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# Infinitely Many Solutions for the Prescribed Boundary Mean Curvature Problem in $\mathbb{B}^N$

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*Abstract.* We consider the prescribed boundary mean curvature problem in $\mathbb{B}^N$ with the Euclidean metric

$$\begin{cases} -\Delta u = 0, \quad u > 0 & \text{in } \mathbb{B}^N, \\ \frac{\partial u}{\partial \nu} + \frac{N-2}{2} u = \frac{N-2}{2} \widetilde{K}(x) u^{2^\# - 1} & \text{on } \mathbb{S}^{N-1}, \end{cases}$$

where $\widetilde{K}(x)$ is positive and rotationally symmetric on $\mathbb{S}^{N-1}$, $2^\# = \frac{2(N-1)}{N-2}$. We show that if $\widetilde{K}(x)$ has a local maximum point, then this problem has infinitely many positive solutions that are not rotationally symmetric on $\mathbb{S}^{N-1}$.

## 1 Introduction

Parallel to the prescribed scalar curvature problem, the prescribed boundary mean curvature problem also plays an important role in conformal geometry. Given an $N$-dimensional ($N \geq 3$) Riemannian manifold $(M, g)$ with boundary, this problem concerns if one can find a new metric $\tilde{g}$ in the conformal class of $g$, such that $(M, \tilde{g})$ has zero scalar curvature and the boundary mean curvature becomes a prescribed function. Denoting $\tilde{g} = u^{4/(N-2)} g$, where $u$ is a positive smooth function, the problem may be addressed by finding a positive solution $u$ of the following equation:

$$\begin{cases} -\dfrac{4(N-1)}{N-2} \Delta_g u + R_g u = 0 & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{N-2}{2} H_g u = \dfrac{N-2}{2} \widetilde{K}(x) u^{2^\# - 1} & \text{on } \partial M, \end{cases}$$

where $2^\# = \frac{2(N-1)}{N-2}$ is the critical exponent of the Sobolev trace embedding. Here $\Delta_g$ is the Laplace–Beltrami operator, $R_g$ is the scalar curvature of $M$, $H_g$ is the mean curvature of $\partial M$, $\nu$ is the outward normal unit vector with respect to the metric $g$, and $\widetilde{K}(x)$ is the prescribed function.

Due to the fact that the embedding $H^1(M) \hookrightarrow L^{2^\#}(\partial M)$ is not compact, the Euler–Lagrange functional $J$ associated with our problem fails to satisfy the Palais–Smale condition. That is, there exists a noncompact sequence along which the functional

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$J$ is bounded and its gradient goes to zero. Therefore, it is not possible to apply the standard variational methods to prove the existence of solutions. Notice that the above problem is a natural analogue to the well-known scalar curvature problems on closed manifolds.

Escobar [12, 14] and Marques [19, 20] studied this problem for the case where $\widetilde{K}(x)$ is a constant. From the existence of solutions, they showed in this case that most compact manifolds with boundary are conformally diffeomorphic to a manifold that resembles the ball in two ways; namely, it has zero scalar curvature and its boundary has constant mean curvature, although very few regions are really conformal to the ball in higher dimensions. For other related results, we refer the reader to [2–4, 7, 13, 16, 17] and the references therein.

In this paper, we prescribe mean curvature on the boundary $\mathbb{S}^{N-1}$ of the unit ball $\mathbb{B}^N$ in $\mathbb{R}^N$ ($N \geq 3$) with Euclidean metric $g_0$. Precisely, we study the problem of finding a conformal metric to $g_0$ whose scalar curvature vanishes in $\mathbb{B}^N$ and where the mean curvature of boundary $\mathbb{S}^{N-1}$ is given by $\widetilde{K}(x)$. This problem is equivalent to solving the following boundary problem:

$$(1.1)\qquad \begin{cases} -\Delta u = 0, \quad u > 0 & \text{in } \mathbb{B}^N, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{N-2}{2} u = \dfrac{N-2}{2} \widetilde{K}(x) u^{2^{\#}-1} & \text{on } \mathbb{S}^{N-1}. \end{cases}$$

Note that Cherrier [8] studied the regularity for this equation. He showed that solutions of (1.1) that are of class $H^1$ are also smooth.

The problem of determining which $\widetilde{K}(x)$ admits a solution to (1.1) has been studied extensively. It is easy to see that a necessary condition for solving the problem is that $\widetilde{K}(x)$ must be positive somewhere. But there are also some obstructions for the existence of solutions, which are said to be of *topological type*. For example, the solution $u$ must satisfy the following Kazdan–Warner condition (see [14]):

$$(1.2)\qquad \int_{\mathbb{S}^{N-1}} \nabla \widetilde{K} \cdot x u^{2^{\#}} \, \mathrm{d}x = 0.$$

Some existence results have been obtained under assumptions involving the Laplacian at the critical points of $\widetilde{K}$. Sufficient conditions in dimensions 3 and 4 are given in [11, 15]. Furthermore in [1], the authors developed a Morse theoretical approach to this problem in the 4-dimensional case providing some multiplicity results under generic conditions on the function $\widetilde{K}$.

Consider the case $\widetilde{K}(x) = 1 + \varepsilon h(x)$ is a perturbation of 1 (or generally a perturbation of some constant). In [6], by a perturbation method, Chang, Xu, and Yang obtained positive solutions by looking for constrained minimizers; more precisely, they proved that if at each critical point $Q$ of $h(x)$, $\Delta_{\mathbb{S}^{N-1}} h(Q) = 0$, then under additional conditions, the above problem has a positive solution for $\varepsilon$ sufficiently small. Furthermore, Cao–Peng [5] constructed a two-peak solution whose maximum points are located near two critical points of $h$ as $\varepsilon \to 0$ under certain assumptions.

It is well known that the unit ball $\mathbb{B}^N$ is conformal to the half-space $\mathbb{R}^N_+$. As in [5], to consider this problem we transfer equation (1.1) to an equation in the half-space

$\mathbb{R}^N_+$. We denote $y = (y_1, \dots, y_N) = (y', y_N) \in \mathbb{B}^N$. By the standard stereographic projection: $\Pi\colon \mathbb{B}^N \to \mathbb{R}^N_+$,

$$\Pi(y', y_N) = \Big( \frac{4y'}{(1+y_N)^2 + |y'|^2}, \frac{2(1 - y_N^2 - |y'|^2)}{(1+y_N)^2 + |y'|^2} \Big),$$

$$\tilde{u}(x) = \frac{4^{\frac{N-2}{2}} u(\Pi^{-1}x)}{\big[(2+x_N)^2 + |x'|^2\big]^{\frac{N-2}{2}}},$$

we see that the function $\tilde{u}(x)$ satisfies

$$\begin{cases} \Delta u = 0, \;\; u > 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} = K(x) u^{2^\#-1} & \text{on } \partial\mathbb{R}^N_+, \\ u \in D^{1,2}(\mathbb{R}^N_+), \end{cases} \tag{1.3}$$

where $D^{1,2}(\mathbb{R}^N_+)$ denotes the completion of $C_0^\infty(\overline{\mathbb{R}^N_+})$ under the norm $\int_{\mathbb{R}^N_+} |\nabla u|^2$, the bounded function $K = \widetilde{K} \circ \Pi^{-1}$.

For the case where $K(x)$ is a positive constant, say 1 for convenience, it is well known from [18] that the only solution to (1.3) has the form

$$U_{\zeta,\Lambda}(x) = (N-2)^{\frac{N-2}{2}} \Big[ \frac{\Lambda}{(1+\Lambda x_N)^2 + \Lambda^2 |\bar{x} - \bar{\zeta}|^2} \Big]^{\frac{N-2}{2}},$$

where both $\Lambda > 0$ and $\bar{\zeta} \in \mathbb{R}^{N-1}$ are arbitrary. Obviously it is radially symmetric in $\partial\mathbb{R}^N_+$ with respect to $\bar{\zeta}$. Here we write $x = (\bar{x}, x_N)$, $\bar{x} \in \mathbb{R}^{N-1}$.

In this paper, we consider the simplest general case, *i.e.*, $K(x) = K(|\bar{x}|) =: K(r)$ is a radially symmetric *positive* function in $\partial\mathbb{R}^N_+$. The Kazdan–Warner condition (1.2) is correspondingly reduced to

$$\int_{\mathbb{R}^{N-1}} K'(r) r u^{2^\#} \, d\bar{x} = \int_{\mathbb{R}^{N-1}} \big( \nabla K(\bar{x}) \cdot \bar{x} \big) u^{2^\#} \, d\bar{x} = 0.$$

Hence by positiveness of $u$, $K'(r)$ cannot have fixed sign in $\mathbb{R}^{N-1}$. Thus it is natural to assume that $K$ is *not monotone*.

The purpose of this paper is to answer the following two questions:

**Q1** Does the existence of a local maximum of $K$ guarantee the existence of solutions to (1.3)?

**Q2** Are there solutions to (1.3) that are non-radially symmetric in $\partial\mathbb{R}^N_+$?

To state the main result, we assume that $K(r)$ satisfies the following condition: *$K(x)$ is positive, bounded and there is a constant $r_0 > 0$, such that*

$$K(r) = K(r_0) - c_0 |r - r_0|^m + O(|r - r_0|^{m+\theta}) \quad \text{for } r \in (r_0 - \delta, r_0 + \delta), \tag{K}$$

where $c_0 > 0$, $\theta > 0$, $\delta > 0$ are some constants and the constant $m$ satisfies $m \in [2, N-2)$. To make sure that such $m$ exists, we consider the problem for $N \geq 5$. Without loss of generality, we assume that $K(r_0) = 1$.

Our main result is stated as follows.

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***Theorem 1.1*** *Suppose that $N \geq 5$. If $K(r)$ satisfies (K), then problem (1.3) has infinitely many solutions, which are non-radial in $\partial\mathbb{R}^N_+$.*

***Remark 1.2*** Combining the results in [15] and [11], we give sufficient conditions for the existence of solutions for all $N \geq 3$.

***Remark 1.3*** The condition (K) is a local condition, while the condition in [1] is global.

***Remark 1.4*** Theorem 1.1 exhibits a new phenomenon for the prescribed boundary mean curvature problem. It suggests that if the critical points of $K$ are not isolated, new solutions to (1.3) may bifurcate.

We formulate the following conjecture in the general case.

***Conjecture*** *If the set $\{x \in \partial\mathbb{R}^N_+ : K(x) = \max_{x\in\partial\mathbb{R}^N_+} K(x)\}$ is an $\ell$-dimensional smooth manifold without boundary, where $1 \leq \ell \leq N-2$, then problem (1.3) admits infinitely many positive solutions.*

Let us outline the main idea in the proof of Theorem 1.1. Let us fix a positive integer $k \geq k_0$, where $k_0$ is a large integer to be determined later. Set $\mu = k^{\frac{N-2}{N-2-m}}$ as the scaling parameter.

Using the transformation $u(y) \mapsto \mu^{-\frac{N-2}{2}} u(\frac{y}{\mu})$, we note that (1.3) is equivalent to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = K(\dfrac{|y|}{\mu}) u^{2^\#-1} & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.4}$$

In this paper, let

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0, \dots, 0\Big), \qquad j = 1, \dots, k,$$

then the approximation solution we choose is

$$W_{r,\Lambda}(y) = \sum_{j=1}^k U_{x_j,\Lambda} = (N-2)^{\frac{N-2}{2}} \sum_{j=1}^k \Big[\frac{\Lambda}{(1+\Lambda y_N)^2 + \Lambda^2|\bar{y} - \bar{x}_j|^2}\Big]^{\frac{N-2}{2}}.$$

We will find the solution with the form $W_{r,\Lambda} + \phi$, furthermore $\phi$ has the following symmetries:

$$\phi(y_1, y_2, \dots, y_{N-1}, y_N) = \phi(y_1, -y_2, \dots, -y_{N-1}, y_N), \tag{1.5}$$

$$\phi(y) = \phi(Q_k y), \qquad Q_k = \left(\begin{array}{cc|c} \cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k} & 0 \\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right), \tag{1.6}$$

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where $I$ denotes the $(N-2)\times(N-2)$ identical matrix. In this paper we always assume that

$$r \in \Big[\mu r_0 - \frac{1}{\mu^{\bar\theta}}, \mu r_0 + \frac{1}{\mu^{\bar\theta}}\Big], \qquad L_0 \le \Lambda \le L_1,$$

where $\bar\theta > 0$ is a small number and $L_1 > L_0 > 0$.

Theorem 1.1 is a direct consequence of the following theorem.

***Theorem 1.5*** *Suppose $N \ge 5$. If $K$ satisfies (K), then there is an integer $k_0 > 0$ such that for any integer $k > k_0$, problem (1.4) has a solution $u_k$ of the form*

$$u_k = W_{r_k,\Lambda_k} + \phi_k,$$

*where $\phi_k$ satisfies (1.5) and (1.6). Moreover, as $k \to \infty$,*

$$\|\phi_k\|_\infty \to 0, \quad r_k \in \Big[\mu r_0 - \frac{1}{\mu^{\bar\theta}}, \mu r_0 + \frac{1}{\mu^{\bar\theta}}\Big], \quad \textit{and} \quad L_0 \le \Lambda_k \le L_1.$$

***Remark 1.6*** Changing back the solutions in Theorem 1.5, we see that the solutions to (1.1) can blow up at an arbitrarily large number of points on $\mathbb{S}^{N-1}$. On the other hand, Escobar–Garcia [15] shows that when $N \ge 4$ and the function $K$ at its critical points vanishes up to order $m$ with $m > N-2$, there is actually at most one possible blow-up point. Thus our existence result means that $m < N-2$ is almost sharp.

