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THE CONNECTION BETWEEN PSEUDO ALMOST PERIODIC FUNCTIONS DEFINED ON TIME SCALES AND ON THE REAL LINE

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Abstract

A necessary and sufficient condition for a continuous function g to be almost periodic on time scales is the existence of an almost periodic function f on $\mathbb{R}$ such that f is an extension of g. Our aim is to study this question for pseudo almost periodic functions. We prove the necessity of the condition for pseudo almost periodic functions. An example is given to show that the sufficiency of the condition does not hold for pseudo almost periodic functions. Nevertheless, the sufficiency is valid for uniformly continuous pseudo almost periodic functions. As applications, we give some results on the connection between the pseudo almost periodic (or almost periodic) solutions of dynamic equations on time scales and of the corresponding differential equations.

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1. Introduction

The theory of time scales was established by Hilger in 1988 (see [2]). This theory unifies continuous and discrete problems and provides a powerful tool for applications to economics, population models and quantum physics, among others. In 2011, Li and Wang [4, 5] introduced the concept of almost periodic function on time scales. As applications, they investigated the existence of almost periodic solutions to functional differential equations and neural networks. Since then, further forms of almost periodicity on time scales have been introduced with applications to dynamical systems, such as pseudo almost periodicity [6], almost automorphy [8], weighted pseudo almost periodicity [7] and weighted piecewise pseudo almost automorphy [10].

Recently, Lizama *et al.* [9] studied the connection between almost periodic functions defined on time scales and on the real line and obtained the following result.

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THEOREM 1.1. *If $\mathbb{T}$ is invariant under translations, a necessary and sufficient condition for a continuous function $g : \mathbb{T} \to \mathbb{E}^n$ to be almost periodic on $\mathbb{T}$ is the existence of an almost periodic function $f : \mathbb{R} \to \mathbb{E}^n$ such that $f(t) = g(t)$ for every $t \in \mathbb{T}$.*

The main purpose of this paper is to find an analogue of Theorem 1.1 for pseudo almost periodic functions. We prove the necessity of the condition in Theorem 1.1 for pseudo almost periodic functions (Theorem 3.4). An example (Example 3.6) shows that the sufficiency of the condition in Theorem 1.1 does not hold for pseudo almost periodic functions, but it is true if the pseudo almost periodic function is uniformly continuous (Theorem 3.7). Our results include Theorem 1.1 (see Remark 3.8(ii)). As applications, we give results on the connection between the pseudo almost periodic (or almost periodic) solutions of dynamic equations on time scales and of the corresponding differential equations (Theorems 4.1 and 4.2).

2. Preliminaries

The concepts and results in this section can be found in [1, 6], or deduced simply from the results given there. Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of positive integers, integers, real numbers and nonnegative real numbers, respectively, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. $\mathbb{E}^n$ denotes the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$, and $BC(\mathbb{R}; \mathbb{E}^n)$ denotes the Banach space of bounded continuous functions $f : \mathbb{R} \to \mathbb{E}^n$ with sup norm $\|\cdot\|_\infty$.

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, we say that t is right-scattered. Otherwise, t is right-dense. Analogously, if $\rho(t) < t$, then t is called left-scattered. Otherwise, t is left-dense.

2.1. Continuity and measurability on $\mathbb{T}$.

DEFINITION 2.1. A function $f : \mathbb{T} \to \mathbb{E}^n$ is said to be continuous on $\mathbb{T}$ if it is continuous at each right-dense point and at each left-dense point. Denote by $BC(\mathbb{T}; \mathbb{E}^n)$ the set of all bounded continuous functions $f : \mathbb{T} \to \mathbb{E}^n$.

For $a, b \in \mathbb{T}$ with $a \le b$, $[a, b], [a, b), (a, b], (a, b)$ are the usual intervals on the real line. The intervals $[a, a), (a, a], (a, a)$ are understood as the empty set, and we write

$$[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}, \quad [a, b)_\mathbb{T} = [a, b) \cap \mathbb{T}, \quad (a, b]_\mathbb{T} = (a, b] \cap \mathbb{T}, \quad (a, b)_\mathbb{T} = (a, b) \cap \mathbb{T}.$$

