Annals of Glaciology 4 1983

COMPARISON BETWEEN DIFFERENT YIELD FUNCTIONS FOR SALINE ICE

by

F. U. Häusler
(Hamburgische Schiffbau-Versuchsanstalt GmbH, Bramfelder Str. 164, Postfach 60 09 29, D-2000 Hamburg 60, West Germany)

ABSTRACT

Data obtained from tests of uniaxial and multiaxial compressive strength on saline ice have been used to determine the coefficients of the Smith yield function, which uses seven parameters, and of the Pariseau yield function, which uses five. Both functions describe orthotropic materials and have been reduced for transverse isotropy. The tests of compressive strength were carried out over the past few years in the ice laboratory of the Hamburgische Schiffbau-Versuchsanstalt (HSVA) on a closed-loop controlled triaxial loading frame with brush-type loading platens. The values for tensile strength have been obtained from data published by other researchers.

The ice-strength values computed by means of the two yield functions were compared with measured ice strengths: the seven-parameter Smith yield function provides reliable results over the whole stress space, while the simpler Pariseau yield function is only applicable within a restricted area of the stress space.

INTRODUCTION

In rock and soil mechanics, multiaxial yield criteria have been used for several decades. Names like Tresca, Coulomb and von Mises are most common in this field. An important stimulus for improving both the theory of yield under multiaxial stress states and the methods of multiaxial testing came from the necessarily high safety requirements at nuclear power plants.

In the field of glaciology several problems can be sensibly treated by means of the multiaxial yield theory, e.g. plastic flow analysis of glaciers, estimation of ice loads on marine structures or evaluation of the bearing capacity of an ice cover. It is mainly the lack of data on multiaxial ice strength which has prevented, up to now, the application of these advanced methods to glaciological and ice-engineering problems. One of the first attempts in this direction was made by Ralston and Reinicke (e.g. Ralston 1977, Reinicke and Ralston 1977). They used ice-strength data reported by Carter and Michel (1971), Frederking (1977) and Jones (1978) to determine yield functions valid for freshwater ice, and applied them to plastic limit analyses. If problems of sea ice are to be examined, e.g. the theoretical estimation of ice loads on a marine structure in the Arctic, it is necessary to know the properties of saline ice. Recently-published data on the multiaxial compressive strength of saline ice (Häusler 1982) combined with results from uniaxial tests in compression and tension on sea ice (Peyton 1966, Weeks and Assur 1969) now make it possible to determine yield functions, even for saline ice. This was one of the objectives of this study. The other was to clarify whether or not it is necessary to use the rather complex Smith yield function for saline ice. As an alternative, the simpler Pariseau yield function was studied.

YIELD FUNCTIONS

A yield function f is used to characterize the elastic-plastic behaviour of a solid material. It describes that part of the stress space, which contains all possible stress states of a special material. The surface of this partial stress space, i.e. where f = 0, characterizes all stress states at which the material ceases to behave elastically and begins to behave plastically. At all stress states inside the partial stress space, i.e. where f < 0, the material behaves elastically. Generally a yield function can be written as

$$f = f(\sigma_{ij}, \varepsilon_{ij}^{p}) \leq 0, \qquad (1)$$

where σ_{ij} is the stress tensor and ε_{ij}^{p} the plastic strain tensor. If the material has never undergone any plastic strains, f is the initial yield function and depends only on the stresses σ_{ij}:

$$f = f(\sigma_{ij}) \leq 0. \qquad (2)$$

In this study only the initial yield function was studied.