We will use the finite reduction method introduced by Wei–Yan [21] to prove Theorem 1.5, in which the authors use $k$, the number of bubbles of the solutions, as the parameter in the construction of bubbles solutions for (1.4). The main difficulty in constructing solution with $k$-bubbles is that we need to obtain a better control of the error terms. Since the maximum norm will not be affected by the number of bubbles, we will carry out the reduction procedure in a space with weighted maximum norm.

Our paper is organized as follows. In Section 2, we obtain some preliminary estimates. In Section 3, we deal with the corresponding linearized and nonlinear problems. In Section 4, we come to the variational reduction procedure. In Section 5, the proof of Theorem 1.5 is given. Finally we give the energy expansion of the approximation solution and list some useful estimates in Appendix A.

Throughout this paper, $C$ is a generic constant independent of $k$ and $\mu$.

## 2 Preliminary Estimates

In this section we will obtain some estimates for later use.

Under the assumption that the solution $u = W_{r,\Lambda} + \phi$, it is not difficult to check that $\phi$ should satisfy the following equation:

$$\text{(2.1)} \qquad \begin{cases} -\Delta\phi = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial\phi}{\partial\nu} - (2^\# - 1)K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^\#-2}\phi = -R(y) + N(\phi) & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

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where the error term $R(y)$ and the nonlinear term $N(\phi)$ are defined by

$$R(y) = \frac{\partial W_{r,\Lambda}}{\partial \nu} - K\Big(\frac{|y|}{\mu}\Big) W_{r,\Lambda}^{2^\#-1},$$

$$N(\phi) = K\Big(\frac{|y|}{\mu}\Big)\Big[(W_{r,\Lambda}+\phi)^{2^\#-1} - W_{r,\Lambda}^{2^\#-1} - (2^\#-1)W_{r,\Lambda}^{2^\#-2}\phi\Big].$$

In what follows, we use the following two important weighted norms

$$\|\phi\|_* = \sup_{y\in\overline{\mathbb{R}}^N_+}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\Big)^{-1}|\phi(y)|,$$

$$\|h\|_{**} = \sup_{y\in\partial\mathbb{R}^N_+}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big)^{-1}|h(y)|$$
$$= \sup_{\bar y\in\mathbb{R}^{N-1}}\Big(\sum_{j=1}^k \frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big)^{-1}|h(y)|,$$

where $0<\tau<\frac{1}{2(N-2)}$ is a fixed small constant.

We need the following two lemmas later.

***Lemma 2.1*** *It holds that, for some small* $0<\sigma<\frac{m}{N-2}(\frac{m}{N-2}-\tau)$,

$$\|R\|_{**} \le C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.$$

**Proof** Define

$$\Omega_j = \Big\{\bar y\in\partial\mathbb{R}^N_+ \;\Big|\; \bar y=(\bar y',\bar y'')\in\mathbb{R}^2\times\mathbb{R}^{N-3},\ \Big\langle\frac{\bar y'}{|\bar y'|},\frac{\bar x_j}{|\bar x_j|}\Big\rangle \ge \cos\frac{\pi}{k}\Big\}.$$

We have

$$R(\bar y) = K\Big(\frac{|\bar y|}{\mu}\Big)\Big(W_{r,\Lambda}^{2^\#-1} - \sum_{j=1}^k U_{x_j,\Lambda}^{2^\#-1}\Big) + \sum_{j=1}^k U_{x_j,\Lambda}^{2^\#-1}\Big(K\Big(\frac{|\bar y|}{\mu}\Big)-1\Big)$$
$$:= J_1 + J_2.$$

From the symmetry, we assume that $\bar y\in\Omega_1$. Then Taylor's theorem gives us

$$|J_1| \le \frac{C}{(1+|\bar y-\bar x_1|)^2}\sum_{j=2}^k\frac{1}{(1+|\bar y-\bar x_j|)^{N-2}} + C\Big(\sum_{j=2}^k\frac{1}{(1+|\bar y-\bar x_j|)^{N-2}}\Big)^{2^\#-1}.$$

Since $|\bar{y}-\bar{x}_j| \ge |\bar{y}-\bar{x}_1|$ and $|\bar{y}-\bar{x}_j| \ge \frac{1}{2}|\bar{x}_j-\bar{x}_1|$ for $\bar{y}\in\Omega_1$, we obtain

$$\begin{aligned}
&\frac{1}{(1+|\bar{y}-\bar{x}_1|)^2}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2}}\\
&\quad\le C\frac{1}{(1+|\bar{y}-\bar{x}_1|)^2}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2-\alpha}}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{\alpha}}\\
&\quad\le C\frac{1}{|\bar{x}_j-\bar{x}_1|^{\alpha}}\frac{1}{(1+|\bar{y}-\bar{x}_1|)^{N-\alpha}}\qquad (0\le\alpha\le N-2).
\end{aligned}$$

Thus, for any $1<\alpha\le N-2$,

$$\frac{1}{(1+|\bar{y}-\bar{x}_1|)^2}\sum_{j=2}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2}}\le\frac{C}{(1+|\bar{y}-\bar{x}_1|)^{N-\alpha}}\Big(\frac{k}{\mu}\Big)^{\alpha}. \tag{2.2}$$

Take $\alpha=\frac{N-2}{2}+\frac{m}{N-2}-\tau\in(1,N-2]$ in (2.2). Then

$$\frac{1}{(1+|\bar{y}-\bar{x}_1|)^2}\sum_{j=2}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2}}\le\frac{C}{(1+|\bar{y}-\bar{x}_1|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.$$

Similarly, for $\bar{y}\in\Omega_1$ and any $1<\alpha\le N-2$, we have

$$\sum_{j=2}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2}}\le\frac{C}{(1+|\bar{y}-\bar{x}_1|)^{N-2-\alpha}}\Big(\frac{k}{\mu}\Big)^{\alpha}.$$

Now we choose $\alpha=\frac{N-2}{N}(\frac{N-2}{2}+\frac{m}{N-2}-\tau)$. It is easy to verify that

$$\alpha-1>\frac{(N-2)^2+4-2(N-2)\tau-2N}{2N}\ge 0,$$

and

$$\alpha<\frac{N-2}{N}\Big(\frac{N-2}{2}+\frac{m}{N-2}\Big)\le\frac{N-2}{N}\cdot\frac{N}{2}<N-2,$$

since $\tau<\frac{1}{2(N-2)}$.

Note also that

$$\frac{Nm\alpha}{(N-2)^2}=\frac{m}{2}+\frac{m^2}{(N-2)^2}-\frac{m\tau}{N-2}\ge\frac{m}{2}+\sigma,$$

since $\tau<\frac{1}{2(N-2)}$. Thus we can directly check that

$$\begin{aligned}
\Big(\sum_{j=2}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_j|)^{N-2}}\Big)^{2^\#-1}&=\frac{C}{(1+|\bar{y}-\bar{x}_1|)^{N-\frac{N\alpha}{N-2}}}\Big(\frac{1}{\mu}\Big)^{\frac{Nm\alpha}{(N-2)^2}}\\
&\le\frac{C}{(1+|\bar{y}-\bar{x}_1|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.
\end{aligned}$$

The same estimates obviously hold for (2.2). Thus, we have proved that

$$\|J_1\|_{**} \le C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.$$

Now, we estimate $J_2$. For $\bar y \in \Omega_1$ and $j > 1$, similarly $|\bar y - \bar x_j| \ge \frac{1}{2}|\bar x_j - \bar x_1|$ indicates that, for $0 \le \alpha \le N$,

$$U^{2^\#-1}_{x_j,\Lambda}(y) \le \frac{C}{(1+|\bar y - \bar x_1|)^{N-\alpha}} \frac{1}{|\bar x_1 - \bar x_j|^{\alpha}},$$

which implies that, for $\alpha = \frac{N-2}{2} + \frac{m}{N-2} - \tau > 1$,

$$\Big|\sum_{j=2}^{k}\Big(K\big(\frac{|\bar y|}{\mu}\big) - 1\Big)U^{2^\#-1}_{x_j,\Lambda}\Big| \le \frac{C}{(1+|\bar y - \bar x_1|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\big(\frac{1}{\mu}\big)^{\frac{m}{2}+\sigma}. \tag{2.3}$$

For $\bar y \in \Omega_1$ and $||\bar y| - \mu r_0| \ge \delta\mu$, where $\delta > 0$ is a fixed constant,

$$\big||\bar y| - |\bar x_1|\big| \ge \big||\bar y| - \mu r_0\big| - \big||\bar x_1| - \mu r_0\big| \ge \frac{1}{2}\delta\mu.$$

As a result, for any $0 \le \alpha \le N$,

$$\begin{aligned}\Big|U^{2^\#-1}_{x_1,\Lambda}\Big(K\big(\frac{|\bar y|}{\mu}\big) - 1\Big)\Big| &\le \frac{C}{(1+|\bar y - \bar x_1|)^{N-\alpha}}\frac{1}{\mu^\alpha}\\ &\le \frac{C}{(1+|\bar y - \bar x_1|)^{N-\frac{m}{2}-\sigma}}\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}\\ &\le \frac{C}{(1+|\bar y - \bar x_1|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.\end{aligned} \tag{2.4}$$

If $\bar y \in \Omega_1$ and $||\bar y| - \mu r_0| \le \delta\mu$, then

$$\begin{aligned}\Big|K\Big(\frac{|\bar y|}{\mu}\Big) - 1\Big| &\le C\Big|\frac{|\bar y|}{\mu} - r_0\Big|^m \le \frac{C}{\mu^m}\Big(\big(\big||\bar y| - |\bar x_1|\big|\big)^m + \big||\bar x_1| - \mu r_0\big|)^m\Big)\\ &\le \frac{C}{\mu^m}\big||\bar y| - |\bar x_1|\big|^m + \frac{C}{\mu^{m+\bar\theta}}.\end{aligned}$$

and

$$\big||\bar y| - |\bar x_1|\big| \le \big||\bar y| - \mu r_0\big| + \big|\mu r_0 - |\bar x_1|\big| \le 2\delta\mu.$$

Consequently it holds that, for any $0 \le \alpha \le m$,

$$\begin{aligned}&\frac{||\bar y| - |\bar x_1||^m}{\mu^m}\frac{1}{(1+|\bar y - \bar x_1|)^N}\\ &= \frac{1}{\mu^\alpha}\frac{1}{(1+|\bar y - \bar x_1|)^{N-\alpha}}\frac{||\bar y| - |\bar x_1||^m}{\mu^{m-\alpha}}\frac{1}{(1+|\bar y - \bar x_1|)^\alpha}\\ &\le \frac{C}{\mu^\alpha}\frac{1}{(1+|\bar y - \bar x_1|)^{N-\alpha}}\frac{||\bar y| - |\bar x_1||^\alpha}{(1+|\bar y - \bar x_1|)^\alpha} \le \frac{C}{\mu^\alpha}\frac{1}{(1+|\bar y - \bar x_1|)^{N-\alpha}}\end{aligned}$$

and

$$\frac{C}{\mu^{m+\bar\theta}}\frac{1}{(1+|\bar y-\bar x_1|)^N}\le\frac{C}{\mu^{\alpha}}\frac{1}{(1+|\bar y-\bar x_1|)^{N-\alpha}}.$$

Thus we obtain, for $||\bar y|-\mu r_0|\le\delta\mu$ and $\alpha=\frac{m}{2}+\sigma$, that

$$\left|U_{x_1,\Lambda}^{2^\#-1}\Big(K\Big(\frac{|\bar y|}{\mu}\Big)-1\Big)\right|\le\frac{C}{(1+|\bar y-\bar x_1|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}. \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we find that

$$\|J_2\|_{**}\le\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.$$

The lemma is proved. ∎

***Lemma 2.2*** *We have* $\|N(\phi)\|_{**}\le C\|\phi\|_*^{2^\#-1}$.

**Proof** Obviously, it holds from Taylor's theorem that

$$|N(\phi)|\le C|\phi|^{2^\#-1}\qquad\text{since } N\ge5>4.$$

Using the inequality

$$\sum_{j=1}^k a_jb_j\le\Big(\sum_{j=1}^k a_j^p\Big)^{\frac1p}\Big(\sum_{j=1}^k b_j^q\Big)^{\frac1q}\qquad\text{for }\frac1p+\frac1q=1,\ a_j,b_j\ge0,$$

we have that

(2.6)

$$\begin{aligned}|N(\phi)|&\le C\|\phi\|_*^{2^\#-1}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac N2-\frac{m}{N-2}+\tau}}\Big)^{2^\#-1}\\&\le C\|\phi\|_*^{2^\#-1}\Big(\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac N2-\frac{m}{N-2}+\tau}}\Big)^{2^\#-1}\\&\le C\|\phi\|_*^{2^\#-1}\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\Big(\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N-2-m}{N-2}+\tau}}\Big)^{\frac{2}{N-2}}\\&\le C\|\phi\|_*^{2^\#-1}\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}},\end{aligned}$$

since without loss of generality we may assume that $\bar y\in\Omega_1$. Then

$$\begin{aligned}\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N-2-m}{N-2}+\tau}}&\le C+\sum_{j=2}^k\frac{1}{|\bar x_1-\bar x_j|^{\frac{N-2-m}{N-2}+\tau}}\\&\le C+\frac{k}{\mu^{\frac{N-2-m}{N-2}+\tau}}\le C.\end{aligned}$$