Denote by $\mathcal{F}_1$ the family of all left closed and right open intervals $[a, b)_\mathbb{T}$ of $\mathbb{T}$. Define a countably additive measure m_1 on the set $\mathcal{F}_1$ that assigns to each interval $[a, b)_\mathbb{T} \in \mathcal{F}_1$ its length, that is,

$$m_1([a, b)_\mathbb{T}) = b - a.$$

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Use m_1 to generate the outer measure m_1^* on $\mathcal{P}(\mathbb{T})$ (all the subsets of $\mathbb{T}$): for $E \in \mathcal{P}(\mathbb{T})$,

$$m_1^*(E) = \begin{cases} \inf\limits_{\mathcal{B}}\Big\{\sum\limits_{i\in I_{\mathcal{B}}}(b_i - a_i)\Big\} \in \mathbb{R}^+ & b \notin E, \\ +\infty & b \in E, \end{cases}$$

where $b = \sup \mathbb{T}$ and $\mathcal{B} = \{\{[a_i, b_i)_{\mathbb{T}} \in \mathcal{F}_1\}_{i\in I_{\mathcal{B}}} : I_{\mathcal{B}} \subset \mathbb{N}, E \subset \bigcup_{i\in I_{\mathcal{B}}}[a_i, b_i)_{\mathbb{T}}\}$. A set $A \subseteq \mathbb{T}$ is said to be Δ-measurable if, for each $E \subseteq \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T}\backslash A)).$$

Let

$$\mathcal{M}(m_1^*) = \{A \subseteq \mathbb{T} : A \text{ is } \Delta\text{-measurable}\}.$$

The Lebesgue Δ-measure, denoted by μ_Δ, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$.

Definition 2.2 [1]. We say that $f : \mathbb{T} \to \bar{\mathbb{R}}$ is Δ-measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ is Δ-measurable.

2.2. Integrals on $\mathbb{T}$. We say that $S : \mathbb{T} \to \mathbb{R}$ is simple if it only takes a finite number of distinct values, $\alpha_1, \ldots, \alpha_n$. Let $A_j = \{t \in \mathbb{T} : S(t) = \alpha_j\}$. Then

$$S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}, \tag{2.1}$$

with $\chi_{A_j} : \mathbb{T} \to \mathbb{R}$ the characteristic function of A_j, that is,

$$\chi_{A_j}(t) := \begin{cases} 1 & t \in A_j, \\ 0 & t \in \mathbb{T}\backslash A_j. \end{cases}$$

Definition 2.3 [1]. Let $E \subseteq \mathbb{T}$ be a Δ-measurable set and let $S : \mathbb{T} \to [0, +\infty]$ be a simple Δ-measurable function given by (2.1). The Lebesgue Δ-integral of S on E is

$$\int_E S(s)\Delta s = \sum_{j=1}^{n} \alpha_j \mu_\Delta(A_j \cap E)$$

(where we use the convention $0 \cdot \infty = 0$).

Definition 2.4 [1]. Let $E \subseteq \mathbb{T}$ be a Δ-measurable set.

(i) Let $f : \mathbb{T} \to [0, +\infty]$ be a Δ-measurable function. The Lebesgue Δ-integral of f on E is defined as

$$\int_E f(s)\Delta s = \sup \int_E S(s)\Delta s,$$

where the supremum is taken over all simple Δ-measurable functions S such that $0 \le S \le f$ in $\mathbb{T}$.

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(ii) Let $f : \mathbb{T} \to \bar{\mathbb{R}}$ be a Δ-measurable function. Then f is Lebesgue Δ-integrable on E if at least one of the elements

$$\int_E f^+(s)\Delta s \quad \text{and} \quad \int_E f^-(s)\Delta s,$$

is finite, where the positive and negative parts of f, f^+ and f^- respectively, are defined as

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\}.$$

In this case, the Lebesgue Δ-integral of f on E is defined as

$$\int_E f(s)\Delta s = \int_E f^+(s)\Delta s - \int_E f^-(s)\Delta s.$$

We note that all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold for the Lebesgue Δ-integrals on $\mathbb{T}$.

Definition 2.5. Let $f = (f_1, f_2, \ldots, f_n) : \mathbb{T} \to \mathbb{R}^n$.

(i) f is Δ-measurable if every $f_i : \mathbb{T} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, is Δ-measurable.

(ii) Let $E \subseteq \mathbb{T}$ be a Δ-measurable set and $f = (f_1, f_2, \ldots, f_n) : \mathbb{T} \to \mathbb{R}^n$ a Δ-measurable function. Then f is Lebesgue Δ-integrable on E if every $f_i : \mathbb{T} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, is Δ-integrable. In this case, the Lebesgue Δ-integral of f on E is

$$\int_E f(s)\Delta s = \Big(\int_E f_1(s)\Delta s, \int_E f_2(s)\Delta s, \ldots, \int_E f_n(s)\Delta s\Big).$$

Definition 2.6. Let $f_j : \mathbb{T} \to \mathbb{R}^n$, $j = 1, 2$.