Many different yield functions have been published in the past. Since ice that is naturally grown in an ice cover is anisotropic, all yield functions describing isotropic materials must lead to more or less imperfect results if applied to this type of ice. In this study two of the more general yield functions describing anisotropic materials have been chosen: the Pariseau and the Smith yield functions (Pariseau 1972, Smith unpublished). Using Smith's notation, the Pariseau yield function can be written as

$$f = a\,(\sigma_x - \sigma_y)^2 + b\,(\sigma_y - \sigma_z)^2 + c\,(\sigma_z - \sigma_x)^2 + d\,\tau_{xy}^2 + e\,\tau_{yz}^2 + f\,\tau_{zx}^2 + (g\,\sigma_x + h\,\sigma_y + k\,\sigma_z) - 1 \leq 0 \qquad (3)$$

and describes an orthotropic material with a linear dependency on the normal stresses. If a quadratic term of the normal stresses is added to Equation (3) the Smith yield function is obtained:

$$f = a\,(\sigma_x - \sigma_y)^2 + b\,(\sigma_y - \sigma_z)^2 + c(\sigma_z - \sigma_x)^2 + d\,\tau^2_{xy} + e\,\tau^2_{yz} + f\,\tau^2_{zx} + (g\,\sigma_x + h\,\sigma_y + k\,\sigma_z) + (l\,\sigma_x + m\,\sigma_y + n\,\sigma_z)^2 - 1 \leqslant 0, \qquad (4)$$

which is able to describe a material which compacts under plastic deformation and which shows a finite hydrostatic strength.

Since ice very often exhibits isotropic properties in the plane of the ice cover, both yield functions can be reduced to transverse isotropy for the purpose of this study. The Pariseau yield function for transversely isotropic materials can be written as

$$f = a\,(\sigma_x - \sigma_y)^2 + b\,[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + d\,\tau^2_{xy} + e\,(\tau^2_{yz} + \tau^2_{zx}) + g\,(\sigma_x + \sigma_y) + k\,\sigma_z - 1 \leqslant 0. \qquad (5)$$

The Smith yield function for transversely isotropic materials can be written as

$$f = a\,(\sigma_x - \sigma_y)^2 + b\,[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + d\,\tau^2_{xy} + e\,(\tau^2_{yz} + \tau^2_{zx}) + g\,(\sigma_x + \sigma_y) + k\,\sigma_z + [l\,(\sigma_x + \sigma_y) + n\,\sigma_z]^2 - 1 \leqslant 0. \qquad (6)$$

The orientation of the axes was chosen so that the x and y axes were in plane with, and the z axis was normal to, the surface of the ice cover.

COEFFICIENTS OF THE YIELD FUNCTIONS

The coefficients of the Pariseau yield function of ice can be determined if five stress states are known at which ice fails, i.e. where f = 0. The failure stress states that were used in this study (compression positive) are T_z: uniaxial tensile strength parallel to the growth direction (normal to the ice cover), C_z: uniaxial compressive strength parallel to the growth direction, T_x: uniaxial tensile strength in plane with the ice cover, C_x: uniaxial compressive strength in plane with the ice cover, and C_{z45}: uniaxial compressive strength under an angle of 45° to the growth direction.

With these five strength values the coefficients of the Pariseau yield function (Equation (5)) (compare with Ralston (1977)) are:

$$a = \frac{1}{2\,C_z\,T_z} - \frac{1}{C_x\,T_x}, \qquad (7.1)$$

$$b = -\frac{1}{2\,C_z\,T_z}, \qquad (7.2)$$

$$d = 2\,(a + 2\,b), \qquad (7.3)$$

$$e = \frac{4}{C_{z45}^2} + \frac{1}{T_x\,C_x} - \frac{2}{C_{z45}}\left(\frac{1}{T_x} + \frac{1}{C_x} + \frac{1}{T_z} + \frac{1}{C_z}\right), \qquad (7.4)$$

$$g = \frac{1}{C_x} + \frac{1}{T_x}, \qquad (7.5)$$

and

$$k = \frac{1}{C_z} + \frac{1}{T_z}. \qquad (7.6)$$

In the case of the Smith yield function two additional strength values are necessary. These are B_{xy}: biaxial compressive strength with $\sigma_x = \sigma_y$ and $\sigma_z = 0$ (no stresses normal to the ice cover) and B_{xz}: biaxial compressive strength with $\sigma_x = \sigma_z$ and $\sigma_y = 0$ (no stresses in one of the two directions in plane with the ice cover).