The lemma is proved. ∎

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## 3 Linearized and Nonlinear Problem

To solve (2.1), we consider the following intermediate nonlinear problem

$$
(3.1)\qquad \begin{cases} -\Delta\phi_k = 0 & \text{in } \mathbb{R}^N_+,, \\ \dfrac{\partial\phi_k}{\partial\nu} - (2^\# - 1)K\big(\dfrac{|y|}{\mu}\big) W_{r,\Lambda}^{2^\#-2}\phi_k \\ \qquad\qquad = R_k + N(\phi_k) + \displaystyle\sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j} & \text{on } \partial\mathbb{R}^N_+, \\ \phi_k \text{ satisfies (1.5) and (1.6)}, \\ \big\langle U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}, \phi_k \big\rangle = 0 \qquad i = 1,\dots,k,\ \ j = 1,2, \end{cases}
$$

for some numbers $c_j$, where $\langle u, v\rangle = \int_{\partial\mathbb{R}^N_+} uv$ and

$$
Z_{i,1} = \frac{\partial U_{x_i,\Lambda}}{\partial r} = U_{x_i,\Lambda} \frac{(N-2)\Lambda^2(\bar{y} - \bar{x}_i)}{(1+\Lambda y_N)^2 + \Lambda^2|\bar{y} - \bar{x}_i|^2} \cdot \frac{\bar{x}_i}{r},
$$
$$
Z_{i,2} = \frac{\partial U_{x_i,\Lambda}}{\partial \Lambda} = U_{x_i,\Lambda} \frac{N-2}{2\Lambda} \cdot \frac{1 - \Lambda^2 y_N^2 - \Lambda^2|\bar{y} - \bar{x}_i|^2}{(1+\Lambda y_N)^2 + \Lambda^2|\bar{y} - \bar{x}_i|^2}.
$$

Let us remark that in general we should also include the translational derivatives of $W_{r,\Lambda}$ in the right-hand side of (3.1). However due to the symmetry assumption on $\phi$, this part of the kernel automatically disappears. This is the main reason for imposing the symmetries (1.5) and (1.6).

Then the following proposition holds.

***Proposition 3.1*** *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $L_0 \leq \Lambda \leq L_1$, $|r - \mu r_0| \leq \frac{1}{\mu^{\bar\theta}}$, where $\bar\theta > 0$ is a fixed small constant, (3.1) has a unique solution $\phi = \phi(r, \Lambda)$, satisfying*

$$
\|\phi\|_* \leq C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}, \qquad |c_j| \leq C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}, \quad j = 1,2.
$$

In order to obtain Proposition 3.1, we first consider the corresponding linearized problem

$$
(3.2)\qquad \begin{cases} -\Delta\phi_k = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial\phi_k}{\partial\nu} - (2^\# - 1)K\big(\frac{|y|}{\mu}\big) W_{r,\Lambda}^{2^\#-2}\phi_k = h + \displaystyle\sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j} & \text{on } \partial\mathbb{R}^N_+, \\ \phi_k \text{ satisfies (1.5) and (1.6)}, \\ \big\langle U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}, \phi_k \big\rangle = 0 \qquad i = 1,\dots,k,\ \ j = 1,2. \end{cases}
$$

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For any fixed $y = (y_1, \dots, y_N) \in \mathbb{R}^N_+$, we denote by $G(x, y)$ the Green's function of the problem

$$\begin{cases} -\Delta G(x, y) = \delta_y & \text{for } x \in \mathbb{R}^N_+, \\ G(x, y) = 0 & \text{for } |x| \to \infty, \\ \dfrac{\partial G}{\partial \nu}(x, y) = 0 & \text{for } x_N = 0. \end{cases}$$

It is not difficult to check that

$$G(x, y) = \frac{1}{\omega_N (N-2)} \Big( \frac{1}{|x - y|^{N-2}} + \frac{1}{|x - y^s|^{N-2}} \Big)$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$ and $y^s$ is the symmetric point of $y$ with respect to $\partial\mathbb{R}^N_+ = \{x : x_N = 0\}$, *i.e.*, $y^s = (\bar{y}, -y_N)$.

***Lemma 3.2*** *Assume that* $\phi_k$ *solves* (3.2) *for* $h = h_k$. *If* $\|h_k\|_{**}$ *goes to zero as* $k$ *goes to infinity, so does* $\|\phi_k\|_*$.

**Proof** We argue by contradiction. Suppose that there are $k \to +\infty$, $h = h_k$, $\Lambda_k \in [L_0, L_1]$, $r_k \in [r_0\mu - \frac{1}{\mu^\theta}, r_0\mu + \frac{1}{\mu^\theta}]$, and $\phi_k$ solving (3.2) for $h = h_k$, $\Lambda = \Lambda_k$, $r = r_k$, with $\|h_k\|_{**} \to 0$, and $\|\phi_k\|_* \geq c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript $k$.

First, we estimate $c_\ell$ ($\ell = 1, 2$). Multiplying (3.2) by $Z_{1,\ell}$ and integrating, we see that $c_j$ satisfies

$$\text{(3.3)} \quad \sum_{j=1}^{2} \sum_{i=1}^{k} \big\langle U^{2^\#-2}_{x_i,\Lambda} Z_{i,j}, Z_{1,\ell} \big\rangle c_j = \int_{\partial\mathbb{R}^N_+} Z_{1,\ell} \frac{\partial \phi}{\partial \nu} - (2^\# - 1) \int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) W^{2^\#-2}_{r,\Lambda} Z_{1,\ell} \phi - \int_{\partial\mathbb{R}^N_+} h Z_{1,\ell}.$$

Using Green's formulas, we have

$$\begin{aligned} &\int_{\partial\mathbb{R}^N_+} Z_{1,\ell} \frac{\partial \phi}{\partial \nu} - (2^\# - 1) \int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) W^{2^\#-2}_{r,\Lambda} Z_{1,\ell} \phi - \int_{\partial\mathbb{R}^N_+} h Z_{1,\ell} \\ &= \int_{\partial\mathbb{R}^N_+} \phi \Big[ \frac{\partial Z_{1,\ell}}{\partial \nu} - (2^\# - 1) K\big(\frac{|y|}{\mu}\big) W^{2^\#-2}_{r,\Lambda} Z_{1,\ell} \Big] - \int_{\partial\mathbb{R}^N_+} h Z_{1,\ell} \\ &:= I_1 + I_2. \end{aligned}$$

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The equation of $Z_{1,\ell}$ indicates that, in $\partial\mathbb{R}^N_+$,

$$\frac{\partial Z_{1,\ell}}{\partial\nu}-(2^\#-1)K\Big(\frac{|y|}{\mu}\Big)W_{r,\Lambda}^{2^\#-2}Z_{1,\ell}=(2^\#-1)Z_{1,\ell}\Big[U_{x_1,\Lambda}^{2^\#-2}-K\Big(\frac{|y|}{\mu}\Big)W_{r,\Lambda}^{2^\#-2}\Big].$$

Note that, because of Lemmas A.3 and A.4,

$$\begin{aligned}
&\Big|\int_{\partial\mathbb{R}^N_+\setminus\Omega_1}\phi Z_{1,\ell}U_{x_1,\Lambda}^{2^\#-2}\Big|\\
&\quad\le C\|\phi\|_*\int_{\partial\mathbb{R}^N_+\setminus\Omega_1}U_{x_1,\Lambda}^{2^\#-1}\sum_{i=1}^k\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\\
&\quad\le C\|\phi\|_*\sum_{i=2}^k\int_{\Omega_i}\frac{1}{(1+|\bar y-\bar x_1|)^N}\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau-1}}\\
&\quad\le C\|\phi\|_*\sum_{i=2}^k\frac{1}{|\bar x_i-\bar x_1|^{\frac{N}{2}-\frac{m}{N-2}}}\int_{\Omega_i}\frac{1}{(1+|\bar y-\bar x_i|)^{N-1+\tau}}=o(1)\|\phi\|_*
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_{\partial\mathbb{R}^N_+\setminus\Omega_1}\phi Z_{1,\ell}W_{r,\Lambda}^{2^\#-2}\Big|\\
&\quad=\Big|\sum_{i=2}^k\int_{\Omega_i}\phi Z_{1,\ell}W_{r,\Lambda}^{2^\#-2}\Big|\\
&\quad\le C\|\phi\|_*\sum_{i=2}^k\int_{\Omega_i}U_{x_1,\Lambda}W_{r,\Lambda}^{2^\#-2}\sum_{j=1}^k\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\\
&\quad\le C\|\phi\|_*\sum_{i=2}^k\frac{1}{|\bar x_i-\bar x_1|^{\frac{N+2}{2}-\frac{N}{N-2}+\frac{2m}{(N-2)^2}}}\int_{\Omega_i}\frac{1}{(1+|\bar y-\bar x_i|)^{N-1+\tau}}=o(1)\|\phi\|_*,
\end{aligned}$$

since $\frac{N+2}{2}-\frac{N}{N-2}+\frac{2m}{(N-2)^2}>1$ for $N\ge 5$. Then we have

$$\begin{aligned}
I_1&=(2^\#-1)\int_{\Omega_1}\phi Z_{1,\ell}U_{x_1,\Lambda}^{2^\#-2}\Big(1-K(\frac{|\bar y|}{\mu})\Big)\,d\bar y\\
&\quad+O\Big\{\int_{\Omega_1}\phi Z_{1,\ell}\Big[U_{x_1,\Lambda}^{2^\#-3}\sum_{i=2}^kU_{x_i,\Lambda}+\Big(\sum_{i=2}^kU_{x_i,\Lambda}\Big)^{2^\#-2}\Big]\,d\bar y\Big\}\\
&\quad+o(1)\|\phi\|_*.
\end{aligned}\tag{3.4}$$

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Direct computations show that

$$
\begin{aligned}
&\Big|\int_{\Omega_1}\phi Z_{1,\ell}U_{x_1,\Lambda}^{2^\#-2}\Big(1-K\Big(\frac{|\bar y|}{\mu}\Big)\Big)\,\mathrm{d}\bar y\Big|\\
&\le C\frac{\|\phi\|_*}{\mu^m}\int_{||\bar y|-\mu r_0|\le\mu^{\frac{m}{N-2}}}||\bar y|-\mu r_0|^mU_{x_1,\Lambda}^{2^\#-1}\sum_{j=1}^k\frac{\mathrm{d}\bar y}{(1+|\bar y-\bar x_j|)^{\frac N2-\frac{m}{N-2}+\tau}}\\
&\quad+C\|\phi\|_*\int_{||\bar y|-\mu r_0|\ge\mu^{\frac{m}{N-2}}}U_{x_1,\Lambda}^{2^\#-1}\sum_{j=1}^k\frac{\mathrm{d}\bar y}{(1+|\bar y-\bar x_j|)^{\frac N2-\frac{m}{N-2}+\tau}}\\
&\le C\frac{\|\phi\|_*}{\mu^m}\int_{||\bar y|-\mu r_0|\le\mu^{\frac{m}{N-2}}}\frac{||\bar y|-\mu r_0|^m\mathrm{d}\bar y}{(1+|\bar y-\bar x_1|)^{N+\frac N2-\frac{m}{N-2}-\frac{N-2-m}{N-2}+\tau}}\\
&\quad+O(\mu^{-\frac{m}{N-2}(\frac{N+2}{2}-\frac{m}{N-2})})\|\phi\|_*\\
&=o(1)\|\phi\|_*.
\end{aligned}
$$

Similar estimates can be obtained for the second term of (3.4). Thus we get that

$$|I_1|=o(1)\|\phi\|_*.$$

In addition, it holds that, using the estimates in the proof of Lemma A.3,

$$
\begin{aligned}
|I_2|&\le C\|h\|_{**}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|\bar y-\bar x_1|)^{N-2}}\sum_{i=1}^k\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\\
&\le C\|h\|_{**}\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|\bar y-\bar x_1|)^{N-2+\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\\
&\quad+C\|h\|_{**}\sum_{i=2}^k\int_{\mathbb{R}^{N-1}}\frac{1}{(1+|\bar y-\bar x_1|)^{N-2}}\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\\
&\le C\|h\|_{**}.
\end{aligned}
$$

