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ON CONTINUATION OF QUASI-ANALYTIC SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS TO COMPACT CONVEX SETS

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Abstract

In the early 70s A. Kaneko studied the problem of continuation of regular solutions of systems of linear partial differential equations with constant coefficients to compact convex sets. We show here that the conditions he obtained for real analytic solutions also hold in the quasi-analytic case. In particular we show that every quasi-analytic solution of the system $p(D)u = 0$ defined outside a compact convex subset K or R^n can be continued as a quasi-analytic solution to K if and only if the system is determined and the $\mathscr{P}$-module $\operatorname{Ext}^1(\operatorname{Coker} p', \mathscr{P})$ has no elliptic component; here $\mathscr{P}$ is the ring of polynomials in n variables, p is a matrix with elements from $\mathscr{P}$ and p' is the transposed matrix. In the scalar case, i.e. when p is a single polynomial, these conditions mean that p has no elliptic factor.

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1. Introduction

While an analytic function of one complex variable may have lots of non-removable isolated singularities (except when it satisfies some a priori local boundedness condition, cf. a classical theorem of Riemann [19]), it was shown by Hartogs [7] that isolated singularities of an analytic function of several complex variables are always removable. This phenomenon has been explained by Bochner [3] from the standpoint of overdetermined systems of differential equations (the Cauchy-Riemann system). A closer examination of the circumstances made by Ehrenpreis, Komatsu, Malgrange and Palamodov [5, 14, 16, 18] revealed that any compact convex set is a removable singularity for any general (i.e. distribution or hyperfunction) solution of an overdetermined system of differential equations with constant coefficients. More precisely, any distribution (hyperfunction) solution of such a system defined outside a convex compact set in R^n can be uniquely continued to that set as a distribution (hyperfunction) solution. Although this property is characteristic for overdetermined systems, an analogous property with only *regular* (e.g. $\mathscr{C}^\infty$ or real analytic) solutions allowed, is not. The first to observe this was probably Grušin [6] who gave a sufficient condition for every isolated singularity of a $\mathscr{C}^\infty$-solution of the scalar equation $p(D)u = 0$ to be removable: every irreducible factor of the polynomial p should have a real simple characteristic direction. Some related results can be found in Palamodov [18, Chapter VIII, §14]. Later, Kaneko [9, 10] succeeded in characterizing the systems, the real analytic solutions of which, have all compact convex sets as removable singularities:

THEOREM 0 [10, Theorem 2.3]. *Let $K \subset U \subset R^n$, K compact and convex, U open. Let $p(D)$ be a matrix of linear partial differential operators with constant coefficients and let p' be its transposed matrix. Let $\mathscr{P}$ be the ring of polynomials in n variables. Then every real analytic solution of $p(D)u = 0$ on $U \setminus K$ can be continued to K as a hyperfunction solution of this system if and only if* $\mathrm{Ext}^1(\mathrm{Coker}\, p', \mathscr{P})$ *has no elliptic component in its primary decomposition. In order that every real analytic solution on $U \setminus K$ be continuable to K as a real analytic solution on U it is necessary and sufficient, that in addition to the condition above, the system is determined (i. e.* $\mathrm{Hom}(\mathrm{Coker}\, p', \mathscr{P}) = 0$).

This result was the starting point of some impressive generalizations developed since then by Kaneko in the direction of micro-local analysis of singularities of systems with variable coefficients, mostly within the framework of real analytic regularity. A recent introduction to the subject with a historical background and all relevant references can be found in the survey lecture [13].

The aim of the present note is to show that Theorem 0 extends, as it stands, to the case of quasi-analytic regularity (but not beyond—cf. Remark 3, Section 3). This is the content of our Theorem 1, formulated in complete analogy with Theorem 2.3 of [10]. In proving the least obvious part of the latter, i.e. the sufficiency of the Ext^1-condition in case of determined systems, Kaneko used hyperfunction theory (in particular the flabbiness property!) and a hyperfunction version of the Fundamental Principle which he developed on this occasion. Later, in [12], he gave an elementary proof for the scalar case, employing the Ehrenpreis' cut-off functions. Our proof of the corresponding part of Theorem 1 is modeled on that of [12]; it is elementary in the sense that, except for some defining notions, it uses neither homological algebra nor hyperfunction theory and it needs only (a part of) the classical Palamodov Fundamental Principle. Decisive for the treatment of the general quasi-analytic case has been the replacement of Carlson's lemma in [12] by the more powerful [2, Theorem 6.3.6]; it is needed in the proof of Lemma 2. It may be interesting to remark in this context that Hörmander [8] recently proved the flabbiness of (what one may perhaps call) the sheaves of infra-hyperfunctions, i.e. the sheaves related to the quasi-analytic classes in the same way the hyperfunctions relate to the analytic class. This result may make it possible, if desired, to construct an infra-hyperfunction proof of Theorem 1 analogous to the proof in [10].

