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LOCAL BIFURCATION OF CRITICAL PERIODS IN VECTOR FIELDS WITH HOMOGENEOUS NONLINEARITIES OF THE THIRD DEGREE

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ABSTRACT. In this paper we study the local bifurcation of critical periods of periodic orbits in the neighborhood of a nondegenerate centre of a vector field with a homogeneous nonlinearity of the third degree. We show that at most three local critical periods bifurcate from a weak linear centre of finite order or from the linear isochrone and at most two local critical periods from the nonlinear isochrone. Moreover, in both cases, there are perturbations with the maximum number of critical periods.

1. **Introduction.** We consider the bifurcation of critical periods of periodic solutions in the neighborhood of a nondegenerate centre of a vector field in the form:

$$\begin{aligned} \dot{x} &= -y + \sum_{j+l=3} b_{jl} x^j y^l \\ \dot{y} &= x + \sum_{j+l=3} c_{jl} x^j y^l. \end{aligned} \tag{1.1}$$

These systems have been studied in the literature; it is known [S] that at most five limit cycles can appear in a nondegenerate Hopf bifurcation at the origin or from a perturbation of a centre. Necessary and sufficient conditions for the centre have been given by Malkin [M] and isochronous systems have been determined by Pleshkan [P]. The study of critical points of the period of an autonomous system with a centre is an important one; it has been studied by several authors [C.S], [G], [C.J]. These three papers consider the one degree of freedom "kinetic + potential" Hamiltonian system with polynomial potentials. [C.J] also addresses the problem of the maximum number of critical periods bifurcating from the centre of a quadratic system and gives a complete answer to that problem: at most two critical periods bifurcate from a weak linear centre of finite order or from the linear isochrone and at most one critical period bifurcates from a nonlinear isochrone.

Here we address the same question for centres of systems with homogeneous nonlinearities of the third degree, using a method similar to the one used for Hopf bifurcation and we give a complete answer: at most three critical periods bifurcate from a weak centre of finite order or from the linear isochrone and at most two critical periods bifurcate

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from a nonlinear isochrone. As in the quadratic case, we identify the centres leading to the maximum number of critical periods.

2. **Preliminaries.** Let $\mathbb{X}(x, y, \lambda)$ be a family of plane analytic vector fields parametrized by $\lambda \in \mathbb{R}^n$ with a nondegenerate centre at the origin. We call $P(r, \lambda)$ the minimum period of the periodic trajectory through a nonzero point $(r, 0)$ of a sufficiently small open interval $J = (-\alpha, \alpha)$ of the x-axis. The following properties of $P(r, \lambda)$ have been proved in [C.J].

1. If we define $P(0, \lambda) = 2\pi$, for $\lambda \in \mathbb{R}^n$, then for $\lambda_* \in \mathbb{R}^n$ there exists an open neighborhood W of λ_* and an open interval U containing $r = 0$ such that $P(r, \lambda)$ is analytic on $U \times W$. Hence, we may write:

$$P(r, \lambda) = 2\pi + \sum_{k=2}^{\infty} p_k(\lambda) r^k, \tag{2.1}$$

for $|r|$ and $|\lambda - \lambda_*|$ sufficiently small and $p_k \in \mathbb{R}[\lambda_1, \ldots, \lambda_n]$, where $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$ is the noetherian ring of polynomials in the variables $\lambda_1, \ldots, \lambda_n$.

2. For $k \geq 1$, $p_{2k+1} \in (p_2, p_4, \ldots, p_{2k})$, the ideal generated by p_{2i}, for $i = 1, \ldots, k$ over $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$. Moreover the first $k > 1$ such that $p_k(\lambda) \neq 0$ is even.

DEFINITION 2.1. Let us define $F(r, \lambda_*) = P(r, \lambda_*) - 2\pi$, for a parameter value λ_*.

1. If $F(0, \lambda_*) = F'(0, \lambda_*) = \cdots = F^{(2k+1)}(0, \lambda_*) = 0$ and $F^{(2k+2)}(0, \lambda_*) \neq 0$, then the origin is a *weak linear centre of finite order k*.

2. If $F^{(k)}(0, \lambda_*) = 0$ for each $k \geq 0$, then the origin is an *isochronous centre*, *i.e.*, all periodic orbits surrounding the origin have the same period.

DEFINITION 2.2. We say that k *local critical periods* bifurcate from the weak centre corresponding to the parameter λ_* if:

1. for every $\alpha > 0$, sufficiently small, there exists a neighborhood W of λ_* such that, for any λ in W, $P(r, \lambda)$ has at most k critical points in $U = (0, \alpha)$.

2. Moreover, any neighborhood of λ_* contains a point λ^1 such that $P(r, \lambda^1)$ has exactly k critical points in $U = (0, \alpha)$.