On the other hand, for any $i\ne 1$, it is easy to check that

$$\Big|\Big\langle U_{x_i,\Lambda}^{2^\#-2}Z_{i,j},Z_{1,\ell}\Big\rangle\Big|\le C\int_{\partial\mathbb{R}^N_+}U_{x_i,\Lambda}^{2^\#-1}U_{x_1,\Lambda}.$$

By Lemma A.3, we may have that

(3.5)

$$
\begin{aligned}
\int_{\partial\mathbb{R}^N_+}U_{x_i,\Lambda}^{2^\#-1}U_{x_1,\Lambda}\mathrm{d}\bar y&\le\int_{\partial\mathbb{R}^N_+}\frac{1}{(1+|\bar y-\bar x_i|)^N}\frac{1}{(1+|\bar y-\bar x_1|)^{N-2}}\\
&\le\frac{C}{|\bar x_i-\bar x_1|^{N-2}}\int_{\partial\mathbb{R}^N_+}\Big[\frac{1}{(1+|\bar y-\bar x_i|)^N}+\frac{1}{(1+|\bar y-\bar x_1|)^N}\Big]\\
&\le\frac{C}{|\bar x_i-\bar x_1|^{N-2}}.
\end{aligned}
$$

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In addition, it is easy to get from the symmetry that, for $j \neq \ell$,

$$\big\langle U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}, Z_{i,\ell} \big\rangle = 0. \tag{3.6}$$

Now we find that the coefficient matrix of the system (3.3) with respect to $(c_1, c_2)$ is nondegenerate. Therefore,

$$|c_\ell| \leq o(1)\|\phi\|_* + C\|h\|_{**} = o(1).$$

We claim that

$$\|\phi\|_{L^\infty(\overline{\mathbb{R}^N_+} \cap |y-x_i| \leq R)} = o(1) \qquad \text{for any } i = 1, \dots, k. \tag{3.7}$$

Indeed, by elliptic regularity we can get a $\widehat{\phi}$ such that $\phi(y - x_i) \to \widehat{\phi}$ in $C^2_{\mathrm{loc}}(\overline{\mathbb{R}^N_+})$ and

$$\begin{cases} -\Delta \widehat{\phi} = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial \widehat{\phi}}{\partial \nu} - (2^\# - 1) U_{0,\Lambda}^{2^\#-2} \phi = 0 & \text{on } \partial \mathbb{R}^N_+, \\ \langle U_{0,\Lambda}^{2^\#-2} Z_{0,j}, \widehat{\phi} \rangle = 0 & j = 1, 2, \end{cases}$$

The nondegeneracy of $U_{0,\Lambda}$ [9, Lemma 2.1] implies that $\widehat{\phi} = 0$, which concludes the claim (3.7).

We next rewrite (3.2) as

$$\begin{aligned} \phi(y) = {} & \int_{\partial \mathbb{R}^N_+} G(y,z) \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^\#-2}(z) Z_{i,j}(z) \mathrm{d}\bar{z} \\ & + \int_{\partial \mathbb{R}^N_+} G(y,z) \Big[ (2^\# - 1) K\Big( \frac{|z|}{\mu} \Big) W_{r,\Lambda}^{2^\#-2}(z) \phi(z) + h(z) \Big] \mathrm{d}\bar{z}, \end{aligned}$$

where $z = (\bar{z}, 0) \in \mathbb{R}^{N-1} \times \{0\}$.

Direct computations show that

$$\begin{aligned} & \Big| \int_{\partial \mathbb{R}^N_+} G(y,z) \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^\#-2}(z) Z_{i,j}(z) \mathrm{d}\bar{z} \Big| \\ & \leq C \sum_{i=1}^{k} \int_{\partial \mathbb{R}^N_+} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_i|)^N} \mathrm{d}\bar{z} \\ & \leq C \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N}{2}}} \leq C \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N}{2} - \frac{m}{N-2} + \tau}}, \end{aligned}$$

where we have used Lemma A.4.

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Also

$$\Big|\int_{\partial\mathbb{R}^N_+} G(y,z)h(z)\mathrm{d}\bar{z}\Big| \le C\|h\|_{**}\int_{\partial\mathbb{R}^N_+}\frac{1}{|y-z|^{N-2}}\sum_{i=1}^k\frac{1}{(1+|z-x_i|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}}\mathrm{d}\bar{z}$$
$$\le C\|h\|_{**}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}},$$

since $\frac{N}{2}-\frac{m}{N-2}+\tau < N-2$ and

$$\Big|\int_{\partial\mathbb{R}^N_+} G(y,z)K(\frac{|z|}{\mu})W_{r,\Lambda}^{2^\#-2}(z)\phi(z)\mathrm{d}\bar{z}\Big|$$
$$\le o(1)\sum_{i=1}^k\int_{B_R(\bar{x}_i)}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{2^\#-2}(z)\mathrm{d}\bar{z}$$
$$+\|\phi\|_*\int_{\partial\mathbb{R}^N_+\setminus\bigcup_{i=1}^k B_R(\bar{x}_i)}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{2^\#-2}(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\mathrm{d}\bar{z}$$
$$\le o(1)\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}$$
$$+\frac{C}{R^{\frac{(m-2)N+4}{(N-2)^2}}}\|\phi\|_*\sum_{i=1}^k\int_{\Omega_i}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+1+\tau}}\mathrm{d}\bar{z}$$
$$\le\Big(o(1)+\frac{C}{R^{\frac{(m-2)N+4}{(N-2)^2}}}\|\phi\|_*\Big)\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}.$$

Now, choosing $R$ large, we obtain that

$$\|\phi\|_*\le C\|h\|_{**}+o(1)=o(1),$$

a contradiction. ■

From Lemma 3.2, using the same argument as in the proof of [10, Proposition 4.1], we can prove the following result.

***Proposition 3.3*** *There exists $k_0>0$ and a constant $C>0$, independent of $k$, such that for all $k\ge k_0$ and all $h\in L^\infty(\mathbb{R}^{N-1})$, problem (3.2) has a unique solution $\phi := L_k(h)$. Besides,*

$$\|L_k(h)\|_*\le C\|h\|_{**},\qquad |c_\ell|\le C\|h\|_{**}.$$

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We are now ready to prove Proposition 3.1.

**Proof of Proposition 3.1** Let us recall that $\mu = k^{\frac{N-2}{N-2-m}}$ and denote

$$E = \Big\{ \phi \in C(\overline{\mathbb{R}^N_+}) : \phi \text{ satisfies (1.5) and (1.6)}, \|\phi\|_* \leq \eta \Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}, \\ \int_{\partial\mathbb{R}^N_+} U^{2^\#-2}_{x_i,\Lambda} Z_{i,j}\phi = 0 \text{ for any } i = 1,\dots,k,\ j = 1,2 \Big\},$$

where $\eta > 0$ is a fixed large constant. Then (2.1) is equivalent to

$$\phi = A(\phi) := L(N(\phi)) + L(R).$$

We will first prove that $A$ is a contraction mapping from $E$ to $E$.

In fact, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
\|\phi\|_* &\leq C\|R\|_{**} + C\|N(\phi)\|_{**} \\
&\leq C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma} + C\eta^{\frac{N}{N-2}}\Big(\frac{1}{\mu}\Big)^{(\frac{m}{2}+\sigma)\frac{N}{N-2}} \\
&\leq C\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}\Big(1 + \eta^{\frac{N}{N-2}}\Big(\frac{1}{\mu}\Big)^{(\frac{m}{2}+\sigma)\frac{2}{N-2}}\Big) \\
&\leq \eta\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.
\end{aligned}$$

Thus $A$ maps $E$ to $E$ itself.

On the other hand, it obviously holds that

$$\|A(\phi_1) - A(\phi_2)\|_* = \|L(N(\phi_1)) - L(N(\phi_2))\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**}.$$

Since $2^\# - 2 < 1$, we have that $|N'(t)| \leq C|t|^{2^\#-2}$. Thus for any $y \in \partial\mathbb{R}^N_+$,

$$\begin{aligned}
C|N(\phi_1) - N(\phi_2)| &\leq C(|\phi_1|^{2^\#-2} + |\phi_2|^{2^\#-2})|\phi_1 - \phi_2| \\
&\leq C(\|\phi_1\|_*^{2^\#-2} + \|\phi_2\|_*^{2^\#-2})\|\phi_1 - \phi_2\|_*\Big(\sum_{j=1}^k \frac{1}{(1+|\bar y - \bar x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\Big)^{2^\#-1} \\
&\leq C\eta^{\frac{2}{N-2}}\Big(\frac{1}{\mu}\Big)^{(\frac{m}{2}+\sigma)\frac{2}{N-2}}\|\phi_1 - \phi_2\|_*\sum_{j=1}^k \frac{1}{(1+|\bar y - \bar x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}} \\
&\leq \frac{1}{2}\|\phi_1 - \phi_2\|_*\sum_{j=1}^k \frac{1}{(1+|\bar y - \bar x_j|)^{\frac{N+2}{2}-\frac{m}{N-2}+\tau}},
\end{aligned}$$

where relation (2.6) has been used. Thus $A$ is a contraction mapping.

It follows from the contraction mapping theorem that there is a unique $\phi \in E$ such that $\phi = A(\phi)$. ■

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## 4 Variational Reduction

After problem (3.1) has been solved, we find a solution to problem (2.1) and hence to the original problem (1.4) if $(r, \Lambda)$ is such that $c_j(r, \Lambda) = 0,\ j = 1, 2$. This problem is in fact variational.

Let $F(r, \Lambda) = I(W_{r,\Lambda} + \phi)$, where $\phi$ is the function obtained in Proposition 3.1 and

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |Du|^2 - \frac{1}{2^\#}\int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big)\,|u|^{2^\#}.$$

***Proposition 4.1*** *Assume that $(r, \Lambda)$ is a critical point of $F(r, \Lambda)$. Then $c_j = 0$ for any $j = 1, 2$.*

**Proof** By (3.5) and (3.6), we first get that

$$\begin{aligned}\sum_{j=1}^{2} c_j \sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} &= \sum_{j=1}^{2} c_j \sum_{i=1}^{k}\sum_{\ell=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j} Z_{\ell,2}\\
&= \sum_{j=1}^{2} c_j \sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j} Z_{i,2} + O\Big(\sum_{j=1}^{2} c_j \sum_{i=1}^{k}\sum_{\ell\neq i}\frac{1}{|\bar{x}_i-\bar{x}_\ell|^{N-2}}\Big)\\
&= c_2 \sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,2}^2 + O\Big(k\mu^{-m}\sum_{j=1}^{2} c_j\Big)\\
&= kc_2\Big[\int_{\partial\mathbb{R}^N_+} U_{x_1,\Lambda}^{2^\#-2} Z_{1,2}^2 + O\Big(\frac{1}{\mu}\Big)^m\Big] + kO(\frac{1}{\mu})^m c_1,\end{aligned}$$

and similarly

$$\sum_{j=1}^{2} c_j \sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}\frac{\partial W_{r,\Lambda}}{\partial r} = kc_1\Big[\int_{\partial\mathbb{R}^N_+} U_{x_1,\Lambda}^{2^\#-2} Z_{1,1}^2 + O\Big(\frac{1}{\mu}\Big)^m\Big] + kO\Big(\frac{1}{\mu}\Big)^m c_2.$$

In addition, since

$$\begin{aligned}\Big|\frac{\partial Z_{i,1}}{\partial\Lambda}\Big| &\le CU_{x_i,\Lambda}\frac{\Lambda^2|\bar{y}-\bar{x}_i|}{(1+\Lambda y_N)^2+\Lambda^2|\bar{y}-\bar{x}_i|^2} \le CU_{x_i,\Lambda},\\
\Big|\frac{\partial Z_{i,2}}{\partial\Lambda}\Big| &\le CU_{x_i,\Lambda},\end{aligned}$$

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it holds that

$$(4.1)\qquad \Big|\sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_i,\Lambda}^{2^\#-2} Z_{i,j}\frac{\partial\phi}{\partial\Lambda}\Big|$$

$$\begin{aligned}
&= \Big|-\sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+}\frac{\partial(U_{x_i,\Lambda}^{2^\#-2}Z_{i,j})}{\partial\Lambda}\phi\Big| \le C\sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+}U_{x_i,\Lambda}^{2^\#-1}|\phi|\\
&\le C\|\phi\|_*\int_{\partial\mathbb{R}^N_+}\sum_{i=1}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_i|)^N}\sum_{\ell=1}^{k}\frac{1}{(1+|\bar{y}-\bar{x}_\ell|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\mathrm{d}\bar{y}\\
&\le C\|\phi\|_*\sum_{i=1}^{k}\int_{\Omega_i}\frac{\mathrm{d}\bar{y}}{(1+|\bar{y}-\bar{x}_i|)^{N-1+\frac{N}{2}-\frac{m}{N-2}-\frac{2(N-2-m)}{N-2}+\tau}}\\
&\le Ck\Big(\frac{1}{\mu}\Big)^{\frac{m}{2}+\sigma}.
\end{aligned}$$

The same estimate holds for $\sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+}U_{x_i,\Lambda}^{2^\#-2}Z_{i,j}\frac{\partial\phi}{\partial r}$.