In [1] Kaneko significantly relaxed the condition on the geometry of the virtual singularity K, replacing its convexity by the connectedness of its complement. We would have done likewise in Theorem 1, were we able to assert the triviality of the cohomology of the quasi-analytic sheaf on $\mathbb{R}^n$; whether this is the case is, however, not known to the author. It would also be interesting to see some of the micro-local real analytic results of Kaneko find their counterparts in the quasi-analytic case.

The reader may find it helpful to read this note together with [9, 10, 12]. The notation used here is that of [9, 10], which, in turn, leans on Palamodov [18].

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2. Preliminaries

$\mathscr{P}$-modules and associated varieties. Let $\mathscr{P}$ be the ring of polynomials in n variables $\zeta_1, \ldots, \zeta_n$ and let $p(\zeta)\colon \mathscr{P}^s \to \mathscr{P}^t$ be a $t \times s$-matrix with polynomial coefficients. Put $M = \operatorname{Coker} p' = \mathscr{P}^s / p'\mathscr{P}^t$ where p' is the transposed matrix of p and let

$$0 \leftarrow M \leftarrow \mathscr{P}^s \overset{p'}{\leftarrow} \mathscr{P}^t \overset{p_1'}{\leftarrow} \mathscr{P}^{t_2} \leftarrow \cdots \tag{1.1}$$

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be a free resolution of the $\mathscr{P}$-module M. Recall that the sequences

$$0 \to \operatorname{Hom}(M, \mathscr{P}) \to \mathscr{P}^s \xrightarrow{p} \mathscr{P}^t$$

and

$$0 \to \operatorname{Ext}^1(M, \mathscr{P}) \to \mathscr{P}^t / p\mathscr{P}^s \xrightarrow{p_1} \mathscr{P}^{t_2}$$

are exact by definition, that M is called *determined* if $\operatorname{Hom}(M, \mathscr{P}) = 0$ and *overdetermined* if, in addition, $\operatorname{Ext}^1(M, \mathscr{P}) = 0$. Recall also that to any finite $\mathscr{P}$-module $\mathscr{Q}$ one may associate a family of irreducible affine algebraic varieties $\{N_\lambda(\mathscr{Q})\}$ in C^n defined as follows: if

$$0 = \mathscr{Q}_0 \cap \cdots \cap \mathscr{Q}_l$$

is a non-redundant primary decomposition of the zero submodule of $\mathscr{Q}$, then $N_\lambda(\mathscr{Q})$ is the zero-set of the annihilator ideal of $\mathscr{Q}_\lambda$, $\lambda = 0, 1, \ldots, l$.

Let us briefly review some well-known facts about projective extentions of complex affine varieties. Consider $\mathbf{C}^n$ imbedded in the complex projective space $\mathbf{P}^n$ by means of a mapping ϕ which to $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$ associates the point in $\mathbf{P}^n$ represented by the homogeneous coordinates $(1, \zeta_1, \ldots, \zeta_n)$. To an affine variety V in $\mathbf{C}^n$ corresponds then the projective extention of V—the smallest projective variety V^* in $\mathbf{P}^n$ containing $\phi(V)$. Of course, V^* and $\phi(V)$ differ (at most) by some points of $\mathbf{P}^n \setminus \phi(\mathbf{C}^n)$ ($=: H_\infty$, the hyperplane at infinity), which are called the points at infinity of V. We recall in passing that a $\mathscr{P}$-module $\mathscr{Q}$ is called *elliptic* [17] if none of the associated varieties $N_\lambda(\mathscr{Q})$, $\lambda = 0, \ldots, l$, has a real point at infinity (i.e. a point at infinity with a real homogeneous coordinate representation). The conic set in $\mathbf{C}^n$ consisting of points $(\zeta_1, \ldots, \zeta_n)$ such that $(0, \zeta_1, \ldots, \zeta_n)$ is a homogeneous coordinate representation of a point at infinity of V is called the *Zariski cone* of V. In the sequel we shall need a description of the asymptotic position of an affine algebraic variety in relation to its Zariski cone. Assume V irreducible. Then, starting from any point on V, one can asymptotically approach the "direction" of any specified complex line in the Zariski cone of V by an irreducible algebraic curve on V; moreover, one can make the approach regular (i.e. complex analytic):