With

$$\xi = \frac{1}{2\;C_z\;T_z} + \frac{1}{B_{xy}}\left(\frac{1}{2\;B_{xy}} - \frac{1}{C_x} - \frac{1}{T_x}\right) \qquad (8.1)$$

and

$$\eta = \frac{1}{C_x\;T_x} + \frac{1}{B_{xz}}\left(\frac{1}{B_{xz}} - \frac{1}{C_x} - \frac{1}{T_x} - \frac{1}{C_z} - \frac{1}{T_z}\right) \qquad (8.2)$$

the coefficients of the Smith yield function (Equation (6)) can be evaluated as follows:

$$a = \frac{1}{2\;C_z\;T_z} - \frac{1}{C_x\;T_x} - \frac{\xi}{2} + \frac{\eta^2}{8\;(\xi + \eta)}, \qquad (9.1)$$

$$b = -\frac{1}{2\;C_z\;T_z} - \frac{\eta^2}{4\;(\xi + \eta)}, \qquad (9.2)$$

$$d = 2\;(a + 2\;b), \qquad (9.3)$$

$$e = \frac{4}{C_{z45}^2} + \frac{1}{T_x\;C_x} - \eta - \frac{2}{C_{z45}}\left(\frac{1}{T_x} + \frac{1}{C_x} + \frac{1}{T_z} + \frac{1}{C_z}\right), \qquad (9.4)$$

$$g = \frac{1}{C_x} + \frac{1}{T_x}, \qquad (9.5)$$

$$k = \frac{1}{C_z} + \frac{1}{T_z}, \qquad (9.6)$$

$$l = \frac{2\xi + \eta}{\sqrt{8(\xi + \eta)}} , \qquad (9.7)$$

and

$$n = \frac{2\eta}{\sqrt{8(\xi + \eta)}} . \qquad (9.8)$$

Values of C_x, C_z, B_{xy} and B_{xz} uniaxial and biaxial compressive strength have been obtained from tests on saline ice having a salinity of 10.6‰ NaCl at the moment of sampling. After testing, the salinity of the meltwater was 7.0 ± 0.9‰ NaCl, and showed no effects due to duration of storage. This ice was prepared under simulated natural conditions at the ice laboratory of the Hamburgische Schiffbau-Versuchsanstalt (HSVA). All samples were taken from the same ice cover. Before testing, they were stored for 7 to 13 weeks at -30°C. Preparation of samples was conducted within this period at -22°C (Häusler 1982). The tests were performed on a triaxial closed-loop controlled loading frame with brush-type loading platens at a temperature T_I of -10°C. The strain-rate in the x-direction (for C_z in z-direction) was $\dot{\varepsilon} = 2.0 \times 10^{-4}\ s^{-1}$ and the relation between the three stresses σ_x, σ_y, σ_z was kept constant during each test (see Table I). The shear strength was acquired from uniaxial compressive strength tests where the load direction was at an angle of 45° to the growth direction C_{z45} (Häusler unpublished). The two tensile strengths T_x and T_z are related to the corresponding compressive strengths

$$T_x = -\frac{1}{3.0} C_x \qquad (10.1)$$

and

$$T_z = -\frac{1}{3.8} C_z . \qquad (10.2)$$

This was necessary because the brush-type loading platens used by Häusler (1982) did not allow tests of tensile strength. The ratios in Equations (10.1) and (10.2) have been estimated graphically from a plot showing the uniaxial compressive and tensile strengths versus sample orientation as determined by Peyton (1966) and Weeks and Assur (1969).

It would have been possible to take other values of the multiaxial compressive strength that were determined in the same study (compare with Table I) instead of T_x and T_z, for example, the ice strength where the ratio between the three stresses was $\sigma_x:\sigma_y:\sigma_z = 3:1:1$. But if only strength values of the pure compression octant of the principal stress space are used to determine the coefficients of the yield function, the resulting tensile strengths would be too high compared with the values obtained by Peyton (1966) and Weeks and Assur (1969).

Another possibility would have been to apply curve-fitting methods to the entire data set of uniaxial and multiaxial compressive strengths (compare with Table I) completed with the tensile strengths from Equations 10.1 and 10.2 and the uniaxial compressive strength under 45° to the growth direction C_{z45}. In the case of the Smith yield function, curve-fitting methods might give reasonable results, but not if applied to the Pariseau yield function, because the latter cannot describe a finite hydrostatic compressive strength if a reasonable tensile strength description is required.