DEFINITION 2.3 [C.J]. For $\lambda_* \in V(p_2, p_4, \ldots, p_{2k}) = \{\lambda \mid p_{2i}(\lambda) = 0, i = 1, \ldots, k\}$ and $p_{2k+2}(\lambda_*) \neq 0$, the period coefficients $p_2, p_4, \ldots, p_{2k}$ of F are said to be *independent with respect to* p_{2k+2} *at* λ_* when the following conditions are satisfied:

1. Every neighborhood of λ_* contains a point λ such that $p_{2k}(\lambda) \cdot p_{2k+2}(\lambda) < 0$.

2. The varieties $V(p_2, p_4, \ldots, p_{2j})$, $j = 1, \ldots, (k-1)$, are such that: if $\lambda \in V(p_2, p_4, \ldots, p_{2j})$, and $p_{2j+2}(\lambda) \neq 0$, then every neighborhood of λ contains a point $\lambda^2 \in V(p_2, p_4, \ldots, p_{2j-2})$ such that $p_{2j}(\lambda^2) \cdot p_{2j+2}(\lambda) < 0$.

The following theorems have been proved by Chicone and Jacobs [C.J].

FINITE ORDER BIFURCATION THEOREM. *From weak centres of finite order k at the parameter value λ_* no more than k local critical periods bifurcate. Moreover, there are* *perturbations with exactly j critical periods for any $j \leq k$, if the coefficients $p_2, p_4, \ldots, p_{2k}$ of F are independent with respect to p_{2k+2} at λ_*.*

ISOCHRONE BIFURCATION THEOREM. *If the vector field $\mathbb{X}$ has an isochronous centre at the origin for the parameter value λ_* and if for each integer $n \geq 1$ the period coefficient p_{2n} is in the ideal $(p_2, p_4, \ldots, p_{2k}, p_{2k+2})$ over $\mathbb{R}\{\lambda_1, \ldots, \lambda_n\}_{\lambda_*}$, the ring of convergent power series at λ_*, then at most k local critical periods bifurcate from this isochronous centre at λ_*.*

Moreover, if $p_2, p_4, \ldots, p_{2k}$ are independent with respect to p_{2k+2} at λ_*, then exactly j local critical periods bifurcate from the centre at λ_* for $j \leq k$.

3. **Vector fields with a homogeneous nonlinearity of the third degree.** We consider a vector field in the Sibirskii's form (S_λ):

$$\dot{x} = -y - (A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3)$$
$$\dot{y} = x + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3,$$

with

$$A_{30} = a_3 + a_2 - a_1; \quad A_{21} = a_6 - 3a_4; \quad A_{12} = 3a_3 - 3a_2 + 2a_1 - a_7; \quad A_{03} = a_4 - a_5,$$
$$B_{30} = a_4 + a_5; \quad B_{21} = 3a_3 + 3a_2 + 2a_1; \quad B_{12} = a_6 - 3a_4; \quad B_{03} = a_3 - a_2 - a_1.$$

We denote by $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{R}^7$ the bifurcation parameter.

The centres are identified in the following theorem.

THEOREM 3.1 (MALKIN [M]). *$(S_\lambda)_{\lambda \in \mathbb{R}^7}$ has a centre at the origin if and only if λ is in the union of the following three sets:*

$$S_I = \{\lambda \in \mathbb{R}^7 \mid a_1 = a_7 = 0\},$$
$$S_{II} = \{\lambda \in \mathbb{R}^7 \mid a_3 = a_5 = a_6 = a_7 = 4(a_4^2 + a_2^2) - a_1^2 = 0\},$$
$$S_{III} = \{\lambda \in \mathbb{R}^7 \mid a_2 = a_5 = a_7 = 0\}.$$

Let us define $S = S_I \cup S_{II} \cup S_{III}$.

DEFINITION 3.2. $(S_\lambda)_{\lambda \in \mathbb{R}^7}$ has a *centre of type I* (respectively *II*, *III*) if the system is nonlinear and $\lambda \in S_I$ (respectively S_{II}, S_{III}).