(i) $f = f_1 + if_2$ is Δ-measurable in $\mathbb{C}^n$ if each $f_j : \mathbb{T} \to \mathbb{R}^n$, $j = 1, 2$, is Δ-measurable.

(ii) Let $E \subseteq \mathbb{T}$ be a Δ-measurable set and $f = f_1 + if_2 : \mathbb{T} \to \mathbb{C}^n$ a Δ-measurable function. Then f is Lebesgue Δ-integrable on E if each $f_j : \mathbb{T} \to \mathbb{R}^n$, $j = 1, 2$, is Δ-integrable. In this case, the Lebesgue Δ-integral of f on E is

$$\int_E f(z)\Delta z = \int_E f_1(s)\Delta s + i\int_E f_2(s)\Delta s.$$

For simplicity, we write Δ-integrable instead of Lebesgue Δ-integrable. It is easy to see that a continuous function is Δ-integrable.

2.3. Pseudo almost periodic functions on $\mathbb{T}$.

Definition 2.7. A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \text{for all } t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.8 [4, 5]. Let $\mathbb{T}$ be invariant under translations. A continuous function $f : \mathbb{T} \to \mathbb{E}^n$ is called almost periodic if for each $\varepsilon > 0$, the set

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{T}\}$$

is relatively dense in Π. That is, given $\varepsilon > 0$, there exists an $l = l(\varepsilon) > 0$ such that each interval of length l contains at least one $\tau = \tau(\varepsilon) \in \Pi$ satisfying

$$|f(t+\tau) - f(t)| < \varepsilon \quad \text{for all } t \in \mathbb{T}.$$

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The set $E(\varepsilon, f)$ is called the ε-translation set of f, and τ is called an ε-translation number of f. Denote by $AP_{\mathbb{T}}(\mathbb{E}^n)$ the set of all almost periodic functions $f : \mathbb{T} \to \mathbb{E}^n$.

Let $f \in BC(\mathbb{T}; \mathbb{E}^n)$. Then f and $|f|$ are Δ-integrable on any Δ-measurable set. Set

$$PAP_0(\mathbb{T}; \mathbb{E}^n) = \Big\{ f \in BC(\mathbb{T}; \mathbb{E}^n) : \lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} |f(s)|\Delta s = 0 \text{ for } t_0 \in \mathbb{T}, r \in \Pi \Big\}.$$

Definition 2.9 [6]. A function $f \in BC(\mathbb{T}; \mathbb{E}^n)$ is called pseudo almost periodic if $f = g + \phi$, where $g \in AP_{\mathbb{T}}(\mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T}; \mathbb{E}^n)$.

Denote by $PAP(\mathbb{T}; \mathbb{E}^n)$ the set of all pseudo almost periodic functions $f : \mathbb{T} \to \mathbb{E}^n$. The functions g and ϕ in Definition 2.9 are called the almost periodic component and the ergodic perturbation of f, respectively. By a standard argument, it is not hard to verify that $PAP(\mathbb{T}; \mathbb{E}^n)$ equipped with the essential sup norm $\|\cdot\|_\infty$ is a Banach space (see [6]) and the decomposition of a pseudo almost periodic function defined on $\mathbb{T}$ is unique, that is, for $f \in PAP(\mathbb{T}; \mathbb{E}^n)$, there exist unique $g \in AP_{\mathbb{T}}(\mathbb{E}^n)$ and $\phi \in PAP_0(\mathbb{T}; \mathbb{E}^n)$ such that $f = g + \phi$.

3. Main results

In this section we investigate the connection between pseudo almost periodic functions defined on $\mathbb{T}$ and $\mathbb{R}$. Let us start with some lemmas.

Lemma 3.1. *Assume $\mathbb{T}$ is invariant under translations and $K = \inf\{|\tau| : \tau \in \Pi, \tau \neq 0\}$. Then $K \in \Pi$ and $\mu(t) \le K$ for $t \in \mathbb{T}$.*

Proof. If there is no right-scattered point in $\mathbb{T}$, then $\mu(t) = 0$ and $K = 0$ and the proof is done. Suppose that $\mathbb{T}$ has at least one right-scattered point. Then $K > 0$. To prove that $K \in \Pi$, it suffices to prove that Π is closed. Let $\tau_n \in \Pi$, $n \in \mathbb{N}$, and $\tau_n \to \tau$ as $n \to \infty$. For any $t \in \mathbb{T}$, we have $t \pm \tau_n \in \mathbb{T}$ and $t \pm \tau_n \to t \pm \tau$ as $n \to \infty$. It follows from the closedness of $\mathbb{T}$ that $t \pm \tau \in \mathbb{T}$. Thus $\tau \in \Pi$ and Π is closed.