Finally we note that

$$0=\frac{\partial F}{\partial r}=\int_{\partial\mathbb{R}^N_+}\sum_{j=1}^{2}c_j\sum_{i=1}^{k}U_{x_i,\Lambda}^{2^\#-2}Z_{i,j}\left(\frac{\partial W_{r,\Lambda}}{\partial r}+\frac{\partial\phi}{\partial r}\right),$$

$$0=\frac{\partial F}{\partial\Lambda}=\int_{\partial\mathbb{R}^N_+}\sum_{j=1}^{2}c_j\sum_{i=1}^{k}U_{x_i,\Lambda}^{2^\#-2}Z_{i,j}\left(\frac{\partial W_{r,\Lambda}}{\partial\Lambda}+\frac{\partial\phi}{\partial\Lambda}\right).$$

Therefore, it is easy for us to get that $c_j=0$ ($j=1,2$) from the nondegeneracy of their coefficient matrix. ∎

***Proposition 4.2*** *We have*

$$\begin{aligned}
F(r,\Lambda)&=I(W_{r,\Lambda})+O\Big(\frac{k}{\mu^{m+2\sigma}}\Big)\\
&=k\Big(A+\frac{B_1}{\Lambda^m\mu^m}+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-|x_1|)^2\\
&\quad-\sum_{i=2}^{k}\frac{B_3}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-|x_1||^{2+\tilde{\sigma}}\Big)\Big),
\end{aligned}$$

*where $B_i>0$, $i=1,2,3$, are some constants and $\tilde{\sigma}>0$ is a small constant.*

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**Proof** Since $DI(W_{r,\Lambda})\phi = 0$, there is $t \in (0,1)$ such that

$$\begin{aligned} F(r,\Lambda) &= I(W_{r,\Lambda}) + \frac{1}{2} D^2 I\big(W_{r,\Lambda} + t\phi\big)(\phi,\phi) \\ &= I(W_{r,\Lambda}) + \frac{1}{2}\int_{\mathbb{R}^N_+} |D\phi|^2 - \frac{2^\# - 1}{2}\int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big)\big(W_{r,\Lambda} + t\phi\big)^{2^\#-2}\phi^2 \\ &= I(W_{r,\Lambda}) - \frac{2^\# - 1}{2}\int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big)\Big[\big(W_r + t\phi\big)^{2^\#-2} - W_r^{2^\#-2}\Big]\phi^2 \\ &\quad + \int_{\partial\mathbb{R}^N_+}\big(N(\phi) - R\big)\phi \\ &= I(W_{r,\Lambda}) + O\Big(\int_{\partial\mathbb{R}^N_+}\big(|\phi|^{2^\#} + |N(\phi)||\phi| + |R||\phi|\big)\Big). \end{aligned}$$

Moreover, it is easy to check that

$$\begin{aligned} &\int_{\partial\mathbb{R}^N_+} |N(\phi)||\phi| \\ &\le C\|N(\phi)\|_{**}\|\phi\|_* \int_{\partial\mathbb{R}^N_+} \sum_{i,j=1}^k \frac{1}{(1+|\bar y - \bar x_j|)^{\frac{N+2}{2} - \frac{m}{N-2} + \tau}} \frac{1}{(1+|\bar y - \bar x_i|)^{\frac{N}{2} - \frac{m}{N-2} + \tau}} d\bar y \\ &\le C\Big(\frac{1}{\mu}\Big)^{m+2\sigma} \sum_{i=1}^k \int_{\partial\mathbb{R}^N_+} \frac{d\bar y}{(1+|\bar y - \bar x_i|)^{N+1-\frac{2m}{N-2}+2\tau-\frac{N-2-m}{N-2}}} \le Ck\Big(\frac{1}{\mu}\Big)^{m+2\sigma}, \end{aligned}$$

as well as $\int_{\partial\mathbb{R}^N_+} |R||\phi| \le Ck\big(\frac{1}{\mu}\big)^{m+2\sigma}$. Similarly, we have

$$\begin{aligned} \int_{\partial\mathbb{R}^N_+} |\phi|^{2^\#} &\le \|\phi\|_*^{2^\#} \int_{\partial\mathbb{R}^N_+} \Big(\sum_{i=1}^k \frac{1}{(1+|\bar y - \bar x_i|)^{\frac{N}{2} - \frac{m}{N-2} + \tau}}\Big)^{2^\#} d\bar y \\ &\le C\|\phi\|_*^{2^\#} \sum_{\ell=1}^k \int_{\Omega_\ell} \frac{d\bar y}{(1+|\bar y - \bar x_\ell|)^{(\frac{N}{2} - \frac{m}{N-2} - \frac{N-2-m}{N-2} + \tau)\frac{2(N-1)}{N-2}}} \\ &\le C\|\phi\|_*^{2^\#} \sum_{\ell=1}^k \int_{\Omega_\ell} \frac{d\bar y}{(1+|\bar y - \bar x_\ell|)^{N-1+\tau}} \\ &\le Ck\|\phi\|_*^{2^\#} \le Ck\Big(\frac{1}{\mu}\Big)^{\frac{m(N-1)}{N-2} + 2^\#\sigma}. \end{aligned} \tag{4.2}$$

From Proposition A.1, we conclude the proof. ■

***Proposition 4.3*** *We have*

$$\frac{\partial F(r,\Lambda)}{\partial\Lambda} = k\Big(-\frac{B_1 m}{\Lambda^{m+1}\mu^m} + \sum_{i=2}^k \frac{B_3(N-2)}{\Lambda^{N-1}|x_1 - x_j|^{N-2}} + O\Big(\frac{1}{\mu^{m+\sigma}} + \frac{1}{\mu^m}\big|\mu r_0 - |x_1|\big|^2\Big)\Big).$$

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**Proof** First we note from (4.1) and Proposition 3.1 that

$$
\begin{aligned}
(4.3)\quad \frac{\partial F(r,\Lambda)}{\partial \Lambda} &= DI(W_{r,\Lambda}+\phi)\Big(\frac{\partial W_{r,\Lambda}}{\partial \Lambda}+\frac{\partial \phi}{\partial \Lambda}\Big)\\
&= DI(W_{r,\Lambda}+\phi)\Big(\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\Big)+\sum_{j=1}^{2}\sum_{i=1}^{k} c_j\Big\langle U_{x_i,\Lambda}^{2^\#-2}Z_{i,j},\frac{\partial \phi}{\partial \Lambda}\Big\rangle\\
&= DI(W_{r,\Lambda}+\phi)\Big(\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\Big)+O(k\mu^{-m-\sigma})\\
&= \frac{\partial}{\partial \Lambda}I(W_{r,\Lambda})-\int_{\partial\mathbb{R}^N_+}K\Big(\frac{|y|}{\mu}\Big)\Big[(W_{r,\Lambda}+\phi)^{2^\#-1}-W_{r,\Lambda}^{2^\#-1}\Big]\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\\
&\quad +O(k\mu^{-m-\sigma}),
\end{aligned}
$$

because the orthogonality of $\phi$ implies

$$
\int_{\mathbb{R}^N_+}\nabla\phi\nabla\frac{\partial W_{r,\Lambda}}{\partial \Lambda}=-\int_{\mathbb{R}^N_+}\phi\Delta\frac{\partial W_{r,\Lambda}}{\partial \Lambda}+\int_{\partial\mathbb{R}^N_+}\phi\frac{\partial}{\partial \nu}\Big(\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\Big)=0.
$$

Next we will deal with the second term in the right side of (4.3). It holds that

$$
\begin{aligned}
&\int_{\partial\mathbb{R}^N_+}K\Big(\frac{|y|}{\mu}\Big)\Big[(W_{r,\Lambda}+\phi)^{2^\#-1}-W_{r,\Lambda}^{2^\#-1}\Big]\frac{\partial W_{r,\Lambda}}{\partial \Lambda}=\\
&\qquad (2^\#-1)\int_{\partial\mathbb{R}^N_+}K(\frac{|y|}{\mu})W_{r,\Lambda}^{2^\#-2}\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\phi+O\Big(\int_{\partial\mathbb{R}^N_+}W_{r,\Lambda}^{2^\#-2}|\phi|^2+|\phi|^{2^\#}\Big).
\end{aligned}
$$

For $\alpha=\frac{N-2-m}{N-2}$, we know that in $\Omega_i$,

$$
\sum_{j\neq i}\frac{1}{(1+|\bar y-\bar x_j|)^{N-2}}\le\frac{1}{(1+|\bar y-\bar x_i|)^{N-2-\alpha}}\sum_{j\neq i}\frac{1}{|\bar x_j-\bar x_i|^{\alpha}},
$$

which leads to

$$
W_{r,\Lambda}^{2^\#-2}\le\frac{C}{(1+|\bar y-\bar x_i|)^{2-\frac{2\alpha}{N-2}}},
$$

$$
\sum_{j=1}^{k}\frac{1}{(1+|\bar y-\bar x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\le\frac{C}{(1+|\bar y-\bar x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau-\alpha}}.
$$

As a result, we find that

$$
\begin{aligned}
\int_{\partial\mathbb{R}^N_+}W_{r,\Lambda}^{2^\#-2}|\phi|^2 &\le C\|\phi\|_*^2\sum_{i=1}^{k}\int_{\Omega_i}\frac{d\bar y}{(1+|\bar y-\bar x_i|)^{2-\frac{2\alpha}{N-2}+N-\frac{2m}{N-2}+2\tau-2\alpha}}\\
&\le C\|\phi\|_*^2\sum_{i=1}^{k}\int_{\Omega_i}\frac{d\bar y}{(1+|\bar y-\bar x_i|)^{N-1+\frac{N^2-6N+2m+8}{(N-2)^2}+2\tau}}\le Ck\mu^{-m-2\sigma}.
\end{aligned}
$$

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A similar estimate also holds for $\int_{\partial\mathbb{R}^N_+} |\phi|^{2^\#}$, which is given by (4.2). Furthermore, from the orthogonality of $\phi$, we have that

$$
\begin{aligned}
&\int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) W_{r,\Lambda}^{2^\#-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi \\
&\quad= \int_{\partial\mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big)\Big(W_{r,\Lambda}^{2^\#-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{i=1}^k U_{x_i,\Lambda}^{2^\#-2}\frac{\partial U_{x_i,\Lambda}}{\partial\Lambda}\Big)\phi \\
&\qquad + \sum_{i=1}^k \int_{\partial\mathbb{R}^N_+}\Big[K\Big(\frac{|y|}{\mu}\Big)-1\Big]U_{x_i,\Lambda}^{2^\#-2}\frac{\partial U_{x_i,\Lambda}}{\partial\Lambda}\phi \\
&\quad= k\int_{\Omega_1} K\Big(\frac{|\bar y|}{\mu}\Big)\Big(W_{r,\Lambda}^{2^\#-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{i=1}^k U_{x_i,\Lambda}^{2^\#-2}\frac{\partial U_{x_i,\Lambda}}{\partial\Lambda}\Big)\phi \\
&\qquad + k\int_{\partial\mathbb{R}^N_+}\Big[K\Big(\frac{|y|}{\mu}\Big)-1\Big]U_{x_1,\Lambda}^{2^\#-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi.
\end{aligned}
$$

Thus we can check that

$$
\begin{aligned}
&\Big|\int_{\Omega_1} K\Big(\frac{|\bar y|}{\mu}\Big)\Big(W_{r,\Lambda}^{2^\#-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda} - \sum_{i=1}^k U_{x_i,\Lambda}^{2^\#-2}\frac{\partial U_{x_i,\Lambda}}{\partial\Lambda}\Big)\phi\Big| \\
&\quad\le C\int_{\Omega_1}\Big(U_{x_1,\Lambda}^{2^\#-2}\sum_{i=2}^k U_{x_i,\Lambda} + \sum_{i=2}^k U_{x_i,\Lambda}^{2^\#-1}\Big)|\phi| \\
&\quad\le C\Big(\frac{1}{\mu}\Big)^{\frac{Nm}{2(N-2)}+\frac{m}{2}+\sigma} \le C\Big(\frac{1}{\mu}\Big)^{m+\sigma},
\end{aligned}
$$

and, using Lemma A.3,

$$
\begin{aligned}
&\Big|\int_{\partial\mathbb{R}^N_+}\Big[K\Big(\frac{|y|}{\mu}\Big)-1\Big]U_{x_1,\Lambda}^{2^\#-2}\frac{\partial U_{x_1,\Lambda}}{\partial\Lambda}\phi\Big| \\
&\le C\|\phi\|_*\frac{1}{\mu^m}\int_{||\bar y|-\mu r_0|\le\mu^{\frac{m}{N-2}}}\frac{||\bar y|-\mu r_0|^m}{(1+|\bar y-\bar x_1|)^N}\sum_{i=1}^k\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}} \\
&\quad + C\|\phi\|_*\int_{||\bar y|-\mu r_0|\ge\mu^{\frac{m}{N-2}}}\frac{1}{(1+|\bar y-\bar x_1|)^N}\sum_{i=1}^k\frac{1}{(1+|\bar y-\bar x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}} \\
&\le \frac{C}{\mu^{m+\sigma}}.
\end{aligned}
$$

