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COUNTABLY COMPACT SPACES AND MARTIN'S AXIOM

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The relationship between compact and countably compact topological spaces has been studied by many topologists. In particular an important question is: "What conditions will make a countably compact space compact?" Conditions which are "covering axioms" have been extensively studied. The best results of this type appear in [**19**]. We wish to examine countably compact spaces which are separable or perfectly normal. Recall that a space is *perfect* if and only if every closed subset is a G_δ, and that a space is *perfectly normal* if and only if it is both perfect and normal. We show that the following statement follows from $MA + \neg CH$ and thus is consistent with the usual axioms of set theory: Every countably compact perfectly normal space is compact. This result is Theorem 3 and can be understood without reading much of what goes before.

A preliminary version of this article, written in 1975, has seen wide circulation. A happy consequence of this is that the original versions of both Theorem 2 and Theorem 3 have been improved. In particular, K. Kunen gave a better proof of Theorem 2 which eliminates the need for complete regularity. Also, P. Zenor, G. Gruenhage and J. Rogers, and independently J. Chaber, significantly simplified the proof of Theorem 3, eliminating the need for complete regularity and the full use of MA. These improvements are incorporated in this article.

We use the usual set-theoretic notation; cardinals are identified with initial ordinals. All spaces are assumed to be T_1. We employ the terminology of cardinal functions as in [**9**]. The *Lindelöf degree* of X is $L(X) = \min\{\kappa :$ every open cover of X has a subcover of cardinality $\leqq \kappa\}$. The *density* of X is $d(X) = \min\{\kappa : X$ has a dense subset of cardinality $\kappa\}$. The *cellularity* of X is $c(X) = \sup\{\kappa : X$ has a discrete subspace of cardinality $\kappa\}$.

The following definitions are found in [**1**]. For $x \in \bar{Z} - Z$, $t(x, Z) = \min\{\lambda :$ there is a subset A of Z such that $|A| = \lambda$ and $x \in \bar{A}\}$. The *tightness* of X is $t(X) = \sup\{t(x, Z) : Z \subseteq X$ and $x \in \bar{Z} - Z\}$.

Martin's Axiom is a set-theoretic proposition implied by $2^{\aleph_0} = \aleph_1$. However, it is also known to be consistent with $2^{\aleph_0} > \aleph_1$; see [**13**]. We are mainly interested in the following two consequences of MA.

PROPOSITION 1. *MA implies both statements* (i) *and* (ii).

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(i) *If X is a regular space such that $c(X) = \aleph_0$ and κ is a regular cardinal less than $2^{\aleph_0}$, then κ is a precaliber for X (i.e. if $\mathscr{U}$ is a family of κ open subsets of X, then there is a subfamily $\mathscr{U}' \subseteq \mathscr{U}$ of cardinality κ such that $\mathscr{U}$ has the finite intersection property).*

(ii) *If $\mathscr{F}$ is a family of less than $2^{\aleph_0}$ subsets of ω such that the intersection of each finite subfamily of $\mathscr{F}$ is infinite, then there exists an infinite $D \subseteq \omega$ such that for each $F \in \mathscr{F}$ we have $D \setminus F$ finite.*

Part (i) of the above proposition is found in [**9**]. Part (ii) is sometimes called Booth's lemma [Bo]. Let us denote statement (ii) by S. One of the several interesting uses of S is the following proposition and its corollary, taken from [**13**] and [**7**].

Proposition 2. *Assume S. If X is a countably compact separable space and $\mathscr{U}$ is an open cover of X with $|\mathscr{U}| < 2^{\aleph_0}$, then there exists a finite subcollection $\mathscr{U}' \subseteq \mathscr{U}$ such that $\{\bar{U} : U \in \mathscr{U}'\}$ covers X.*

Corollary. *Assume S. If X is a countably compact, separable, regular space with $L(X) < 2^{\aleph_0}$. Then X is compact.*

As shown in [**5**], a countably compact, separable, regular space may not be compact. In fact, the following lemma easily leads to many such examples.