Now for every $t \in \mathbb{T}$ we have $t + K \in \mathbb{T}$ since $\mathbb{T}$ is invariant under translations. It follows from the definition of the forward jump operator σ that $\sigma(t) \le t + K$, and this means that $\mu(t) \le K$. □

Lemma 3.2. *Suppose that $t_0 \in \mathbb{T}$, $K = \inf\{|\tau| : \tau \in \Pi, \tau \neq 0\} > 0$ and $J = [t_0, t_0 + K)_{\mathbb{T}}$. Then there are at most countably many right-scattered points $\{t_i\}_{i\in I} \subset J$, $I \subseteq \mathbb{N}$ and*

$$\sum_{i\in I} \mu(t_i) \le K. \tag{3.1}$$

Moreover, $t_{ij} := t_i + jK$, $i \in I$, $j \in \mathbb{Z}$, are all the right-scattered points in $\mathbb{T}$ and $\mu(t_{ij}) = \mu(t_i)$.

Proof. Clearly, $K > 0$ implies that $\mathbb{T} \neq \mathbb{R}$. It follows from [1, Lemma 3.1] that there are at most countably many right-scattered points $\{t_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, in $J := [t_0, t_0 + K)_{\mathbb{T}}$. Then

$$\sum_{i\in I} \mu(t_i) = \sum_{i\in I} \mu_\Delta(\{t_i\}) = \mu_\Delta\Big(\bigcup_{i\in I} \{t_i\}\Big) \le \mu_\Delta(J) = K.$$

For any right-scattered point $t \in \mathbb{T}$, there exists $j_t \in \mathbb{Z}$ such that $t - j_t K \in J$. In addition, since $K \in \Pi$ by Lemma 3.1, it follows from [9, Lemma 2.6] that $t - j_t K$ is also a right-scattered point, so $t - j_t K = t_i$ for some $i \in I$. This means that $t_{ij} = t_i + jK$, $i \in I$, $j \in \mathbb{Z}$, are all the right-scattered points in $\mathbb{T}$. Moreover, by the translation invariance of μ (see [9, Corollary 2.8]), we have $\mu(t_{ij}) = \mu(t_i + jK) = \mu(t_i)$. □

REMARK 3.3. We will use the following facts in the proof of our main results. They can be obtained by basic calculations and we omit the details.

(i) Let $f \in BC(\mathbb{R}; \mathbb{E}^n)$, $t_0, r_0 \in \mathbb{R}$, $r_0 > 0$. Suppose that for $n \in \mathbb{N}$,

$$\lim_{n\to+\infty} \frac{1}{2nr_0} \int_{t_0-nr_0}^{t_0+nr_0} |f(s)|\, ds = a.$$

Then for $r \in \mathbb{R}$,

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{r}^{r} |f(s)|\, ds = a.$$

(ii) Let $\mathbb{T}$ be invariant under translations, $f \in BC(\mathbb{T}; \mathbb{E}^n)$, $t_0 \in \mathbb{T}$, $r_0 \in \Pi$, $r_0 > 0$. Suppose that for $n \in \mathbb{N}$,

$$\lim_{n\to+\infty} \frac{1}{2nr_0} \int_{t_0-nr_0}^{t_0+nr_0} |f(s)|\Delta s = a.$$

Then for $r \in \Pi$,

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} |f(s)|\Delta s = a.$$

We are now in a position to give our first main result.

THEOREM 3.4. *Let $\mathbb{T}$ be invariant under translations and $f \in PAP(\mathbb{T}; \mathbb{E}^n)$. Then there exists $g \in PAP(\mathbb{R}; \mathbb{E}^n)$ such that $g(t) = f(t)$ for $t \in \mathbb{T}$.*

PROOF. If $\mathbb{T} = \mathbb{R}$, the conclusion is obvious. Assume that $\mathbb{T} \neq \mathbb{R}$. Let $f = f_1 + f_2$ with $f_1 \in AP_{\mathbb{T}}(\mathbb{E}^n)$ and $f_2 \in PAP_0(\mathbb{T}; \mathbb{E}^n)$. By Theorem 1.1, there exists an almost periodic function $g_1 : \mathbb{R} \to \mathbb{E}^n$ such that $f_1(t) = g_1(t)$ for $t \in \mathbb{T}$. So we only need to prove that there exists $g_2 \in PAP_0(\mathbb{R}, \mathbb{E}^n)$ such that $f_2(t) = g_2(t)$ for $t \in \mathbb{T}$.