Thus we complete the proof using Proposition A.2. ■

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## 5 Proof of Theorem 1.5

Since

$$|x_j - x_1| = 2|x_1| \sin\frac{(j-1)\pi}{k}, \quad j = 2, \dots, k,$$

we have

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{1}{(2|x_1|)^{N-2}} \sum_{j=2}^{k} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}$$

$$= \begin{cases} \frac{2}{(2|x_1|)^{N-2}} \sum_{j=2}^{\frac{k}{2}} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}} + \frac{1}{(2|x_1|)^{N-2}} & \text{if } k \text{ is even,} \\ \frac{2}{(2|x_1|)^{N-2}} \sum_{j=2}^{[\frac{k}{2}]} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}} & \text{if } k \text{ is odd.} \end{cases}$$

But

$$0 < c' \le \frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \le c'', \quad j = 2, \dots, \left[\frac{k}{2}\right].$$

So, there is a constant $B_4 > 0$, such that

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{B_4 k^{N-2}}{|x_1|^{N-2}} + O\Big(\frac{k}{|x_1|^{N-2}}\Big).$$

Thus, we obtain

$$F(r,\Lambda) = k\Big(A + \frac{B_1}{\Lambda^m \mu^m} + \frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0 - r)^2 - \frac{B_3 B_4 k^{N-2}}{\Lambda^{N-2} r^{N-2}} + O\Big(\frac{1}{\mu^{m+\sigma}} + \frac{1}{\mu^m}|\mu r_0 - r|^3 + \frac{k}{r^{N-2}}\Big)\Big),$$

and

$$\frac{\partial F(r,\Lambda)}{\partial \Lambda} = k\Big(-\frac{B_1 m}{\Lambda^{m+1}\mu^m} + \frac{B_3 B_4 (N-2) k^{N-2}}{\Lambda^{N-1} r^{N-2}} + O\Big(\frac{1}{\mu^{m+\sigma}} + \frac{1}{\mu^m}|\mu r_0 - r|^2 + \frac{k}{r^{N-2}}\Big)\Big).$$

Let $\Lambda_0$ be the solution of

$$-\frac{B_1 m}{\Lambda^{m+1}} + \frac{B_3 B_4 (N-2)}{\Lambda^{N-1} r_0^{N-2}} = 0,$$

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that is

$$\Lambda_0 = \Big( \frac{B_3 B_4 (N-2)}{B_1 m r_0^{N-2}} \Big)^{\frac{1}{N-2-m}}.$$

Define

$$D = \Big\{ (r, \Lambda) : r \in \Big[ \mu r_0 - \frac{1}{\mu^{\bar\theta}}, \mu r_0 + \frac{1}{\mu^{\bar\theta}} \Big], \Lambda \in \Big[ \Lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar\theta}}, \Lambda_0 + \frac{1}{\mu^{\frac{3}{2}\bar\theta}} \Big] \Big\},$$

where $\bar\theta > 0$ is a small constant.

The existence of a critical point of $F(r, \Lambda)$ in D can be proved just as in [21, Prop. 3.3, Prop. 3.4]. We omit the details.

It remains to prove that the solution we found for $v_\mu = W_{r,\Lambda} + \phi$ is positive. Testing the equation for $v_\mu$ (1.4) against $v_\mu^- = \min\{v_\mu, 0\}$ itself, it holds that

$$\int_{\mathbb{R}^N_+} |\nabla v_\mu^-|^2 = \int_{\partial \mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) (v_\mu^-)^{2^\#}.$$

Moreover the trace theorem tells us that

$$\int_{\partial \mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) (v_\mu^-)^{2^\#} \le C \Big( \int_{\mathbb{R}^N_+} |\nabla v_\mu^-|^2 \Big)^{\frac{2^\#}{2}}.$$

Combining the above two inequalities, we get that

$$\int_{\partial \mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) (v_\mu^-)^{2^\#} \ge C \qquad \text{or} \qquad v_\mu^- \equiv 0 \quad \text{on } \partial \mathbb{R}^N_+. \tag{5.1}$$

On the other hand, we know that $|v_\mu^-| \le |\phi|$, since $W_{r,\Lambda} > 0$. Thus, by (4.2) it holds that

$$\int_{\partial \mathbb{R}^N_+} K\Big(\frac{|y|}{\mu}\Big) (v_\mu^-)^{2^\#} \le C \int_{\partial \mathbb{R}^N_+} |\phi|^{2^\#} \le C \Big(\frac{1}{\mu}\Big)^{2^\#(\frac{m}{2}+\sigma)} = o(1).$$

On account of (5.1) again it must hold that $v_\mu^- \equiv 0$ on $\partial \mathbb{R}^N_+$, which implies that $v_\mu \ge 0$ on $\partial \mathbb{R}^N_+$. Therefore $v_\mu$ must be positive, because it is harmonic in $\mathbb{R}^N_+$.

## A Appendix

In this appendix, we assume that

$$x_j = \Big( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \Big) \qquad j = 1, \dots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and $r \in \big[ r_0 \mu - \frac{1}{\mu^{\bar\theta}}, r_0 \mu + \frac{1}{\mu^{\bar\theta}} \big]$ for some small $\bar\theta > 0$.

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## A.1 Energy Expansion of the Approximate Solution

In this section, we will calculate $I(W_{r,\Lambda})$.

Let us recall that

$$\mu = k^{\frac{N-2}{N-2-m}},$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |Du|^2 - \frac{1}{2^\#}\int_{\partial\mathbb{R}^N_+} K\big(\frac{|y|}{\mu}\big)\,|u|^{2^\#},$$

$$U_{x_j,\Lambda}(y) = \big(N-2\big)^{\frac{N-2}{2}}\Big[\frac{\Lambda}{(1+\Lambda y_N)^2+\Lambda^2|\bar y-\bar x_j|^2}\Big]^{\frac{N-2}{2}},$$

$$W_{r,\Lambda}(y) = (N-2)^{\frac{N-2}{2}}\sum_{j=1}^{k}\Big[\frac{\Lambda}{(1+\Lambda y_N)^2+\Lambda^2|\bar y-\bar x_j|^2}\Big]^{\frac{N-2}{2}}.$$

***Proposition A.1*** *We have*

$$I(W_{r,\Lambda}) = k\Big[A+\frac{B_1}{\Lambda^m\mu^m}+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-r)^2 - \sum_{i=2}^{k}\frac{B_3}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^{2+\tilde\sigma}\Big)\Big],$$

*where $A$, $B_i$ $(i=1,2,3)$ are some positive constants depending only on $N$, $r=|x_1|$ and $\tilde\sigma>0$ is a small constant.*

**Proof** First let us calculate $\int_{\mathbb{R}^N}|DW_{r,\Lambda}|^2$. It is easy to get that, for $j=1,\dots,k$,

$$A_N := \int_{\partial\mathbb{R}^N_+} U^{2^\#}_{x_j,\Lambda} = (N-2)^{N-1}\int_{\mathbb{R}^{N-1}}\frac{\mathrm{d}z}{(1+|z|^2)^{N-1}}. \tag{A.1}$$

By using the symmetry, we claim that

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\partial\mathbb{R}^N_+} U^{2^\#-1}_{x_i,\Lambda}U_{x_j,\Lambda} &= k\sum_{j=2}^{k}\int_{\mathbb{R}^{N-1}} U^{2^\#-1}_{x_1,\Lambda}U_{x_j,\Lambda}\\ &= k\Big[\sum_{j=2}^{k}\frac{C_{3N}}{\Lambda^{N-2}|\bar x_1-\bar x_j|^{N-2}}+O\Big(\sum_{j=2}^{k}\frac{\ln\Lambda|\bar x_i-\bar x_1|}{\Lambda^{N-1}|\bar x_1-\bar x_j|^{N-1}}\Big)\Big],\end{aligned} \tag{A.2}$$

where

$$C_{3N} = (N-2)^{N-1}\int_{\mathbb{R}^{N-1}}\frac{\mathrm{d}z}{(1+|z|^2)^{\frac{N}{2}}}.$$

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In fact, denote that $d_j = |\bar{x}_1 - \bar{x}_j|$, then Taylor's expansion tells us that, in $B_{\frac{d_j}{2}}(\bar{x}_1) \subset \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$ and for large $d_j$,

$$\text{(A.3)}\quad \Big(\frac{1}{1+\Lambda^2|\bar{y}-\bar{x}_j|^2}\Big)^{\frac{N-2}{2}} = \Big(\frac{1}{1+\Lambda^2|\bar{x}_1-\bar{x}_j|^2}\Big)^{\frac{N-2}{2}} + O\Big(\frac{|\bar{y}-\bar{x}_1|}{\Lambda^{N-2}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big) = \frac{1}{\Lambda^{N-2}|\bar{x}_1-\bar{x}_j|^{N-2}} + O\Big(\frac{|\bar{y}-\bar{x}_1|}{\Lambda^{N-2}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big) + O\Big(\frac{1}{\Lambda^{N}|\bar{x}_1-\bar{x}_j|^{N}}\Big).$$

Thus,

$$\int_{B_{\frac{d_j}{2}}(\bar{x}_1)} U_{x_1,\Lambda}^{2^\#-1}U_{x_j,\Lambda} = \frac{(N-2)^{N-1}}{\Lambda^{N-2}|\bar{x}_1-\bar{x}_j|^{N-2}}\int_{\mathbb{R}^{N-1}}\frac{dz}{(1+|z|^2)^{\frac{N}{2}}} + O\Big(\frac{\ln\Lambda|\bar{x}_1-\bar{x}_j|}{\Lambda^{N-1}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big).$$

In $B_{\frac{d_j}{2}}(\bar{x}_j)$, since $|\bar{y}-\bar{x}_1| \geq \frac{|\bar{x}_1-\bar{x}_j|}{2}$ and $|\bar{y}-\bar{x}_1| \geq |\bar{y}-\bar{x}_j|$, it is easy to know that

$$\Big(\frac{1}{1+\Lambda^2|\bar{y}-\bar{x}_1|^2}\Big)^{\frac{N}{2}} \leq \Big(\frac{1}{1+\frac{\Lambda^2}{4}|\bar{x}_1-\bar{x}_j|^2}\Big)^{\frac{N-1}{2}}\Big(\frac{1}{1+\Lambda^2|\bar{y}-\bar{x}_j|^2}\Big)^{\frac{1}{2}},$$

therefore we have

$$\int_{B_{\frac{d_j}{2}}(\bar{x}_j)} U_{x_1,\Lambda}^{2^\#-1}U_{x_j,\Lambda} = O\Big(\frac{\ln\Lambda|\bar{x}_1-\bar{x}_j|}{\Lambda^{N-1}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big).$$

In $\mathbb{R}^{N-1}\setminus B_{\frac{d_j}{2}}(\bar{x}_1)\cup B_{\frac{d_j}{2}}(\bar{x}_j)$, it holds that

$$\int_{\mathbb{R}^{N-1}\setminus B_{\frac{d_j}{2}}(\bar{x}_1)\cup B_{\frac{d_j}{2}}(\bar{x}_j)} U_{x_1,\Lambda}^{2^\#-1}U_{x_j,\Lambda} = O\Big(\frac{1}{\Lambda^{N-1}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big).$$

From (A.1) and (A.2), we finally obtain that

$$\text{(A.4)}\quad \int_{\mathbb{R}^N}|DW_{r,\Lambda}|^2 = \sum_{j=1}^{k}\sum_{i=1}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_j,\Lambda}^{2^\#-1}U_{x_i,\Lambda} = k\Big(\int_{\partial\mathbb{R}^N_+} U_{0,1}^{2^\#} + \sum_{j=2}^{k}\int_{\partial\mathbb{R}^N_+} U_{x_1,\Lambda}^{2^\#-1}U_{x_j,\Lambda}\Big) = k\Big[A_N + \sum_{j=2}^{k}\frac{C_{3N}}{\Lambda^{N-2}|\bar{x}_1-\bar{x}_j|^{N-2}} + O\Big(\sum_{j=2}^{k}\frac{\ln\Lambda|\bar{x}_1-\bar{x}_j|}{\Lambda^{N-1}|\bar{x}_1-\bar{x}_j|^{N-1}}\Big)\Big].$$

Let

$$\Omega_j = \Big\{ \bar{y} : \ \bar{y} = (\bar{y}', \bar{y}'') \in \mathbb{R}^2 \times \mathbb{R}^{N-3} = \partial \mathbb{R}^N_+, \ \Big\langle \frac{\bar{y}'}{|\bar{y}'|}, \frac{x_j}{|x_j|} \Big\rangle \geq \cos\frac{\pi}{k} \Big\}.$$

Then, from Taylor's expansion we obtain that

$$\text{(A.5)} \quad \begin{aligned} &\int_{\partial \mathbb{R}^N_+} K\Big(\frac{|\bar{y}|}{\mu}\Big) |W_{r,\Lambda}|^{2^{\#}} \\ &= k \int_{\Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) |W_{r,\Lambda}|^{2^{\#}} \\ &= k\Big[ \int_{\Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) U^{2^{\#}}_{x_1,\Lambda} + 2^{\#} \int_{\Omega_1} K\big(\frac{|y|}{\mu}\big) \sum_{i=2}^{k} U^{2^{\#}-1}_{x_1,\Lambda} U_{x_i,\Lambda} \\ &\quad + O\Big( \int_{\Omega_1} U^{2^{\#}-2}_{x_1,\Lambda} \Big( \sum_{i=2}^{k} U_{x_i,\Lambda} \Big)^2 \Big) + O\Big( \int_{\Omega_1} \Big( \sum_{i=2}^{k} U_{x_i,\Lambda} \Big)^{2^{\#}} \Big) \Big]. \end{aligned}$$