Lemma 1. *If X is a countable compact extremally disconnected regular space then every infinite subset of X has at least two limit points.*

Proof. Let X be as in the statement of the lemma and suppose $A = \{a_n : n < \omega\}$ is an infinite subset of X with only one limit point y. We may assume that for all $m \neq n$, $y \neq a_n \neq a_m$. Now for each $m < \omega$,

$$\{a_n : m < n < \omega\} \cup \{y\}$$

is closed, hence we can define a sequence of open sets $\{U_m : m < \omega\}$ by induction. Let U_0 be an open neighbourhood of a_0 such that $\bar{U}_0 \cap (\{a_n : 0 < n < \omega\} \cup \{y\}) = \emptyset$. Let U_m be an open neighbourhood of a_m such that for all $n < m$, $U_n \cap U_m = \emptyset$ and $\bar{U}_m \cap (\{a_n : m < n < \omega\} \cup \{y\}) = \emptyset$. Let

$$U = \bigcup \{U_{2k} : k < \omega\}.$$

U is open, and since X is extremally disconnected, so is $\bar{U}$. Thus $y \in \bar{U}$ is an open neighbourhood of y which is disjoint from $\{a_{2k+1} : k < \omega\}$, contradicting X countably compact.

Example. Let $p \in \beta N - N$ such that the smallest local basis for p has cardinality $2^{\aleph_0}$. The existence of such points was proven in [**15**]. $\beta N - \{p\}$ is a countably compact, separable, completely regular space with Lindelöf degree $2^{\aleph_0}$. $\beta N - \{p\}$ is countably compact since every infinite subset of βN has at least two limit points, so that every infinite subset of $\beta N - \{p\}$ must have at least one. $\beta N - \{p\}$ must have Lindelöf degree $\leqq 2^{\aleph_0}$, since βN has a basis of cardinality $2^{\aleph_0}$. If $L(\beta N - \{p\}) = \kappa < 2^{\aleph_0}$, then a straightforward argument shows that there exists a collection of open subsets β, such that $|\beta| = \kappa$ and $\bigcup \beta = \{p\}$. Since βN is compact, β would give rise to a local basis of cardinality κ, contradicting our choice of p.

In [**14**] Ostaszewski constructs a space θ with the aid of $\diamond$, a combinatorial principle consistent with the usual axioms of set theory [**4**]. The space θ is regular, hereditarily separable, countably compact, and not compact. Furthermore, θ is perfectly normal. However, we will show that with MA and $2^{\aleph_0} > \aleph_1$, we can prove that every countably compact, perfectly normal space is compact. Let us begin by examining countably compact, perfectly normal spaces.

PROPOSITION 3. *If X is a countably compact, perfect space, then $s(X) = \aleph_0$. Furthermore, if X is also regular, then $t(X) = \aleph_0$.*

For the proof see [**14**] or [**17**]. The following proposition is in [**9**].

PROPOSITION 4. *Suppose X is a topological space.*

(i) *X is not hereditarily Lindelöf if and only if there is a subspace $A \subseteq X$ and a well-ordering $A = \{a_\alpha : \alpha < \omega_1\}$, such that for all $\beta < \omega_1 \{a_\alpha : \alpha < \beta\}$ is open in A. The subspace A is called a right-separated subspace (of cardinality $\aleph_1$).*

(ii) *X is not hereditarily separable if and only if there is a subspace $B \subseteq X$ and a well-ordering $B = \{b_\alpha : \alpha < \omega_1\}$, such that for all $\beta < \omega_1 \{b_\alpha : \alpha < \beta\}$ is closed in B. The subspace B is called a left-separated subspace (of cardinality $\aleph_1$).*

The following proposition is very useful. The proof appears in [**8**].

PROPOSITION 5. *If A is a right-separated space with $s(A) = \aleph_0$, then A is hereditarily separable.*

A similar proof shows that if B is a left-separated space with $s(B) = \aleph_0$, then B is hereditarily Lindelöf. From Proposition 5 we can show the following.