We now turn to the computation of the period function P for the parameter value λ_* in S. The computer algebra system "Mathematica" was used for most of our computations. First, eliminating t from system $(S_\lambda)_{\lambda \in \mathbb{R}^7}$ and changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we get:

$$\frac{dr}{d\theta} = \frac{r^3 w_1(\theta, \lambda)}{1 + r^2 w_2(\theta, \lambda)} \tag{3.1}$$

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where

$$w_1(\theta,\lambda) = (a_1 - a_2 - a_3)\cos^4\theta + (4a_4 + a_5 - a_6)\cos^3\theta\sin\theta + (6a_2 + a_7)\cos^2\theta\sin^2\theta$$
$$+ (a_6 - 4a_4 + a_5)\cos\theta\sin^3\theta + (a_3 - a_2 - a_1)\sin^4\theta,$$
$$w_2(\theta,\lambda) = (a_4 + a_5)\cos^4\theta + (4a_3 + 4a_2 + a_1)\cos^3\theta\sin\theta + (2a_6 - 6a_4)\cos^2\theta\sin^2\theta$$
$$+ (4a_3 - 4a_2 + a_1 - a_7)\cos\theta\sin^3\theta + (a_4 - a_5)\sin^4\theta.$$

This solution $r = r(\theta,\lambda)$ of equation (3.1) with initial conditions $r(0,\lambda) = r_0 > 0$ may be locally represented as a convergent power series in r_0:

$$r(\theta,\lambda) = \sum_{k=1}^{\infty} u_k(\theta,\lambda) r_0^k. \tag{3.2}$$

Because of the symmetry we have $u_{2j}(\theta,\lambda) \equiv 0, j \geq 1$.

Substituting (3.2) into (3.1), we obtain the coefficients $u_k(\theta,\lambda)$, $k \geq 1$, by successive integration; the first four coefficients $u_1(\theta,\lambda)$, $u_3(\theta,\lambda)$, $u_5(\theta,\lambda)$, $u_7(\theta,\lambda)$ are calculated by integrating the following differential equations:

$$u_1' = 0, \quad u_1(0,\lambda) = 1,$$
$$u_3' = w_1(\theta,\lambda), \quad u_3(0,\lambda) = 0,$$
$$u_5' = -w_1(\theta,\lambda)[w_2(\theta,\lambda) - 3u_3], \quad u_5(0,\lambda) = 0,$$
$$u_7' = w_1(\theta,\lambda)[w_2^2(\theta,\lambda) - 5w_2(\theta,\lambda)u_3 + 3(u_3^2 + u_5)], \quad u_7(0,\lambda) = 0.$$

Let us denote by γ_r the closed trajectory through $(r,0)$.

The period function $P(r,\lambda)$ is given by:

$$P(r,\lambda) = \int_{\gamma_r} dt$$
$$= \int_0^{2\pi} \frac{1}{1 + r^2 w_2(\theta,\lambda)}\, d\theta,$$

and may be represented by the following formula:

$$P(r,\lambda) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\lambda) r^{2k},$$

for r sufficiently small and λ in the neighborhood of the point λ_* corresponding to a centre, *i.e.*, λ in S.

The first four coefficients p_2, p_4, p_6, p_8, have the following expressions:

$$p_2(\lambda) = -\int_0^{2\pi} w_2(\theta,\lambda)\, d\theta,$$
$$p_4(\lambda) = \int_0^{2\pi} w_2(w_2 - 2u_3)\, d\theta,$$
$$p_6(\lambda) = \int_0^{2\pi} \left(-w_2(u_3^2 + 2u_5) + w_2^2(4u_3 - w_2)\right) d\theta,$$
$$p_8(\lambda) = \int_0^{2\pi} \left(-2w_2(u_3u_5 + u_7) + 2w_2^2(3u_3^2 + 2u_5) + w_2^3(w_2 - 6u_3)\right) d\theta.$$

We prove the following theorem:

THEOREM 3.3. *1. A weak centre of type I has order at most one. Any such centre of order one has perturbations with exactly one critical period.*

2. A weak centre of type II has order at most one. No local critical period can bifurcate from it, except if it belongs to the intersection of S_{II} and S_{III} in which case one critical period bifurcates from it.

3. A weak centre of type III has order at most three. For any such centre of order $k \leq 3$ and each $j \leq k$, there exist perturbations with exactly j critical periods.

PROOF. 1. For a centre of type I we get:

$$p_2(\lambda) = -\frac{\pi}{2}a_6.$$

If $a_6 = 0$, then we have

$$p_4(\lambda) = \frac{3\pi}{2}(a_2^2 + 4a_3^2 + a_4^2 + a_5^2).$$

Hence $p_4 = 0$ if and only if $a_i = 0, i = 2, 3, 4, 5$. Thus the corresponding weak centre has order at most one. By the Finite Order Bifurcation Theorem, at most one critical period can bifurcate from it. Moreover, a perturbation with one local critical period is given by a system in the form:

$$\begin{aligned}\dot{x} &= -y - [(a_3 + a_2)x^3 + (\varepsilon_6 - 3a_4)x^2y + (3a_3 - 3a_2)xy^2 + (a_4 - a_5)y^3]\\ \dot{y} &= x + (a_4 + a_5)x^3 + (3a_3 + 3a_2)x^2y + (\varepsilon_6 - 3a_4)xy^2 + (a_3 - a_2)y^3,\end{aligned}$$

if $\varepsilon_6 \geq 0$ is sufficiently small. In this case we have $p_2 \cdot p_4 < 0$.