For $t_0 \in \mathbb{T}$, by Lemma 3.2, there are at most countably many right-scattered points $\{t_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, in $J := [t_0, t_0 + K)_{\mathbb{T}}$ such that (3.1) holds, and $t_{ij} = t_i + jK, i \in I, j \in \mathbb{Z}$, are all the right-scattered points in $\mathbb{T}$ with $\mu(t_{ij}) = \mu(t_i)$. Let $\varrho_i \in (0, \mu(t_i))$. Then (3.1) implies that

$$\varrho := \sum_{i\in I} \varrho_i < K. \tag{3.2}$$

Define $g : \mathbb{R} \to \mathbb{E}^n$ by

$$g_2(t) = \begin{cases} f_2(t) & t \in \mathbb{T}, \\ f_2(t_{ij}) & t \in (t_{ij}, \sigma(t_{ij}) - \varrho_i/2^{|j|}], \\ -\dfrac{2^{|j|}}{\varrho_i}(t - \sigma(t_{ij}))(f_2(t_{ij}) - f_2(\sigma(t_{ij}))) + f_2(\sigma(t_{ij})) & t \in (\sigma(t_{ij}) - \varrho_i/2^{|j|}, \sigma(t_{ij})), \end{cases}$$

for $i \in I$, $j \in \mathbb{Z}$. It is easy to see that g_2 is well defined on $\mathbb{R}$, and is bounded and continuous. Now it suffices to prove that $g_2 \in PAP_0(\mathbb{R};\mathbb{E}^n)$. By the definition of g_2,

$$\begin{aligned}\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|g_2(s)|\,ds &= \frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|g_2(s)|\,ds + \frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}\int_{t_{ij}}^{\sigma(t_{ij})}|g_2(s)|\,ds\\ &= \frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|f_2(s)|\,ds + \frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}\int_{t_{ij}}^{\sigma(t_{ij})}|g_2(s)|\,ds.\end{aligned} \tag{3.3}$$

Split the interval of integration $(t_{ij},\sigma(t_{ij}))$ into two parts, $(t_{ij},\sigma(t_{ij})-\rho_i/2^{|j|}]$ and $(\sigma(t_{ij})-\rho_i/2^{|j|},\sigma_i(t_{ij}))$. By (3.2) and the definition of g_2,

$$\begin{aligned}&\frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}\int_{t_{ij}}^{\sigma(t_{ij})}|g_2(s)|\,ds\\ &\quad\le \frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}\Big(|f_2(t_{ij})|\Big(\mu(t_{ij})-\frac{\varrho_i}{2^{|j|}}\Big)+\frac{\varrho_i}{2^{|j|+1}}(|f_2(t_{ij})|+|f_2(\sigma(t_{ij})|)\Big)\Big)\\ &\quad\le \frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}|f_2(t_{ij})|\mu(t_{ij})+\frac{\varrho\|f_2\|_\infty}{2nK}\sum_{j=-n}^{n-1}\frac{1}{2^{|j|+1}}\\ &\quad\le \frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}|f_2(t_{ij})|\mu(t_{ij})+\frac{\|f_2\|_\infty}{n}.\end{aligned} \tag{3.4}$$

By substituting (3.4) into (3.3) and using [1, Theorem 5.2],

$$\begin{aligned}&\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|g_2(s)|\,ds\\ &\quad\le \frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|f_2(s)|\,ds+\frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}|f_2(t_{ij})|\mu(t_{ij})+\frac{\|f_2\|_\infty}{n}\\ &\quad= \frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|f_2(s)|\Delta s+\frac{\|f_2\|_\infty}{n}\to 0 \quad \text{as } n\to+\infty,\end{aligned}$$

since $f_2 \in PAP_0(\mathbb{T};\mathbb{E}^n)$. Then it follows from Remark 3.3(i) that for $r\in\mathbb{R}$,

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}|g_2(s)|\,ds=0,$$

which implies that $g_2 \in PAP_0(\mathbb{R};\mathbb{E}^n)$. □

Remark 3.5. In the important case $\mathbb{T}=h\mathbb{Z}$ for some $h>0$, g can be constructed linearly as follows:

$$g(t)=\begin{cases} f(hn) & t=hn, n\in\mathbb{Z},\\ \Big(1-\dfrac{t-nh}{h}\Big)f(nh)+\dfrac{t-nh}{h}f(nh+h) & t\in(nh,nh+h), n\in\mathbb{Z}.\end{cases}$$

It is easy to verify that g is pseudo almost periodic on $\mathbb{R}$ provided that $f\in PAP(h\mathbb{Z};\mathbb{E}^n)$.