First, let us estimate the remainders. Note that for $\bar{y} \in \Omega_1$, it holds that $|\bar{y} - \bar{x}_i| \geq |\bar{y} - \bar{x}_1|$ and $|\bar{y} - \bar{x}_i| \geq \frac{1}{2}|\bar{x}_i - \bar{x}_1|$. Thus we know, for any $0 < \alpha < N-2$, that

$$\sum_{i=2}^{k} U_{x_i,\Lambda} \leq \frac{C}{(1 + |\bar{y} - \bar{x}_1|)^{N-2-\alpha}} \sum_{i=2}^{k} \frac{1}{|\bar{x}_i - \bar{x}_1|^{\alpha}},$$

and it is not difficult to check, for any $\alpha > 1$, that

$$\sum_{j=1}^{k} \frac{1}{|\bar{x}_1 - \bar{x}_j|^{\alpha}} = \sum_{j=1}^{k} \frac{1}{r^{\alpha} \sin^{\alpha} \frac{(j-1)\pi}{k}} = O\Big( \Big(\frac{k}{\mu}\Big)^{\alpha} \Big) = O\Big( \frac{1}{\mu^{\frac{m\alpha}{N-2}}} \Big).$$

If we select the constant $\alpha$ with $\frac{(N-2)}{2} < \alpha = \frac{m+\sigma}{m} \cdot \frac{N-2}{2} < \frac{N-1}{2}$ $(N \geq 5)$, then

$$\text{(A.6)} \quad \int_{\Omega_1} U^{2^{\#}-2}_{x_1,\Lambda} \Big( \sum_{i=2}^{k} U_{x_i,\Lambda} \Big)^2 \leq C \Big(\frac{k}{\mu}\Big)^{2\alpha} \int_{\Omega_1} \frac{1}{(1 + |\bar{y} - \bar{x}_1|)^{2+2(N-2-\alpha)}} = O\Big(\frac{1}{\mu^{m+\sigma}}\Big).$$

In addition, we may also choose $\alpha$ independently such that

$$\frac{(N-2)^2}{2(N-1)} < \alpha = \frac{m+\sigma}{m} \cdot \frac{(N-2)^2}{2(N-1)} < \frac{N-2}{2}, \quad (N \geq 5)$$

and then acquire that

$$\int_{\Omega_1} \Big( \sum_{i=2}^{k} U_{x_i,\Lambda} \Big)^{2^{\#}} = O\Big(\frac{1}{\mu^{m+\sigma}}\Big).$$

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Next we will calculate the second term in (A.5). It is easy to show as in (A.2) that

$$
\begin{aligned}
\text{(A.7)} \quad & \int_{\Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^\#-1} U_{x_i,\Lambda} \\
&= \int_{\Omega_1} \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^\#-1} U_{x_i,\Lambda} + \int_{\Omega_1} \Big(K\Big(\frac{|\bar{y}|}{\mu}\Big) - 1\Big) \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^\#-1} U_{x_i,\Lambda} \\
&= \sum_{i=2}^{k} \frac{C_{3N}}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} + O\Big(\frac{1}{\mu^{m+\sigma}}\Big).
\end{aligned}
$$

Finally, the first term in (A.5) can be written as

(A.8)

$$
\begin{aligned}
&\int_{\Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) U_{x_1,\Lambda}^{2^\#} \\
&= \int_{\{||\bar{y}|-\mu r_0| \le \mu\delta\} \cap \Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) U_{x_1,\Lambda}^{2^\#} + O\Big(\frac{k^{N-1}}{\Lambda^{2N-2}\mu^{N-1}}\Big) \\
&= \int_{\{|\bar{y}-\mu r_0| \le \mu\delta\} \cap \Omega_1} U_{x_1,\Lambda}^{2^\#} - \frac{c_0}{\mu^m} \int_{\{||\bar{y}|-\mu r_0| \le \mu\delta\} \cap \Omega_1} ||\bar{y}| - \mu r_0|^m U_{x_1,\Lambda}^{2^\#} \\
&\quad + O\Big(\mu^{-m-\theta} \int_{\{||\bar{y}|-\mu r_0| \le \mu\delta\} \cap \Omega_1} ||\bar{y}| - \mu r_0|^{m+\theta} U_{x_1,\Lambda}^{2^\#}\Big) + O\Big(\frac{k^{N-1}}{\Lambda^{2N-2}\mu^{N-1}}\Big) \\
&= A_N - \frac{c_0}{\mu^m} \int_{\partial\mathbb{R}^N_+} ||\bar{y}| - \mu r_0|^m U_{x_1,\Lambda}^{2^\#} \mathrm{d}\bar{y} + O\Big(\frac{1}{\mu^{m+\theta}}\Big) + O\Big(\frac{k^{N-1}}{\Lambda^{2N-2}\mu^{N-1}}\Big) \\
&= A_N - \frac{c_0}{\mu^m} \int_{\partial\mathbb{R}^N_+} ||\bar{y} - \bar{x}_1| - \mu r_0|^m U_{0,\Lambda}^{2^\#} \mathrm{d}\bar{y} + O\Big(\frac{1}{\mu^{m+\theta}} + \frac{k^{N-1}}{\mu^{N-1}}\Big).
\end{aligned}
$$

But,

$$
\begin{aligned}
&\frac{1}{\mu^m} \int_{\partial\mathbb{R}^N_+ \setminus B_{\frac{|\bar{x}_1|}{2}}(0)} ||\bar{y} - \bar{x}_1| - \mu r_0|^m U_{0,\Lambda}^{2^\#} \mathrm{d}\bar{y} \\
&\le C \int_{\partial\mathbb{R}^N_+ \setminus B_{\frac{|\bar{x}_1|}{2}}(0)} \Big(\frac{|\bar{y}|^m}{\mu^m} + 1\Big) \frac{\mathrm{d}\bar{y}}{(1 + \Lambda^2|\bar{y}|^2)^{N-1}} \le \frac{C}{\mu^{N-1}}.
\end{aligned}
$$

On the other hand, if $\bar{y} \in B_{\frac{|\bar{x}_1|}{2}}(0)$, $\bar{y} = (\bar{y}_1, \bar{y}^*)$, $\bar{y}^* = (\bar{y}_2, \dots, \bar{y}_{N-1})$, then $|\bar{x}_1| - \bar{y}_1 \ge |\bar{x}_1|/2 > 0$. So, as $|\bar{x}_1|$ becomes large,

$$
|\bar{y} - \bar{x}_1| = |\bar{x}_1| - \bar{y}_1 + O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1| - \bar{y}_1}\Big) = |\bar{x}_1| - \bar{y}_1 + O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big).
$$

As a result, Taylor's expansion says, for $m > 2$, that

$$\begin{aligned}\Big||\bar{y}-\bar{x}_1|-\mu r_0\Big|^m &= \Big||\bar{x}_1|-\bar{y}_1+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)-\mu r_0\Big|^m\\
&= |\bar{y}_1|^m+m|\bar{y}_1|^{m-2}\bar{y}_1\Big[\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big]\\
&\quad+\frac{1}{2}m(m-1)|\bar{y}_1|^{m-2}\Big[\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big]^2\\
&\quad+O\Big(|y_1|^{m-2-\widehat{\sigma}}\Big|\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big|^{2+\widehat{\sigma}}\Big)\\
&\quad+O\Big(\Big|\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big|^m\Big),\end{aligned}$$

where $\widehat{\sigma}=\min\{m-2,1\}$; while for $m=2$,

$$\begin{aligned}\Big||\bar{y}-\bar{x}_1|-\mu r_0\Big|^m &= \Big||\bar{x}_1|-\bar{y}_1+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)-\mu r_0\Big|^m\\
&= |\bar{y}_1|^m+m|\bar{y}_1|^{m-2}\bar{y}_1\Big[\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big]\\
&\quad+\frac{1}{2}m(m-1)|\bar{y}_1|^{m-2}\Big[\mu r_0-|\bar{x}_1|+O\Big(\frac{|\bar{y}^*|^2}{|\bar{x}_1|}\Big)\Big]^2.\end{aligned}$$

Thus, using

$$\int_{B_{\frac{|\bar{x}_1|}{2}}(0)}\frac{|\bar{y}_1|^{m-2}\bar{y}_1}{(1+\Lambda^2|\bar{y}|^2)^{N-1}}\mathrm{d}\bar{y}=0,$$

whether $m > 2$ or $m = 2$, we obtain that, since $m < N-2$,

$$\begin{aligned}\text{(A.9)}\quad &\frac{1}{\mu^m}\int_{\partial\mathbb{R}^N_+}||y-x_1|-\mu r_0|^m U_{0,\Lambda}^{2^\#}\\
&=\frac{1}{\mu^m}\int_{B_{\frac{|\bar{x}_1|}{2}}}||y-x_1|-\mu r_0|^m U_{0,\Lambda}^{2^\#}+O\Big(\frac{1}{\mu^{N-1}}\Big)\\
&=\frac{1}{\mu^m}\int_{\partial\mathbb{R}^N_+}|\bar{y}_1|^m U_{0,\Lambda}^{2^\#}\mathrm{d}\bar{y}+\frac{m(m-1)}{2\mu^m}\int_{\partial\mathbb{R}^N_+}|\bar{y}_1|^{m-2}(\mu r_0-|x_1|)^2U_{0,\Lambda}^{2^\#}\mathrm{d}\bar{y}\\
&\quad+O\Big(\frac{1}{\mu^m}|\mu r_0-r|^{2+\tilde{\sigma}}+\frac{1}{\mu^{N-1}}\Big)\\
&=\frac{C_{1N}}{\Lambda^m\mu^m}+\frac{C_{2N}}{\Lambda^{m-2}\mu^m}(\mu r_0-|x_1|)^2+O\Big(\frac{1}{\mu^m}|\mu r_0-r|^{2+\tilde{\sigma}}+\frac{1}{\mu^{N-1}}\Big),\end{aligned}$$

where $\tilde{\sigma}$ is chosen to be a small positive constant,

$$C_{1N}=(N-2)^{N-1}\int_{\mathbb{R}^{N-1}}\frac{|\bar{y}_1|^m\mathrm{d}\bar{y}}{(1+|\bar{y}|^2)^{N-1}},$$

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and

$$C_{2N} = \frac{m(m-1)(N-2)^{N-1}}{2} \int_{\mathbb{R}^{N-1}} \frac{|\bar{y}_1|^{m-2}}{(1+|\bar{y}|^2)^{N-1}} \mathrm{d}\bar{y}.$$

Thus, from (A.5)–(A.9) we conclude

$$\text{(A.10)} \quad \int_{\mathbb{R}^{N-1}} K\Big(\frac{|\bar{y}|}{\mu}\Big) |W_{r,\Lambda}|^{2^\#} \\ = k\Bigg[ A_N - \frac{C_{1N}}{\Lambda^m \mu^m} - \frac{C_{2N}}{\Lambda^{m-2}\mu^m}(\mu r_0 - |x_1|)^2 + 2^\# \sum_{i=2}^{k} \frac{C_{3N}}{\Lambda^{N-2}|x_1 - x_j|^{N-2}} + \\ O\Big( \frac{1}{\mu^m} |\mu r_0 - r|^{2+\bar{\sigma}} + (\frac{1}{\mu})^{m+\sigma} \Big) \Bigg].$$

The proposition now follows from (A.4) and (A.10) by setting $A = (\frac{1}{2} - \frac{1}{2^\#})A_N$, $B_1 = \frac{C_{1N}}{2^\#}$, $B_2 = \frac{C_{2N}}{2^\#}$, and $B_3 = \frac{C_{3N}}{2}$. ■

***Proposition A.2*** *We have*

$$\frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} = k\Big[ -\frac{mB_1}{\Lambda^{m+1}\mu^m} + \sum_{i=2}^{k} \frac{B_3(N-2)}{\Lambda^{N-1}|x_1 - x_j|^{N-2}} \\ + O\Big( \frac{1}{\mu^{m+\sigma}} + \frac{1}{\mu^m} |\mu r_0 - r|^2 \Big) \Big],$$

*where $B_i$ $(i = 1, 3)$ are the same positive constants as in Proposition A.1.*

**Proof** The proof of this proposition is similar to that of Proposition A.1. So we just sketch it.

It is not difficult to get

$$\text{(A.11)} \quad \frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} = k\Big[ \frac{1}{2} \sum_{i=2}^{k} \frac{\partial}{\partial \Lambda} \int_{\partial \mathbb{R}^N_+} U^{2^\#-1}_{x_1,\Lambda} U_{x_i,\Lambda} - \frac{1}{2^\#} \frac{\partial}{\partial \Lambda} \int_{\Omega_1} K\Big(\frac{|\bar{y}|}{\mu}\Big) W^{2^\#}_{r,\Lambda} \Big].$$