2. For a centre of type II, we have

$$\begin{aligned}p_2(\lambda) &= 0,\\ p_4(\lambda) &= \frac{\pi}{2}(a_2^2 + a_4^2).\end{aligned}$$

Then we may write

$$P(r, \lambda) = 2\pi + \frac{\pi}{2}(a_2^2 + a_4^2)r^4 + \cdots.$$

As in [C.J], since p_4 is positive definite, positive zeros of $P'(r, \lambda)$ are zeros of

$$H(r, \lambda) = \begin{cases} \frac{P'(r,\lambda)}{4r^3p_4(\lambda)} & \text{if } \lambda \neq 0\\ 1 & \text{if } \lambda = 0,\end{cases}$$

which is continuous and nonzero on some compact neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^7$. Hence there is no critical period bifurcating except in $S_{II} \cap S_{III}$.

3. We now turn to the case of a centre of type III and compute the period coefficients. Each coefficient, still denoted by p_{2n}, is reduced modulo the ideal of the previous coefficients.

$$\begin{aligned}p_2(\lambda) &= -\frac{\pi}{2}a_6,\\ p_4(\lambda) &= -\frac{\pi}{4}\big((a_1 - 6a_3)(a_1 + 4a_3) - 6a_4^2\big),\\ p_6(\lambda) &= \frac{5\pi}{8}a_3a_4(a_1 + 24a_3),\\ p_8(\lambda) &= \frac{5\pi}{1152}(a_1 - 66a_3)(a_1 - 6a_3)(a_1 + 4a_3)(a_1 + 24a_3).\end{aligned}$$

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For further purposes we define

$$q_8(\lambda) = (a_1 - 6a_3)(a_1 + 4a_3)(a_1 + 24a_3).$$

Let us note that, under the condition $p_2(\lambda) = p_4(\lambda) = p_6(\lambda) = 0$,

$$p_8(\lambda) = 0 \Leftrightarrow q_8(\lambda) = 0.$$

The analysis of the different ways to make $p_2 = p_4 = p_6 = 0$ leads to the values of the finite order bifurcation points and the isochrone points using the results of Pleshkan [P]. Indeed:

1. For $\lambda_* \in S_{III}$ and satisfying:

$$a_6 = a_3 = a_1^2 - 6a_4^2 = 0,$$

we get

$$p_2 = p_4 = p_6 = 0 \text{ and } p_8 = \frac{5\pi}{32}a_4^4,$$

with $a_4 \neq 0$.

In this case the origin is a weak centre of order at most three and, from the finite order bifurcation theorem, at most three local critical periods bifurcate from the origin. Moreover, three critical periods bifurcate from the origin in a perturbation of the form:

$$\begin{aligned}\dot{x} &= -y - [(\varepsilon_3 - \hat{a}_1)x^3 + (\varepsilon_6 - 3a_4)x^2y + (3\varepsilon_3 + 2\hat{a}_1)xy^2 + a_4y^3]\\ \dot{y} &= x + a_4x^3 + (3\varepsilon_3 + 2\hat{a}_1)x^2y + (\varepsilon_6 - 3a_4)xy^2 + (\varepsilon_3 - \hat{a}_1)y^3,\end{aligned}$$

with $\hat{a}_1 = a_1 + \varepsilon_1 + \delta_1(\varepsilon_3)$ and $|\varepsilon_6| \ll |\varepsilon_1| \ll |\varepsilon_3| \ll |a_4|$. We choose ε_3 and $\delta_1(\varepsilon_3)$ such that $p_6 < 0$ and $p_2 = p_4 = 0$; *i.e.*, δ_1 is the "small" root of the equation:

$$\delta_1^2 + 2(a_1 - \varepsilon_3)\delta_1 - 2a_1\varepsilon_3 - 24\varepsilon_3^2 = 0.$$

The sign of ε_3 is such that $\varepsilon_3 a_4 a_1 < 0$. Then ε_1 is chosen such that $p_4 > 0$; this is achieved when $\varepsilon_1 a_1 < 0$. Finally we choose $\varepsilon_6 > 0$, which leads to $p_2 < 0$. Such a perturbation λ of λ_* yields that every neighborhood $U = [0, \alpha)$ of $r = 0$ contains some points $0 < r_1 < r_2 < r_4$ such that $F(0, \lambda) = 0$ and $F(r_1, \lambda) < 0$, $F(r_2, \lambda) > 0$, $F(r_3, \lambda) < 0$, $F(r_4, \lambda) > 0$ by continuity of $F(r, \lambda)$. Then $P'(r, \lambda) = 0$ has at least three different solutions in U. Thus, as claimed, three critical periods bifurcate from the origin. A similar perturbation with $(3 - j)$ of the ε_i being zero gives a system with $j \leq 3$ local critical periods.