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We now consider the converse problem of Theorem 3.4 just as Theorem 1.1 does. That is, if $f \in PAP(\mathbb{R}; \mathbb{E}^n)$, is the restriction function $g := f|_{\mathbb{T}} : \mathbb{T} \to \mathbb{E}^n$ in $PAP(\mathbb{T}; \mathbb{E}^n)$? Unfortunately, the following example shows that it is not.

Example 3.6 [11]. Let $\mathbb{T} = \mathbb{Z}$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-n^2|t-n|} - e^{-1} & t \in [n - 1/n^2, n + 1/n^2],\ n = 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $f \in PAP_0(\mathbb{R}; \mathbb{R})$ and

$$f(n) = \begin{cases} 1 - e^{-1} & n = 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{f(n)\}_{n\in\mathbb{Z}}$ is not in $PAP_0(\mathbb{Z}; \mathbb{R})$.

However, if $f : \mathbb{R} \to \mathbb{E}^n$ is uniformly continuous, the converse of Theorem 3.4 is true.

Theorem 3.7. *Assume that $\mathbb{T}$ is invariant under translations. If $f \in PAP(\mathbb{R}; \mathbb{E}^n)$ is uniformly continuous, then the restriction function $g := f|_{\mathbb{T}} : \mathbb{T} \to \mathbb{E}^n$ is in $PAP(\mathbb{T}; \mathbb{E}^n)$.*

Proof. If $\mathbb{T} = \mathbb{R}$, the conclusion is obvious. Assume that $\mathbb{T} \neq \mathbb{R}$. Let $f = f_1 + f_2$ with $f_1 \in AP(\mathbb{R}; \mathbb{E}^n)$, $f_2 \in PAP_0(\mathbb{R}; \mathbb{E}^n)$. Then

$$g = f|_{\mathbb{T}} = f_1|_{\mathbb{T}} + f_2|_{\mathbb{T}} := g_1 + g_2.$$

It follows from Theorem 1.1 that $g_1 \in AP_{\mathbb{T}}(\mathbb{E}^n)$. Thus, it remains to prove that $g_2 \in PAP_0(\mathbb{T}; \mathbb{E}^n)$.

For $t_0 \in \mathbb{T}$, by Lemma 3.2, there are at most countably many right-scattered points $\{t_i\}_{i\in I}$, $I \subseteq \mathbb{N}$, in $J := [t_0, t_0 + K)_{\mathbb{T}}$ such that (3.1) holds, and $t_{ij} = t_i + jK$, $i \in I$, $j \in \mathbb{Z}$, are all the right-scattered points in $\mathbb{T}$ with $\mu(t_{ij}) = \mu(t_i)$. Given $\varepsilon > 0$, by (3.1), there exists a subset $\{n_1, n_2, \ldots, n_m\} \subset I$ such that

$$\sum_{i\in I'} \mu(t_i) < \varepsilon \quad \text{with } I' = I \backslash \{n_1, n_2, \ldots, n_m\}. \tag{3.5}$$

We note that if I is a finite set, we can choose $I' = \emptyset$.

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Now by (3.5) and [1, Theorem 5.2],

$$\begin{aligned}
&\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|g_2(s)|\Delta s\\
&\quad=\frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|g_2(s)|\,ds+\frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i\in I}\mu(t_{ij})|g_2(t_{ij})|\\
&\quad=\frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|f_2(s)|\,ds+\frac{1}{2nK}\sum_{j=-n}^{n-1}\Big(\sum_{i\in I\setminus I'}+\sum_{i\in I'}\Big)\mu(t_i)|f_2(t_{ij})|\\
&\quad\le\frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|f_2(s)|\,ds+\frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i=1}^{m}\mu(t_{n_i})|f_2(t_{n_ij})|+\frac{\|f_2\|_\infty}{K}\varepsilon\\
&\quad:=\Upsilon_1+\Upsilon_2+\frac{\|f_2\|_\infty}{K}\varepsilon. \qquad (3.6)
\end{aligned}$$

Obviously,

$$\Upsilon_1=\frac{1}{2nK}\int_{[t_0-nK,t_0+nK]_{\mathbb{T}}}|f_2(s)|\,ds\le\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|f_2(s)|\,ds. \qquad (3.7)$$