The coefficients of both yield functions calculated by means of five and seven strength values mentioned above are listed in Table II. It is important to note that the coefficients l and n in the Smith yield function have been found to be imaginary.

COMPARISON OF RESULTS

The coefficients listed in Table I have been used to evaluate four curves on both of the two yield surfaces in the principal stress space. At each of the four curves the ratio between the principal stresses

TABLE I. MULTIAXIAL STRENGTHS OF SALINE ICE WITH SALINITY OF 10.6‰ NaCl AT THE MOMENT OF SAMPLING, AT ICE TEMPERATURE T_I = -10°C AND STRAIN-RATE $\dot{\varepsilon} = 2.0 \times 10^{-4}\ s^{-1}$. MEASURED VALUES (HÄUSLER 1981) AND CALCULATED VALUES FROM SMITH AND PARISEAU YIELD CRITERIONS

Ratio of principal stresses			Measured strengths (MPa)			Calculated strengths using Smith yield criterion (MPa)			Calculated strengths using Pariseau yield criterion (MPa)		
σ_x	σ_y	σ_z	σ_x	σ_y	σ_z	σ_x	σ_y	σ_z	σ_x	σ_y	σ_z
1	0	0	2.06	0	0	2.06	0	0	2.06	0	0
0	0	1	0	0	10.05	0	0	10.04	0	0.	10.06
1	1/3	0	3.46	1.19	0	3.99	1.33	0	4.63	1.54	0
1	2/3	0	6.98	4.72	0	7.79	5.19	0	16.22	10.81	0
1	1	0	9.36	9.40	0	9.38	9.38	0	52.16	52.16	0
1	0	1/3	2.10	0	0.73	2.21	0	0.74	2.18	0	0.73
1	1/3	1/3	3.74	1.28	1.26	4.52	1.51	1.51	5.05	1.68	1.68
1	2/3	1/3	8.23	5.50	2.78	9.81	6.54	3.27	20.26	13.51	6.75
1	1	1/3	17.12	17.07	5.79	12.41	12.41	4.14	122.27	122.27	40.76
1	0	2/3	2.73	0	1.84	2.36	0	1.57	2.28	0	1.52
1	1/3	2/3	5.45	1.86	3.65	5.03	1.68	3.35	5.38	1.79	3.59
1	2/3	2/3	8.15	5.51	5.44	12.25	8.17	8.17	23.52	15.86	15.86
1	1	2/3	11.62	11.63	7.73	16.72	16.72	11.15	509.83	509.83	339.89
1	0	1	2.47	0	2.48	2.48	0	2.48	2.36	0	2.36
1	1/3	1	4.49	1.54	4.49	5.48	1.83	5.48	5.58	1.86	5.58
1	2/3	1	8.65	5.83	8.65	14.94	9.96	14.94	24.68	16.45	24.68
1	1	1	14.20	14.25	14.18	22.72	22.72	22.72	∞	∞	∞

TABLE II. COEFFICIENTS OF SMITH AND PARISEAU YIELD FUNCTIONS REDUCED FOR TRANSVERSELY ISOTROPIC SALINE ICE (S = 10.6% NaCl AT THE MOMENT OF SAMPLING, $T_I = -10°C$, $\dot{\varepsilon} = 2.0 \times 10^{-4}\ s^{-1}$).

Coefficient	Smith yield function	Pariseau yield function
a MPa^{-2}	0.647	0.688
b MPa^{-2}	0.0106	0.0188
e MPa^{-2}	3.12	3.08
g MPa^{-1}	-0.971	-0.971
k MPa^{-1}	-0.279	-0.279
l MPa^{-1}	0.222	-
n MPa^{-1}	-0.128	-

σ_x and σ_z was kept constant with σ_z = (0, 1/3, 2/3, 1) σ_x. The projections of the four curves onto the σ_x-σ_y-plane are shown in Figure 1. Because of the symmetry of the x-y plane only one half of each set of curves was drawn: the curves belonging to the Pariseau yield surface are shown in the lower right half, while those belonging to the Smith yield surface are reflected at the $\sigma_x = \sigma_y$ line and are shown in the upper left half. In addition to the theoretical curves the measured strengths are shown. Here the points with a constant σ_x-σ_z ratio are connected by thin dashed lines. All the measured strengths are listed in Table I together with the corresponding yield stresses calculated by means of the two yield functions studied here.