2. For $\lambda_* \in S_{III}$ and

$$a_6 = a_4 = a_1 - 6a_3 = 0,$$

we get $p_2 = p_4 = p_6 = p_8 = 0$.

The corresponding system has the following form:

$$\begin{aligned}\dot{x} &= -y + 5a_3x^3 - 15a_3xy^2\\ \dot{y} &= x + 15a_3x^2y - 5a_3y^3,\end{aligned}$$

and it satisfies Cauchy-Riemann conditions, *i.e.*, it can also be written

$$dt = \frac{dz}{iz + 5a_3z^3}$$
$$= dz\Big(\frac{-i}{z} + O(z)\Big),$$

where $z = x + iy$. By the residue theorem the period is constant [P].

3. For $\lambda_* \in S_{III}$ and

$$a_6 = a_3 = a_1^2 - 6a_4^2 = 0$$

we obtain again $p_2 = p_4 = p_6 = p_8 = 0$. In polar coordinates (ρ, θ), we get

$$\dot{\theta} = 1,$$

and again the corresponding system is an isochrone [P].

4. For $\lambda_* \in S_{III}$ and

$$a_6 = a_1 + 24a_3 = a_4^2 - 100a_3^2 = 0,$$

again $p_2 = p_4 = p_6 = p_8 = 0$.

Changing first to coordinates $x_1 = x - y$; $y_1 = x + y$, and then to coordinates

$$u = \frac{x_1}{\sqrt{(1 - 15a_3y_1^2)^3}}; \quad v = \frac{y_1 - 10a_3y_1^3}{\sqrt{(1 - 15a_3y_1^2)^3}}, \quad \text{for } a_4 = 10a_3,$$
$$u = \frac{x_1 + 10a_3x_1^3}{\sqrt{(1 + 15a_3x_1^2)^3}}; \quad v = \frac{y_1}{\sqrt{(1 + 15a_3x_1^2)^3}}, \quad \text{for } a_4 = -10a_3,$$

the corresponding system is reduced to the linear isochrone [P]. ∎

Hence Theorem 3.3. is proved. Moreover, the isochronous centres have been identified. The results obtained agree with those of Pleshkan. Let us recall them in the theorem below:

THEOREM 3.4. *$(S_\lambda)_{\lambda\in\mathbb{R}^7}$ is an isochrone system if and only if λ lies in one of the following two sets:*

$$I_1 = \{\lambda \in \mathbb{R}^7 \mid a_2 = a_5 = a_7 = a_6 = a_4 = (a_1 - 6a_3)(a_1 + 4a_3) = 0\}$$
$$I_2 = \{\lambda \in \mathbb{R}^7 \mid a_2 = a_5 = a_7 = a_6 = a_1 + 24a_3 = a_4^2 - 100a_3^2 = 0\}.$$

Let us note that each I_k is included in S_{III}, for $k = 1, 2$; moreover no isochrone lies in S_I or S_{II} except the linear system. Hence, a nonlinear isochrone can only be perturbed into centres of type *III*. However, near the linear isochrone, we must consider perturbations into each one of the sets S_I, S_{II}, S_{III}. This is done in the following two theorems.

THEOREM 3.5. *A perturbation of the linear isochrone inside centres of type I (respectively of type II) has at most one (respectively has no) critical period.*

PROOF. As in the proof of Theorem 3.3, P being the period function, we consider

$$G(r,\lambda) = \frac{\left(\frac{P'(r,\lambda)}{r}\right)'}{r} = 8p_4(\lambda) + O(r),$$

with

$$p_4(\lambda) = \frac{3\pi}{2}(a_2^2 + 4a_3^2 + a_4^2 + a_5^2).$$

i.e., p_4 is positive definite. Then we may define

$$H(r,\lambda) = \begin{cases} \frac{G(r,\lambda)}{8p_4(\lambda)} & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0. \end{cases}$$

which does not vanish in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^7$. Hence $G(r,\lambda)$ has no zero in that neighborhood, yielding at most one zero of P'.

The result in case of perturbation inside S_{II} is an obvious consequence of part 2 in the proof of Theorem 3.3. ∎

THEOREM 3.6. *We consider (S_{λ_*}) with $\lambda_* \in S_{III}$ and λ in the neighborhood of λ_*.*

1. Every coefficient $p_{2n}(\lambda)$ belongs to the ideal $I = (p_2, p_4, p_6, p_8)$ over the noetherian ring $\mathbb{R}[a_1, a_3, a_4, a_6]$.