Since f is uniformly continuous, it follows from the uniform continuity of f_1 that f_2 is uniformly continuous, and there exists $\delta'>0$ such that $|f_2(t)-f_2(s)|<\varepsilon$ whenever $|t-s|<\delta'$. Let $\delta:=\min\{\delta',\mu(t_{n_1}),\dots,\mu(t_{n_m})\}$. Then by (3.1),

$$\begin{aligned}
\Upsilon_2&\le\frac{1}{2nK}\sum_{j=-n}^{n-1}\sum_{i=1}^{m}\Big(\frac{1}{\delta}\int_{t_{n_ij}}^{t_{n_ij}+\delta}(|f_2(t_{n_ij})-f_2(s)|+|f_2(s)|)\,ds\Big)\mu(t_{n_i})\\
&\le\frac{\varepsilon}{2nK}\sum_{j=-n}^{n-1}\sum_{i=1}^{m}\mu(t_{n_i})+\frac{1}{2nK\delta}\sum_{j=-n}^{n-1}\sum_{i=1}^{m}\Big(\int_{t_{n_ij}}^{t_{n_ij}+\delta}|f_2(s)|\,ds\Big)\mu(t_{n_i})\\
&\le\frac{\varepsilon}{2nK}2nK+\frac{1}{2nK\delta}\sum_{j=-n}^{n-1}\sum_{i=1}^{m}\Big(\int_{t_{n_ij}}^{t_{n_ij}+\mu(t_{n_ij})}|f_2(s)|\,ds\Big)K\\
&\le\varepsilon+\frac{1}{2n\delta}\int_{t_0-nK}^{t_0+nK}|f_2(s)|\,ds. \qquad (3.8)
\end{aligned}$$

Since $f_2\in PAP_0(\mathbb{R};\mathbb{E}^n)$, there exists $N>0$ such that for $n>N$,

$$\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|f_2(s)|\,ds<\Big(1+\frac{K}{\delta}\Big)^{-1}\varepsilon.$$

Now, from (3.6), (3.7) and (3.8), for $n>N$,

$$\begin{aligned}
\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|g_2(s)|\Delta s&\le\Big(1+\frac{K}{\delta}\Big)\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|f_2(s)|\,ds+\Big(1+\frac{\|f_2\|_\infty}{K}\Big)\varepsilon\\
&<\Big(2+\frac{\|f_2\|_\infty}{K}\Big)\varepsilon.
\end{aligned}$$

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That is,

$$\lim_{n\to+\infty}\frac{1}{2nK}\int_{t_0-nK}^{t_0+nK}|g_2(s)|\Delta s=0.$$

By Remark 3.3(ii), for $r\in\Pi$,

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}|g_2(s)|\Delta s=0,$$

which implies that $g_2\in PAP_0(\mathbb{T};\mathbb{E}^n)$. □

REMARK 3.8.

(i) We note that the special case $\mathbb{T}=\mathbb{Z}$ of Theorem 3.7 is the same as [3, Proposition 2.3].

(ii) If f is almost periodic, it is automatically uniformly continuous. So Theorem 1.1 follows from Theorems 3.4 and 3.7.

4. Applications

As applications of our main results, we present some results on the connection between the pseudo almost periodic (or almost periodic) solutions of dynamic equations on time scales and of the corresponding differential equations.

Let A be a nonsingular square matrix of order n and $h>0$. Set

$$\hat{A}=\frac{e^{Ah}-I}{h},\quad C=\frac{e^{Ah}}{h}A^{-1}+\frac{I-e^{Ah}}{h^2}A^{-2},\quad D=\frac{1}{h}A^{-1}+\frac{I-e^{Ah}}{h^2}A^{-2}.$$

We assume that C is nonsingular and $\|C^{-1}D\|<1$. For a pseudo almost periodic function $g:h\mathbb{Z}\to\mathbb{E}^n$, we define, for $n\in h\mathbb{Z}$,

$$\bar{f}(n)=C^{-1}\sum_{i=0}^{\infty}(C^{-1}D)^i g(n+ih),$$

$$f(t)=\begin{cases}\bar{f}(n) & t=n,\\ \left(1-\dfrac{t-n}{h}\right)\bar{f}(n)+\dfrac{t-n}{h}\bar{f}(n+h) & t\in(n,n+h).\end{cases}$$

It is easy to check that $\bar{f}$ and f are well defined and pseudo almost periodic, and the following equality holds:

$$g(n)=C\bar{f}(n)-D\bar{f}(n+h),\quad n\in h\mathbb{Z}. \tag{4.1}$$

With these assumptions, we have the following result.