Note that

$$\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\Big|_{\partial \mathbb{R}^N_+} = \frac{(N-2)}{2\Lambda} \frac{1 - \Lambda^2 |\bar{y} - \bar{x}_j|^2}{1 + \Lambda^2 |\bar{y} - \bar{x}_j|^2} U_{x_j,\Lambda}\Big|_{\partial \mathbb{R}^N_+},$$

hence

$$\text{(A.12)} \quad \frac{\partial}{\partial \Lambda} \int_{\partial \mathbb{R}^N_+} U^{2^\#-1}_{x_1,\Lambda} U_{x_i,\Lambda} = \frac{\partial}{\partial \Lambda} \int_{\partial \mathbb{R}^N_+} U^{2^\#-1}_{x_1,\Lambda} \frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}} |\bar{x}_i - \bar{x}_1|^{N-2}} \\ + \frac{\partial}{\partial \Lambda} \int_{\partial \mathbb{R}^N_+} U^{2^\#-1}_{x_1,\Lambda} \Big( U_{x_i,\Lambda} - \frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}} |\bar{x}_i - \bar{x}_1|^{N-2}} \Big).$$

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In $B_{\frac{d_i}{2}}(\bar{x}_1)$, recalling (A.3) and using

$$\frac{\partial}{\partial\Lambda}\Big(U_{x_i,\Lambda}-\frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}|\bar{x}_i-\bar{x}_1|^{N-2}}\Big)=O\Big(\frac{|\bar{y}-\bar{x}_1|}{\Lambda^{\frac{N}{2}}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big)+O\Big(\frac{1}{\Lambda^{\frac{N+4}{2}}|\bar{x}_i-\bar{x}_1|^{N}}\Big),$$

we have that

$$\frac{\partial}{\partial\Lambda}\int_{B_{\frac{d_i}{2}}(\bar{x}_1)}U_{x_1,\Lambda}^{2^\#-1}\Big(U_{x_i,\Lambda}-\frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}|\bar{x}_i-\bar{x}_1|^{N-2}}\Big)=O\Big(\frac{\ln\Lambda|\bar{x}_i-\bar{x}_1|}{\Lambda^{N}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big).$$

As in the proof of Proposition A.1, it is also easy to check that

$$\frac{\partial}{\partial\Lambda}\int_{B_{\frac{d_i}{2}}(\bar{x}_i)}U_{x_1,\Lambda}^{2^\#-1}\Big(U_{x_i,\Lambda}-\frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}|\bar{x}_i-\bar{x}_1|^{N-2}}\Big)=O\Big(\frac{\ln\Lambda|\bar{x}_i-\bar{x}_1|}{\Lambda^{N}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big),$$

$$\frac{\partial}{\partial\Lambda}\int_{\partial\mathbb{R}^N_+\setminus B_{\frac{d_i}{2}}(\bar{x}_i)\cup B_{\frac{d_i}{2}}(\bar{x}_1)}U_{x_1,\Lambda}^{2^\#-1}\Big(U_{x_i,\Lambda}-\frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}|\bar{x}_i-\bar{x}_1|^{N-2}}\Big)=O\Big(\frac{1}{\Lambda^{N}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big).$$

Thus from (A.12) we get that

$$\begin{aligned}\frac{\partial}{\partial\Lambda}\int_{\partial\mathbb{R}^N_+}U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}&=\frac{\partial}{\partial\Lambda}\int_{\partial\mathbb{R}^N_+}U_{x_1,\Lambda}^{2^\#-1}\frac{(N-2)^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}|\bar{x}_i-\bar{x}_1|^{N-2}}+O\Big(\frac{\ln\Lambda|\bar{x}_i-\bar{x}_1|}{\Lambda^{N}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big)\\&=-\frac{(N-2)C_{3N}}{\Lambda^{N-1}|\bar{x}_i-\bar{x}_1|^{N-2}}+O\Big(\frac{\ln\Lambda|\bar{x}_i-\bar{x}_1|}{\Lambda^{N}|\bar{x}_i-\bar{x}_1|^{N-1}}\Big).\end{aligned}$$

As for the terms in the right side of (A.11), direct computations show that

$$\begin{aligned}&\frac{\partial}{\partial\Lambda}\int_{\Omega_1}K\Big(\frac{|\bar{y}|}{\mu}\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\&=\frac{\partial}{\partial\Lambda}\int_{\Omega_1}U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}+\frac{\partial}{\partial\Lambda}\int_{\Omega_1}\Big(K\Big(\frac{|\bar{y}|}{\mu}\Big)-1\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\&=\frac{\partial}{\partial\Lambda}\Big(\int_{\partial\mathbb{R}^N_+}-\int_{\partial\mathbb{R}^N_+\setminus\Omega_1\cup B_{\frac{d_i}{2}}(\bar{x}_i)}-\int_{B_{\frac{d_i}{2}}(\bar{x}_i)}\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\&\quad+\frac{\partial}{\partial\Lambda}\int_{\Omega_1}\Big(K\Big(\frac{|\bar{y}|}{\mu}\Big)-1\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\&=-\frac{(N-2)C_{3N}}{\Lambda^{N-1}|\bar{x}_i-\bar{x}_1|^{N-2}}+O\Big(\Big(\frac{k}{\mu}\Big)^{N-2+\sigma}\Big).\end{aligned}$$

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The last equality is due to the fact that, because of the condition on the function $K$,

$$\begin{aligned}
&\frac{\partial}{\partial\Lambda}\int_{\Omega_1}\Big(K\Big(\frac{|\bar y|}{\mu}\Big)-1\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\
&=\frac{\partial}{\partial\Lambda}\Big(\int_{\Omega_1\cap\{\,|\,|\bar y|-\mu r_0|\,\le\mu^{1-\sigma}\}}+\int_{\Omega_1\cap\{\,|\,|\bar y|-\mu r_0|\,\ge\mu^{1-\sigma}\}}\Big)U_{x_1,\Lambda}^{2^\#-1}U_{x_i,\Lambda}\\
&=O\Big(\frac{1}{\mu^{m\sigma}\Lambda^{N-1}|\bar x_i-\bar x_1|^{N-2}}\Big)+O\Big(\frac{\ln\Lambda|\bar x_i-\bar x_1|}{\Lambda^N|\bar x_i-\bar x_1|^{N-1}}\Big).
\end{aligned}$$

By similar estimates as in (A.8) and (A.9), we have that

$$\begin{aligned}
&\frac{\partial}{\partial\Lambda}\int_{\Omega_1}K\big(\frac{|\bar y|}{\mu}\big)U_{x_1,\Lambda}^{2^\#}\\
&=\frac{\partial}{\partial\Lambda}\int_{\{||\bar y|-\mu r_0|\le\mu\delta\}\cap\Omega_1}K\Big(\frac{|\bar y|}{\mu}\Big)U_{x_1,\Lambda}^{2^\#}+O\Big(\frac{k^{N-1}}{\mu^{N-1}}\Big)\\
&=\frac{\partial}{\partial\Lambda}\int_{\{|\bar y-\mu r_0|\le\mu\delta\}\cap\Omega_1}U_{x_1,\Lambda}^{2^\#}-\frac{c_0}{\mu^m}\frac{\partial}{\partial\Lambda}\int_{\{||\bar y|-\mu r_0|\le\mu\delta\}\cap\Omega_1}||y|-\mu r_0|^mU_{x_1,\Lambda}^{2^\#}\\
&\quad+O\Big(\frac{1}{\mu^{m+\theta}}+\frac{k^{N-1}}{\mu^{N-1}}\Big)\\
&=-\frac{c_0}{\mu^m}\frac{\partial}{\partial\Lambda}\int_{\mathbb{R}^{N-1}}||\bar y|-\mu r_0|^mU_{x_1,\Lambda}^{2^\#}\mathrm{d}\bar y+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{k^{N-1}}{\mu^{N-1}}\Big)\\
&=-\frac{c_0}{\mu^m}\frac{\partial}{\partial\Lambda}\int_{\mathbb{R}^{N-1}}||\bar y-\bar x_1|-\mu r_0|^mU_{0,\Lambda}^{2^\#}\mathrm{d}\bar y+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{k^{N-1}}{\mu^{N-1}}\Big).\\
&=\frac{mC_{1N}}{\Lambda^{m+1}\mu^m}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^2\Big).
\end{aligned}$$

The remaining estimates of this proposition are similar to the previous one. We omit the details. ■

## A.2 Basic Estimates

For each fixed $i$ and $j$, $i\neq j$, consider the following function

$$g_{ij}(y)=\frac{1}{(1+|y-x_j|)^\alpha}\frac{1}{(1+|y-x_i|)^\beta},$$

where $\alpha>0$ and $\beta>0$ are two constants.

Then we have the following lemma whose proof can be found in [21, Appendix B].

***Lemma A.3*** *For any constant* $0\le\sigma\le\min(\alpha,\beta)$, *there is a constant* $C>0$, *such that*

$$g_{ij}(y)\le\frac{C}{|x_i-x_j|^\sigma}\Big[\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big].$$

***Lemma A.4*** *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that for any $y \in \overline{\mathbb{R}^N_+}$,*

$$\int_{\partial \mathbb{R}^N_+} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{1+\sigma}} \,\mathrm{d}\bar{z} \le \frac{C}{(1+|y|)^{\sigma}},$$

*where $z = (\bar{z}, 0) = \mathbb{R}^{N-1} \times \{0\} \in \partial \mathbb{R}^N_+$.*

The result is well known. Readers may refer to [21, Appendix B] to find almost the same proof.

***Lemma A.5*** *Suppose that $N \ge 5$. Then for any $y \in \overline{\mathbb{R}^N_+}$, we have that*

$$\int_{\partial \mathbb{R}^N_+} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{2^\#-2}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}-\frac{m}{N-2}+\tau}} \mathrm{d}\bar{z} \le \sum_{i=1}^{k} \frac{C}{(1+|y-x_i|)^{\frac{N-2}{2}-\frac{m}{N-2}+\tau}}.$$

**Proof** Note that for any $\beta \ge \frac{N-2-m}{N-2}$ and fixed $\ell$, as $k \to \infty$

$$\sum_{i \neq \ell} \frac{1}{|x_i - x_\ell|^{\beta}} = \frac{1}{2^{\beta}} \sum_{i \neq \ell} \frac{1}{r^{\beta} \sin^{\beta} \frac{|i-\ell|\pi}{k}} \le \frac{Ck^{\beta}}{\mu^{\beta}} \sum_{i=1}^{k} \frac{1}{i^{\beta}} \le \begin{cases} \dfrac{Ck^{\beta}}{\mu^{\beta}} = O(\mu^{-\frac{m\beta}{N-2}}) & \beta > 1, \\ \dfrac{Ck^{\beta} \ln k}{\mu^{\beta}} = O(\mu^{-\frac{m\beta}{N-2}} \ln \mu) & \beta = 1, \\ \dfrac{Ck}{\mu^{\beta}} = O(\mu^{-(\beta - \frac{N-2-m}{N-2})}) & \beta < 1. \end{cases}$$

In $\Omega_\ell$, we have $|z - x_j| = |\bar{z} - \bar{x}_j| \ge |z - x_\ell|$ and $|z - x_j| \ge |x_j - x_\ell|$ for any $j \neq \ell$. Thus for any $\frac{N-2-m}{N-2} \le \alpha \le N-2$, it holds

$$\sum_{j \neq \ell} \frac{1}{(1+|z-x_j|)^{N-2}} \le \frac{1}{(1+|z-x_\ell|)^{N-2-\alpha}} \sum_{j \neq \ell} \frac{1}{|x_j - x_\ell|^{\alpha}}.$$

Thus in $\Omega_\ell$ we have

$$W_{r,\Lambda}^{2^\#-2}(z) \le \frac{C}{(1+|z-x_\ell|)^{2-\frac{2\alpha}{N-2}}},$$

$$\sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}} \le \frac{C}{(1+|z-x_\ell|)^{\frac{N}{2}-\frac{m}{N-2}+\tau-\alpha}}.$$

As a result, we find for $z \in \Omega_\ell$ that

$$W_{r,\Lambda}^{2^\#-2}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}} \le \frac{C}{(1+|z-x_\ell|)^{\frac{N+2}{2}-\frac{m}{N-2}+1-\frac{N\alpha}{N-2}+\tau}}.$$

It gives that, for $\alpha = \frac{N-2-m}{N-2}$, since $\partial\mathbb{R}^N_+ = \bigcup_{i=1}^{k}\Omega_i$,

$$\begin{aligned}
&\int_{\partial\mathbb{R}^N_+}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{2^\#-2}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}\,\mathrm{d}\bar{z}\\
&\le \sum_{i=1}^{k}\frac{C}{(1+|y-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\frac{(m-2)N+4}{(N-2)^2}+\tau}}\\
&\le \sum_{i=1}^{k}\frac{C}{(1+|y-x_i|)^{\frac{N}{2}-\frac{m}{N-2}+\tau}}.
\end{aligned}$$

■

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