2. At most three critical periods bifurcate from the origin of the linear isochrone in the family S_{III} and at most two critical periods bifurcate from the nonlinear isochrone in S_{III}.

Moreover, in both cases, there exist perturbations with the maximum number of critical periods.

PROOF: PART 1. Throughout the proof, we may use the following expressions of the period coefficients as a basis of the ideal I, since reducing them modulo the previous ones and getting rid of the constant factors does not affect the ideal they generate over $\mathbb{R}[a_1, a_3, a_4, a_6]$.

$$\begin{aligned} p_2 &= a_6, \\ p_4 &= (a_1 - 6a_3)(a_1 + 4a_3) - 6a_4^2, \\ p_6 &= a_3a_4(a_1 + 24a_3), \\ p_8 &= (a_1 - 66a_3)(a_1 + 24a_3)(a_1 - 6a_3)(a_1 + 4a_3). \end{aligned}$$

We must show that $p_{2n} \in I = (p_2, p_4, p_6, p_8)$. Here we present a direct proof. The referee suggested us a computer aided proof using Macaulay, which can be found in the Appendix.

For every $n \geq 1$, p_{2n} is a polynomial element of $\mathbb{R}[a_1, a_3, a_4, a_6]$; then we may write:

$$p_{2n} = a_6T_2(a_1, a_3, a_4, a_6) + R_2(a_1, a_3, a_4),$$

where a_6T_2 is the sum of all terms from p_{2n} containing a_6.

Recalling that

$$a_4^2 = \frac{1}{6}\big((a_1 - 6a_3)(a_1 + 4a_3) - p_4\big),$$

we can write:

$$R_2(a_1, a_3, a_4) = p_4S_4(a_1, a_3, a_4) + a_4S_6(a_1, a_3) + S_8(a_1, a_3).$$

Now, for $n \geq 5$, we have that p_{2n} is a polynomial of degree ≥ 5.

Moreover, from the expression of p_8 we have

$$a_1^4 = p_8 - a_3U_8(a_1, a_3).$$

Hence

$$a_4S_6(a_1, a_3) = a_4p_8V_8(a_1, a_3) + a_3a_4V_6(a_1, a_3).$$

Therefore

$$p_{2n} = p_2T_2 + p_4S_4 + a_3a_4V_6(a_1, a_3) + a_4p_8V_8(a_1, a_3) + S_8(a_1, a_3).$$

We know that $p_{2n} = 0$ for $\lambda_* \in I_1$, yielding that $(a_1 - 6a_3)(a_1 + 4a_3)$ divides S_8, *i.e.*,

$$S_8(a_1, a_3) = (a_1 - 6a_3)(a_1 + 4a_3)W_8(a_1, a_3).$$

The same argument with $\lambda_* \in I_2$, *i.e.*, $a_6 = a_1 + 24a_3 = (a_4 - 10a_3)(a_4 + 10a_3) = 0$ leads to

$$\begin{aligned} 10a_3^2V_6(-24a_3, a_3) + 600W_8(-24a_3, a_3) &= 0 \\ -10a_3^2V_6(-24a_3, a_3) + 600W_8(-24a_3, a_3) &= 0, \end{aligned}$$

from which we get simultaneously:

$$\begin{aligned} V_6(-24a_3, a_3) &= 0 \\ W_8(-24a_3, a_3) &= 0, \end{aligned}$$

with $a_3 \neq 0$.

Hence $a_1 + 24a_3$ must divide the polynomials $V_6(a_1, a_3)$ and $W_8(a_1, a_3)$, *i.e.*,

$$\begin{aligned} V_6(a_1, a_3) &= (a_1 + 24a_3)H_6(a_1, a_3) \\ W_8(a_1, a_3) &= (a_1 + 24a_3)H_8(a_1, a_3). \end{aligned}$$

Finally we may rewrite:

$$\begin{aligned} p_{2n} &= p_2T_2 + p_4S_4 + a_3a_4(a_1 + 24a_3)H_6(a_1, a_3) + a_4p_8V_8 \\ &\quad + (a_1 - 6a_3)(a_1 + 4a_3)(a_1 + 24a_3)H_8(a_1, a_3) \\ &= p_2T_2 + p_4S_4 + p_6H_6 + a_4p_8V_8 + q_8H_8(a_1, a_3). \end{aligned}$$

