-1}}$$

$$|y_2 - 1 + |t|| \leq L_2 + R_2 + C_q \epsilon^{(q+1)^{-1}},$$

where

$$L_1 = \begin{cases} |A_1 - 1| \exp\left(\dfrac{-1 - t}{\epsilon}\right), & q = 0, \\ \dfrac{|A_1 - 1|}{\left[1 + \dfrac{q}{\epsilon\sqrt{q+1}} |A_1 - 1|^q (1 + t)\right]^{q^{-1}}}, & q \geq 1, \end{cases}$$

$$R_1 = \begin{cases} |B_1 - 1| \exp\left(\dfrac{-1 + t}{\epsilon}\right), & q = 0, \\ \dfrac{|B_1 - 1|}{\left[1 + \dfrac{q}{\epsilon\sqrt{q+1}} |B_1 - 1|^q (1 - t)\right]^{q^{-1}}}, & q \geq 1, \end{cases}$$

$$L_2 = \begin{cases} |A_2| \exp\left(\dfrac{-1 - t}{\epsilon}\right), & q = 0, \\ \dfrac{|A_2|}{\left[1 + \dfrac{q}{\epsilon\sqrt{q+1}} |A_2|^q (1 + t)\right]^{q^{-1}}}, & q \geq 1, \end{cases}$$

$$R_2 = \begin{cases} |B_2| \exp\left(\dfrac{-1 + t}{\epsilon}\right), & q = 0, \\ \dfrac{|B_2|}{\left[1 + \dfrac{q}{\epsilon\sqrt{q+1}} |B_2|^q (1 - t)\right]^{q^{-1}}}, & q \geq 1, \end{cases}$$

where C_q is positive, computable constant independent of ϵ. (The result is indicated in the following figures.)

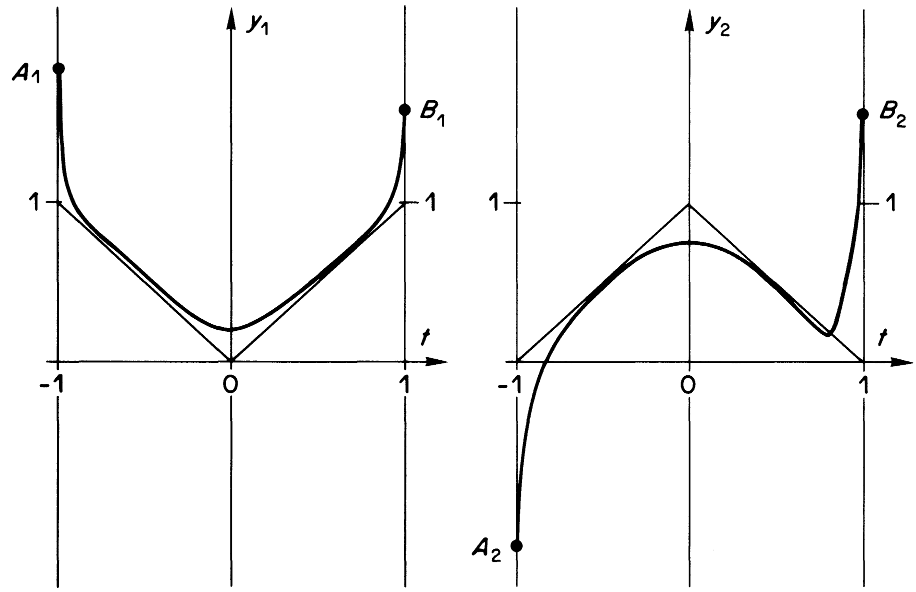

Figure 1

References

1. N. I. Brish, *On Boundary Value Problems for the Equation* $\epsilon y'' = f(x, y, y')$ *for small* ϵ, Dokl. Akad. Nauk SSSR **95** (1954), pp. 429–432.

2. P. Hebets and M. Laloy, *Étude de problèmes aux limites par la méthod des sur-et sous-solutions*, Lecture notes, Catholic University of Louvain, 1974.

3. S. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

4. Yu. B. Boglaev, *The two-point problem for a class of ordinary differential equations with a small parameter coefficient of the derivative*, USSR Comp. Math. and Math. Phys. **10** (1970), 4, pp. 191–204.

5. K. W. Chang and F. A. Howes, *Nonlinear Singular Perturbation Phenomena, Theory and Appl.* Springer-Verlag Pub. 1984.

6. M. A. O'Donnell, *Boundary and Corner Layer Behavior in Singularly Perturbed Semilinear Systems of Boundary Value Problems*, SIAM J. Math. Anal. (To appear).

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