THEOREM 1. *If there exists a countably compact, perfectly normal space which is not compact, then there exists a separable one.*

Proof. Suppose X is a countably compact, perfectly normal space which is not Lindelöf. There exists an uncountable right-separated subspace $A \subseteq X$. By Proposition 5, A is separable. Hence $\bar{A}$ is a countably compact, perfectly normal, separable space. Since A is not Lindelöf, $\bar{A}$ is perfectly normal, $\bar{A}$ cannot be Lindelöf.

However, if there exists a countably compact, perfectly normal, non-compact space, it need not be separable. For example, if we assume $\diamond$ we can construct both Ostaszewski's space θ, as in [**12**], and a compact Souslin line L, as in [**4**]. The disjoint union of θ and L is a countably compact, perfectly normal space which is neither compact nor separable. However, as Proposition 3 and the next theorem show, if we assume MA and $2^{\aleph_0} > \aleph_1$, every countably compact, perfectly normal space is hereditarily separable. This generalizes a well-known theorem of I. Juhász [**9**].

THEOREM 2. *Assume MA and* $2^{\aleph_0} > \aleph_1$. *If X is a countably compact regular space with* $s(X) = \aleph_0$, *then X is hereditarily separable.*

Proof. We suppose that X is not hereditarily separable and derive a contradiction. By restricting the argument for Theorem 1 in [1] to the countable case we can conclude that since X is countably compact and regular with $s(X) = \aleph_0$, $t(X) = \aleph_0$.

Let $\{y_\alpha : \alpha < \omega_1\}$ be a left-separated subspace of X and let

$$Y_\alpha = \operatorname{cl}\{y_\gamma : \gamma < \alpha\}.$$

Let $Y = \bigcup\{Y_\alpha : \alpha < \omega_1\}$; Y is closed since $t(X) = \aleph_0$. We now deal only with the subspace Y.

For each $\alpha < \omega_1$, let U_α be an open neighbourhood of y_α such that $\bar{U}_\alpha \cap Y_\alpha = \emptyset$. Since $s(X) = \aleph_0$, $c(Y) = \aleph_0$ and hence by Proposition 1 (i), Y has $\aleph_1$ as a precaliber. So let S be an uncountable subset of ω_1 such that $\{U_\alpha : \alpha \in S\}$ has the finite intersection property. Since Y is countably compact, for each $\beta \in S$ we can define a non-empty F_β as follows:

$$F_\beta = \bigcap\{\bar{U}_\alpha : \alpha \leqq \beta \text{ and } \alpha \in S\}.$$

We can now inductively pick an increasing sequence of ordinals $\{\beta_\mu : \mu < \omega_1\}$ and a sequence of points $\{x_\mu : \mu < \omega_1\}$ such that $\{x_\nu : \nu < \mu\} \subseteq Y_{\beta_\mu}$ but $x_\mu \in F_{\beta\mu}$. Then $\{x_\mu : \mu < \omega_1\}$ is left-separated by $\{Y_{\beta\mu} : \mu < \omega_1\}$ and right-separated by $\{F_{\beta\mu} : \mu < \omega_1\}$ and is hence discrete. This contradicts that $s(X) = \aleph_0$.

The next theorem shows that it is consistent with the usual axioms of set theory that every countably compact, perfectly normal space is compact. This answers the question asked in [**4**; **6**; **2**; and **7**].

THEOREM 3. *Assume S and* $2^{\aleph_0} > \aleph_1$. *Every countably compact, perfect, regular space is compact.*

Proof. Suppose X is a countably compact, perfect, regular space which is not Lindelöf. X contains a right-separated subspace Y of cardinality $\aleph_1$. Let $Y = \{Y_\alpha : \alpha < \omega_1\}$. For each $\beta < \omega_1$, there is a set U_β open in X such that $\{y_\alpha : \alpha \leqq \beta\} \subseteq U_\beta$ and $U_\beta \cap \{y_\alpha > \beta\} = \emptyset$.