Let us note that q_8 and H_8 are homogeneous polynomials in a_1, a_3 of degree 3 and $n-3$ respectively; that allows us to write

$$H_8(a_1,a_3) = Ca_1^{n-3} + a_3\hat{H}_8(a_1,a_3).$$

From the expressions of p_4, p_6 and q_8 we have

$$a_3(a_1+24a_3)p_4 + 6a_4p_6 = a_3q_8.$$

Therefore a_3q_8 belongs to the ideal I; moreover, $p_8 = (a_1-66a_3)q_8$ implies that a_1q_8 is also an element of I; thus $q_8H_8(a_1,a_3)$ belongs to I over $\mathbb{R}[a_1,a_3,a_4,a_6]$ and can be written in the form:

$$q_8H_8(a_1,a_3) = p_4\hat{S}_4 + p_6\hat{S}_6 + p_8\hat{S}_8,$$

which, substituted in the above expression of p_{2n}, leads to

$$p_{2n} = p_2\tilde{S}_2 + p_4\tilde{S}_4 + p_6\tilde{S}_6 + p_8\tilde{S}_8,$$

where $\tilde{S}_k$, $k=2,4,6,8$, belong to $\mathbb{R}[a_1,a_3,a_4,a_6]$.

Hence the period coefficients p_{2n} for $n\geq 1$ are in the ideal I over $\mathbb{R}[a_1,a_3,a_4,a_6]$.

PROOF: PART 2. Combining Part 1 with the Isochrone Bifurcation Theorem we may conclude that at most three local critical periods bifurcate from the linear isochrone into the family S_{III}.

A perturbation with three critical periods is obtained in the following way: for $\varepsilon>0$, the system

$$\begin{aligned}\dot{x} &= -y+\sqrt{6}\varepsilon x^3+3\varepsilon x^2y-2\sqrt{6}\varepsilon xy^2-\varepsilon y^3\\ \dot{y} &= x+\varepsilon x^3+2\sqrt{6}\varepsilon x^2y-3\varepsilon xy^2-\sqrt{6}\varepsilon y^3,\end{aligned}$$

has a weak center of order 3. A small perturbation constructed as in part 1 of Theorem 3.3. has three critical periods.

Next we are concerned with perturbations of a nonlinear isochrone (a nonlinear isochrone never belongs to the intersection of two strata): we must consider perturbations $\lambda=\lambda_*+\delta$ into S_{III}, where λ_* is a nonlinear isochrone point, say, λ_* is nonzero and belongs to I_k, for $k=1,2,$.

First, let us assume that λ_* satisfies the condition: $a_6=a_4=a_1-6a_3=0$, with a_3 nonzero (the other case $a_6=a_4=a_1+4a_3=0$ is similar). The components of λ are of the forms: $\hat{a}_1=a_1+\delta_1$; $\hat{a}_3=a_3+\delta_3$; $\hat{a}_4=\delta_4$; $\hat{a}_6=\delta_6$. The result will follow from the Isochrone Bifurcation Theorem if we show that $p_8(\lambda)$ belongs to the ideal $J=\big(p_2(\lambda),p_4(\lambda),p_6(\lambda)\big)$ over $\mathbb{R}\{\hat{a}_1,\hat{a}_3,\hat{a}_4,\hat{a}_6\}_{\lambda_*}$ the noetherian ring of convergent power series at λ_* and $p_{2k}(\lambda)$, $k=1,2,3,4$, are the corresponding perturbed period coefficients. Computing the coefficients reduced modulo the previous ones, with $\delta_3<|a_3|$, it follows that:

$$\begin{aligned}\hat{p}_8(\lambda)-(\hat{a}_1-66\hat{a}_3)(\hat{a}_1+24\hat{a}_3)\hat{p}_4 &= 6\hat{a}_4^2(\hat{a}_1-66\hat{a}_3)(\hat{a}_1+24\hat{a}_3)\\ &= \frac{6\hat{a}_4(\hat{a}_1-66\hat{a}_3)\hat{p}_6}{\hat{a}_3}.\end{aligned}$$

Then we get:

$$\begin{aligned}\hat{p}_8(\lambda) &= (\hat{a}_1 - 66\hat{a}_3)(\hat{a}_1 + 24\hat{a}_3)\hat{p}_4(\lambda) + \frac{6\hat{a}_4\hat{p}_6(\lambda)(\hat{a}_1 - 66\hat{a}_3)}{\hat{a}_3}\\ &= M(a_3,\delta_1,\delta_3)\hat{p}_4(\lambda) + \frac{6\delta_4(-60a_3+\delta_1-66\delta_3)}{a_3}\frac{1}{1+\frac{\delta_3}{a_3}}\hat{p}_6(\lambda)\\ &= M(a_3,\delta_1,\delta_3)\hat{p}_4(\lambda) + N(a_3,\delta_1,\delta_3,\delta_4)\Big[\sum_{n=0}^{\infty}(-1)^n\Big(\frac{\delta_3}{a_3}\Big)^n\Big]\hat{p}_6(\lambda),\end{aligned}$$

where

$$M(a_3,\delta_1,\delta_3) = (-60a_3+\delta_1-66\delta_3)(30a_3+\delta_1+24\delta_3),$$

and

$$N(a_3,\delta_1,\delta_3,\delta_4) = \frac{6\delta_4(-60a_3+\delta_1-66\delta_3)}{a_3}.$$