THEOREM 4.1. *Assume that all the eigenvalues of A are negative. If $u:\mathbb{R}\to\mathbb{E}^n$ is a pseudo almost periodic solution of*

$$x'(t)=Ax(t)+f(t),\quad t\in\mathbb{R}, \tag{4.2}$$

then the restriction $u|_{h\mathbb{Z}}$ of u to the time scale $h\mathbb{Z}$ is a pseudo almost periodic solution of

$$x^{\Delta}(t)=\hat{A}x(t)+g(t),\quad t\in h\mathbb{Z}. \tag{4.3}$$

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Proof. For simplicity, in the following calculation, we write $\sum_{k=k_0}^{n} \phi(k)$, where $k_0, k, n \in h\mathbb{Z}$ to mean $\sum_{j=j_0}^{m} \phi(hj)$, where $j_0, m \in \mathbb{Z}$ and $k_0 = hj_0, n = hm$.

Since all the eigenvalues of A are negative, it is well known that the unique bounded solution of (4.2) is given by the function

$$u(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s)\,ds.$$

Now we compute

$$\begin{aligned}
u(n) &= \int_{-\infty}^{n} e^{A(n-s)} f(s)\,ds \\
&= \sum_{k=-\infty}^{n-h} \int_{k}^{k+h} e^{A(n-s)} \Big[\Big(1 - \frac{s-k}{h}\Big)\bar{f}(k) + \frac{s-k}{h}\bar{f}(k+h)\Big]\,ds \\
&= \sum_{k=-\infty}^{n-h} \Big(\int_{k}^{k+h} e^{A(n-s)}\,ds\Big)\bar{f}(k) + \sum_{k=-\infty}^{n-h} \Big(\int_{k}^{k+h} e^{A(n-s)} s\,ds\Big)\frac{\bar{f}(k+h) - \bar{f}(k)}{h} \\
&\quad - \sum_{k=-\infty}^{n-h} k\Big(\int_{k}^{k+h} e^{A(n-s)}\,ds\Big)\frac{\bar{f}(k+h) - \bar{f}(k)}{h} \\
&= -A^{-1}e^{An} \sum_{k=-\infty}^{n-h} (e^{-A(k+h)} - e^{-Ak})\bar{f}(k) \\
&\quad - A^{-1}e^{An} \sum_{k=-\infty}^{n-h} ((k+h)e^{-A(k+h)} - ke^{-Ak})\frac{\bar{f}(k+h) - \bar{f}(k)}{h} \\
&\quad - A^{-2}e^{An} \sum_{k=-\infty}^{n-h} (e^{-A(k+h)} - e^{-Ak})\frac{\bar{f}(k+h) - \bar{f}(k)}{h} \\
&\quad + A^{-1}e^{An} \sum_{k=-\infty}^{n-h} k(e^{-A(k+h)} - e^{-Ak})\frac{\bar{f}(k+h) - \bar{f}(k)}{h} \\
&= -A^{-1}\bar{f}(n) - A^{-2}e^{An} \sum_{k=-\infty}^{n-h} (e^{-A(k+h)} - e^{-Ak})\frac{\bar{f}(k+h) - \bar{f}(k)}{h}
\end{aligned}$$

after exploiting the telescoping sum. Then, by (4.1), for $n \in h\mathbb{Z}$,

$$\begin{aligned}
u^{\Delta}(n) - \hat{A}u(n) &= \frac{u(n+h) - u(n)}{h} - \frac{e^{Ah} - I}{h}u(n) = \frac{u(n+h) - e^{Ah}u(n)}{h} \\
&= \frac{1}{h}A^{-1}(e^{Ah}\bar{f}(n) - \bar{f}(n+h)) - \frac{I - e^{Ah}}{h^2}A^{-2}(\bar{f}(n+h) - \bar{f}(n)) \\
&= C\bar{f}(n) - D\bar{f}(n+h) = g(n).
\end{aligned}$$

That is, the restriction $u|_{h\mathbb{Z}}$ of u to $h\mathbb{Z}$ is a solution of (4.3).

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Note that $\|u'\|_\infty \leq \|A\|\|u\|_\infty + \|f\|_\infty$. Thus u is uniformly continuous and, by Theorem 3.7, $u|_{h\mathbb{Z}}$ is pseudo almost periodic. □

For the case of almost periodicity, namely, when $g : h\mathbb{Z} \to \mathbb{E}^n$ is almost periodic and $f, \bar{f}$ are defined as above, Theorem 1.1 and the same arguments as in the proof of Theorem 4.1 yield the following result.

THEOREM 4.2. *Assume that all the eigenvalues of A are negative. Then if $u : \mathbb{R} \to \mathbb{E}^n$ is an almost periodic solution of*

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R},$$

the restriction $u|_{h\mathbb{Z}}$ of u to the time scale $h\mathbb{Z}$ is an almost periodic solution of

$$x^{\Delta}(t) = \hat{A}x(t) + g(t), \quad t \in h\mathbb{Z}.$$

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