Since X is regular, for each $\beta < \omega_1$ we can pick an open neighbourhood V_β of y_β such that $\bar{V}_\beta \subseteq U_\beta$. No countable subcollection of $\{\bar{V}_\beta : \beta < \omega_1\}$ can cover an uncountable subset of Y.

Since X is perfect and $\bigcup\{V_\beta : \beta < \omega_1\}$ is open, there is a countable collection of closed sets $\{F_n : n < \omega\}$ such that $\bigcup\{V_\beta : \beta < \omega_1\} = \bigcup\{F_n : n < \omega\}$. So there is an $m < \omega$ such that $Y \cap F_m$ is uncountable.

Let $E = \overline{Y \cap F_m}$. Since E is a closed subset of F_m, E is countably compact and regular. By Propositions 3 and 5, Y is hereditarily separable, so that E is separable. Thus a contradiction to Proposition 2 is achieved, completing the proof of the theorem.

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COROLLARY. *Assume MA and* $2^{\aleph_0} > \aleph_1$. *Every countably compact, perfectly normal space is compact.*

Recall that Ostaszewski's space θ is constructed using and is countably compact perfectly normal and not compact. It should be noted that, using $\diamond$, a space is contructed in [**18**] which is countably compact, locally compact, perfect, completely regular, hereditarily separable and not normal. It is also of interest to note that J. Chaber [**3**] has proven the following theorem: every countably compact space with a G_δ — diagonal is compact.

The proposition S is a "combinatorial" consequence of MA, in the sense of [**11**]. Thus, as is shown in [**11**], there is a model of set theory in which there is a Souslin line, but S holds, hence every countably compact, perfect, regular space is compact. Hence, although it is shown in [**16**] that the existence of a Souslin line implies the existence of a normal, hereditarily separable, non-Lindelöf space; the existence of a Souslin line does not imply the existence of Ostaszewski's space B. F. Tall has noticed the following application of Theorem 3.

THEOREM 4. *Assume MA and* $2^{\aleph_0} > \aleph_1$. *If* X *is a countably compact space such that* X^2 *is hereditarily normal, then* X *is compact.*

Proof. We can assume X is infinite and thus contains a countably infinite subset A. Since X is countably compact, A cannot be discrete; hence A contains a countable subset which is not closed. By a theorem in [**10**], X is perfectly normal. Thus, by Theorem 3, X is compact.

This leads to what may be the most elementarily-stated mathematical result, which is consistent with and independent of the usual axioms of set theory.

THEOREM 5. *It is consistent with and independent of the axioms of set theory that there exists a countably compact non-compact space* X *such that* $X \times [0, 1]$ *is hereditarily normal.*

Proof. The independence is shown in Theorem 4. To prove the consistency let θ be Ostaszewski's space, as in [**14**], and consider the space $\theta \times [0, 1]$. Since a perfectly normal space is hereditarily normal, the following lemma completes the proof.

LEMMA 2. X *is perfectly normal if and only if* $X \times [0, 1]$ *is perfectly normal.*

Proof. The necessity is evident. To show the sufficiency, note that X is countably paracompact and hence $X \times [0, 1]$ is normal, and thus it only remains to show that $X \times [0, 1]$ is perfect.

Let $\{B_n : n < \omega\}$ be a basis for $[0, 1]$. Let $\{U_\alpha : \alpha < \kappa\}$ be a basis for X. Let V be an arbitrary open subset of $X \times [0,1]$. For each $n < \omega$, let

$A_n = \{\alpha \in \kappa : U_\alpha \times B_n \subseteq V\}$. Thus

$$V = \bigcup \{\bigcup \{U_\alpha \times B_n : \alpha \in A_n\} : n < \omega\} = \bigcup \{\bigcup \{U_\alpha : \alpha \in A_n\} \times B_n : n < \omega\}.$$