Therefore $p_8(\lambda)$ is in the ideal J over $\mathbb{R}\{\hat{a}_1,\hat{a}_3,\hat{a}_4,\hat{a}_6\}_{\lambda_*}$, leading to the claim.

For λ_* in I_2, *i.e.*, satisfying the condition $a_6 = a_1+24a_3 = a_4 \pm 10a_3 = 0$ with a_3 and a_4 nonzero, the perturbation λ has components: $\hat{a}_1 = a_1+\delta_1$, $\hat{a}_3 = a_3+\delta_3$, $\hat{a}_4 = a_4+\delta_4$, $\hat{a}_6 = \delta_6$; computing the corresponding period coefficients as in the preceding case and assuming $\delta_3 < |a_3|$ and $\delta_4 < |a_4|$ we obtain:

$$\begin{aligned}\hat{p}_6(\lambda) &= a_3a_4\Big(1+\frac{\delta_3}{a_3}\Big)\Big(1+\frac{\delta_4}{a_4}\Big)(\delta_1+24\delta_3),\\ \hat{p}_8(\lambda) &= (-90a_3+\delta_1-66\delta_3)(-30a_3+\delta_1-6\delta_3)(-20a_3+\delta_1+4\delta_3)(\delta_1+24\delta_3)\\ &= G(a_3,a_4,\delta_1,\delta_3)\frac{1}{a_3a_4}\frac{1}{1+\frac{\delta_3}{a_3}}\frac{1}{1+\frac{\delta_4}{a_4}}\hat{p}_6(\lambda),\end{aligned}$$

and then

$$\hat{p}_8(\lambda) = G(a_3,a_4,\delta_1,\delta_3)\frac{1}{a_3a_4}\Big[\sum_{n=0}^{\infty}(-1)^n\Big(\frac{\delta_3}{a_3}\Big)^n\Big]\Big[\sum_{n=0}^{\infty}(-1)^n\Big(\frac{\delta_4}{a_4}\Big)^n\Big]\hat{p}_6(\lambda),$$

where $G(a_3,a_4,\delta_1,\delta_3) = (-90a_3+\delta_1-66\delta_3)(-30a_3+\delta_1-6\delta_3)(-20a_3+\delta_1+4\delta_3)$. Again $p_8(\lambda)$ is in the ideal J over the ring $\mathbb{R}\{\hat{a}_1,\hat{a}_3,\hat{a}_4,\hat{a}_6\}_{\lambda_*}$.

Hence in both cases at most two local critical periods bifurcate from the nonlinear isochrones; as shown in the preceding cases we may construct a perturbation with the maximum number of critical periods. ∎

APPENDIX. An alternative proof of the first part of Theorem 3.6 using Macaulay was suggested to us by the referee. There we must prove that every homogeneous polynomial p of degree $n \geq 5$ which vanishes on the variety of the ideal $I = (p_2,p_4,p_6,p_8)$ is in I. The polynomial p belongs to the radical J of the ideal I. This radical is computed using Macaulay: $\operatorname{rad} I = J = \big(p_2,p_4,a_4(a_1+24a_3)\big)$. The reduction of J modulo I is generated by the single polynomial $q = a_4(a_1+24a_3)$. Since p is in J, p can be represented as $p = i + hq$, where i is in I and h is homogeneous of degree at least $(n-2)$. But, this means that each monomial in h is homogeneous of degree at least 3. Since it is clear that any monomial containing either a_6 or a_3 when multiplied by g is in I, one only needs to check that all the products of the form $a_1^j a_4^k q$, with $j+k = 3$, belong to I conclude that hq is in I. This can be checked by a finite number of ideal membership tests in Macaulay.

CONCLUSION. The preceding results allow the following observations:

1. Exactly as in the quadratic case, the maximum number of local critical periods corresponds to centres of systems with symmetry axis.

2. The maximum number of local critical periods from centres of finite order is strictly larger than from nonlinear isochronous centres. This is not the case for limit cycles in a Hopf bifurcation.

3. The maximum number of local critical periods bifurcating from a weak centre or from an isochrone is less than the maximum number of limit cycles bifurcating from a weak focus or a centre.

4. There are systems for which the independence condition is not satisfied.

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