It is obvious that in all octants of the principal stress space where tension stresses occur, both yield functions (Pariseau and Smith) give nearly the same results. In the pure compression octant of the principal stress space only the Smith yield function describes the measured strengths sufficiently well. The Pariseau yield function only gives satisfactory results in the vicinity of the uniaxial strengths, i.e. up to stresses not exceeding twice the uniaxial compressive strengths.

CONCLUSIONS

It is possible to describe the yield behaviour of saline ice by means of the Smith yield function using seven parameters sufficiently well. The agreement between strengths computed with this yield function and measured values is rather good over the whole stress space. If interest is restricted to stress states on a rather low hydrostatic stress level even the Pariseau yield function using five parameters can be applied to saline ice with sufficient accuracy.

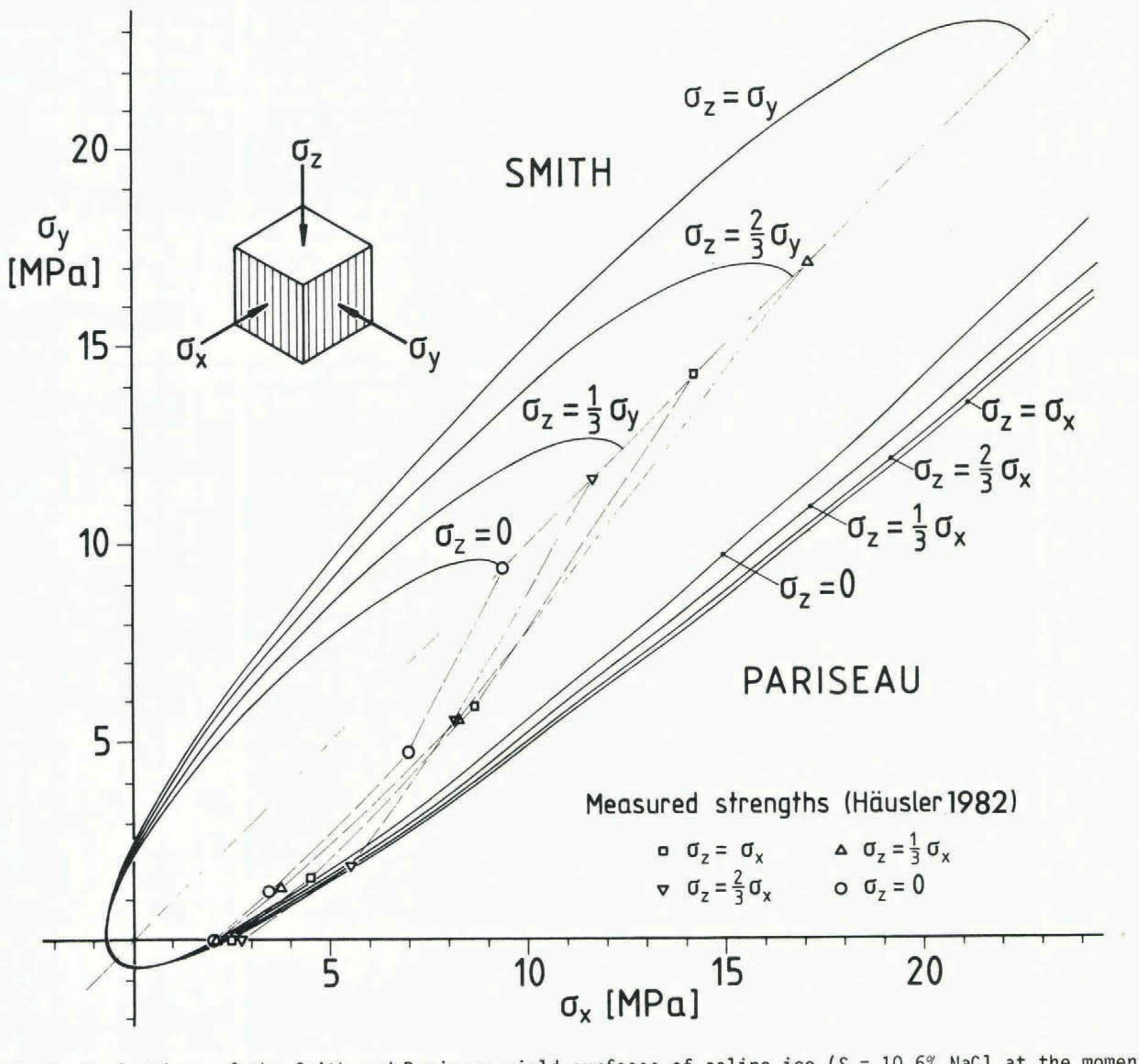

Fig.1. Projections of the Smith and Pariseau yield surfaces of saline ice (S = 10.6% NaCl at the moment of sampling, $T_I = -10°C$, $\dot{\varepsilon} = 2.0 \times 10^{-4}\ s^{-1}$) and of corresponding measured ice strengths onto the σ_x-σ_y-plane of the principal stress space.

https://doi.org/10.3189/S0260305500005310 Published online by Cambridge University Press

Nearly the same results are obtained with both functions, particularly in the octants of the principal stress space where tension stresses exist.

The yield functions determined in this study do not represent the best fit to all measured strengths. For practical applications it may be more appropriate to determine the coefficients of the yield functions by means of curve-fitting methods applied to a certain range of interest than to choose whether the Pariseau or the Smith yield function should be used.

ACKNOWLEDGMENTS