Since $\bigcup \{U_\alpha : \alpha \in A_n\}$ is open in X,

$$\bigcup \{U_\alpha : \alpha \in A_n\} = \bigcup \{F_m^n : m < \omega\}$$

where each F_m^n is closed in X. Also, $B_n = \bigcup \{E_k^n : k < \omega\}$ where each E_k^n is closed in $[0, 1]$. Hence

$$\bigcup \{U_\alpha : \alpha \in A_n\} \times B_n = \bigcup \{F_m^n \times E_k^n : m < \omega, k < \omega\}.$$

Thus

$$V = \bigcup \{ \bigcup \{F_m^n \times E_k^n : m < \omega, k < \omega\} : n < \omega\} = \bigcup \{F_m^n \times E_k^n : m < \omega; k < \omega, n < \omega\},$$

which is an F_σ. Therefore $X \times [0, 1]$ is perfect and the lemma is proved.

Using a theorem of P. Zenor in [**20**], we can prove results similar to Theorem 4 and Theorem 5, replacing "hereditarily normal" with "hereditarily countably paracompact". Using Katetov's theorem in [**10**] as well, we have the following corollary to Lemma 2.

COROLLARY. *The following are equivalent*:
(i) *X is perfectly normal.*
(ii) *$X \times [0, 1]$ is perfectly normal.*
(iii) *$X \times [0, 1]$ is hereditarily normal.*
(iv) *$X \times [0, 1]$ is hereditarily countably paracompact.*

REFERENCES

1. A. V. Arhangel'skii, *On bicompacta hereditarily satisfying Souslin's condition*, Soviet Math. Dokl. *12* (1971), 1253–1247.
2. M. P. Berri, J. R. Porter and R. M. Stephenson, *A survey of minimal topological spaces*, Proceedings of the Indian Topological Conference in Kanpur (Academic Press, N.Y. 1971).
3. J. Chaber, *Conditions which imply compactness in countably compact spaces*, preprint.
4. K. J. Devlin, *Aspects of constructibility*, Lecture Notes in Mathematics *354* (Springer Verlag, 1973).
5. S. P. Franklin and M. Rajagopalan, *Some examples in topology*, Trans. Amer. Math. Soc. *155* (1971), 305–314.
6. P. Fletcher and W. F. Lindgren, *Some unsolved problems concerning countably compact spaces*, Rocky Mountain J. Math. *5* (1975), 95–106.
7. S. H. Hechler, *On some weakly compact spaces and their products*, Gen. Top. Appl. *5* (1975), 83–93.
8. A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. *29* (1967), 343–356.
9. I. Juhász, *Cardinal functions in topology*, Math. Centre Tract No. 34, Amsterdam (1971).
10. M. Katetov, *Complete normality of Cartesian products*, Fund. Math. *35* (1948), 271–274.
11. K. Kunen and F. D. Tall, *Between Martin's axiom and Souslin's hypothesis*, preprint.

12. D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic *2* (1970), 143–178.
13. V. I. Malhyin and B. E. Sapirovskii, *Martin's axiom and properties of topological spaces*, Soviet Math. Dokl. *14* (1973), 1746–1751.
14. A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, preprint.
15. B. Pospisil, *On bicompact spaces*, Publ. Fac. Sci. Univ. Masaryk *270* (1939), 3–16.
16. M. E. Rudin, *A normal hereditarily separable non-Lindelöf space*, Ill. J. Math. *16* (1972), 621–626.
17. R. M. Stephenson, Jr., *Discrete subsets of perfectly normal spaces*, Proc. Amer. Math. Soc. *34* (1972), 621–626.
18. M. L. Wage, *Countable paracompactness, normality and Moore spaces*, preprint.
19. H. H. Wicke and J. M. Worrell, Jr., *Point-countability and compactness*, preprint.
20. P. Zenor, *Countable paracompactness in product spaces*, Proc. Amer. Math. Soc. *30* (1971), 199–201.

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