The author acknowledges the financial support of the Bundesministerium für Forschung und Technologie which made this study possible and is grateful to the colleagues at HSVA who contributed to this work.

REFERENCES

Carter D, Michel B 1971 Lois et mécanismes de l'apparente fracture fragile de la glace de rivière et de lac. *Université Laval. Département de Génie Civil. Faculté des Sciences. Section Mécanique des Glaces. Rapport* S-22

Frederking R M W 1977 Plane-strain compressive strength of columnar-grained and granular-snow ice. *Journal of Glaciology* 18(80): 505-516

Häusler F U 1982 Multiaxial compressive strength tests on saline ice with brush-type loading platens. *In IAHR. International Association for Hydraulic Research. International symposium on ice, Québec, Canada, 1981. Proceedings Vol 2*: 526-539

Häusler F U Unpublished. Dreidimensionales Bruchkriterium für Meer-Eis. (Report E 113/81, Hamburgische Schiffbau-Versuchsanstalt GmbH, Hamburg, September 1981)

Jones S J 1978 Triaxial testing of polycrystalline ice. *In Proceedings of the Third International Conference on Permafrost... 1978, Edmonton, Alberta, Canada, Vol 1:* 670-674

Pariseau W G 1972 Plasticity theory for anisotropic rocks and soils. *In* Gray K E (*ed*) *10th Annual Symposium on Rock Mechanics*. Baltimore, Port City Press

Peyton H R 1966 Sea ice strength. *University of Alaska. Geophysical Institute. Report Series* UAG-R-182

Ralston T D 1977 Yield and plastic deformation in ice crushing failure. *International Association of Hydrological Sciences Publication* 124 (Symposium of Seattle 1977 - *Sea Ice Processes and Models*): 234-245

Reinicke K M, Ralston T D 1977 Plastic limit analysis with an anisotropic, parabolic yield function. *International Journal of Rock Mechanics and Mining Sciences (and Geomechanics Abstracts)* 14: 147-154

Smith M B Unpublished. A parabolic yield condition for anisotropic rocks and soils. (PhD thesis, Rice University, Houston, 1974)

Weeks W F, Assur A 1969 Fracture of lake and sea ice. *CRREL Research Report* 269