PROPOSITION 1 ([17, §4]). *Let V be an irreducible affine algebraic variety in $\mathbf{C}^n$. For any pair of points ζ_0 in V and ζ in the Zariski cone of V, there exists an irreducible algebraic curve Γ on V through ζ_0 and a regular algebraic mapping γ: $\Delta \to \Gamma$, defined on the outside of a disc around the origin in the complex plane slit along the negative imaginary axis, with values in Γ, such that, for some $\delta > 0$,*

$$\left| \frac{\gamma(s)}{s} - \zeta \right| = o(s^{-\delta}), \qquad s \to \infty, \qquad s \in \Delta;$$

in other words, $\gamma(s) = s \cdot \zeta + \eta(s)$, *where the components of* η *are regular algebraic functions in* Δ *and, for some* $\rho < 1$,

$$\lim_{\substack{s \in \Delta \\ s \to \infty}} \frac{\eta(s)}{s^{\rho}} = 0. \tag{2.2}$$

PROOF. This rather well-known result will be crucial for Lemma 1 in the next section. Since the proof is not hard (but not easy to find in standard references) we give a brief outline referring for details to [17] and the references given there. See also [1, Chapter 3, §A.1].

It is known that any two points on a projective irreducible algebraic variety can be joined by an irreducible curve on the variety. Accordingly, we may join the point at infinity p corresponding to ζ with the point $\phi(\zeta_0)$ by a curve Γ^* on the projective extension of V. Let $\Gamma = \phi^{-1}(\Gamma^*)$ be the corresponding affine curve on V. Using a normalization of Γ^* one can parametrize a neighbourhood of p in a branch of Γ^* by a regular algebraic function σ defined in a small disc around the origin in a complex plane, with $\sigma(0) = p$. Obviously, Γ^* meets H_∞ only in a finite number of points, hence, shrinking the domain of σ if necessary, we may assume that, except for $\sigma(0)$, the values of σ stay in $\phi(V)$. By expressing σ in a convenient local affine coordinate system, composing it with ϕ^{-1} and conveniently changing its parameter, one finally obtains the desired parametrization γ of a branch of Γ, where, to begin with, one can only say that $\eta(s) = o(s)$; the sharper, final estimate (2.2) is obtained by expanding each of the components of η in Puiseux series around $s = \infty$.

NOETHERIAN OPERATORS. Let $\mathscr{E}'[K]$ be the set of distributions on $\mathbf{R}^n$ with support in the compact convex set K. The Fourier transform of $u \in \mathscr{E}'[K]$ is the entire function

$$\mathbf{C}^n \ni \zeta \to \hat{u}(\zeta) = u(e^{-i\langle \cdot, \zeta \rangle});$$

denote the set of all such $\hat{u}$ by $\widehat{\mathscr{E}'[K]}$. The matrix $p(\zeta)$ gives rise to the natural mapping $p(\zeta)$: $\widehat{\mathscr{E}'[K]}^s \to \widehat{\mathscr{E}'[K]}^t$ the range of which is described by the so-called Fundamental Principle for $\mathscr{E}'[K]$ (see [18, Chapter IV, §4, 2°] or [1, Chapter 8, §4]):

PROPOSITION 2. *There exists a family of* (*matrix-*) *differential operators* $d = \{\partial^\lambda = \partial^\lambda(\zeta, D_\zeta), \lambda = 0, 1, \dots, l\}$ *with polynomial coefficients* (*the so-called Noetherian Operator associated with the matrix* p) *with the following property*: *the conditions*

$$\partial^\lambda(\zeta, D_\zeta) f \big|_{N_\lambda(\mathscr{P}^t/p\mathscr{P}^s)} = 0, \qquad \lambda = 0, 1, \dots, l,$$

are necessary and sufficient for $f \in \widetilde{\mathscr{E}'[K]}^t$ *to be in the image of* $p(\zeta)$: $\widetilde{\mathscr{E}'[K]}^s \to \widetilde{\mathscr{E}'[K]}^t$; *here* $D = (D_{\zeta_1}, \ldots, D_{\zeta_n})$, $D_{\zeta_j} = -i\frac{\partial}{\partial \zeta_j}$.

QUASI - ANALYTIC CLASSES. Let $L = (L_0, L_1, \ldots)$, $L_0 = 1$, be a logarithmically convex sequence of positive numbers and let U be an open set in $\mathbb{R}^n$. By $\mathscr{C}^L(U)$ we denote the subspace of $\mathscr{C}^\infty(U)$ consisting of functions f which on every $V \subset\subset U$ satisfy

$$|D^\alpha f(x)| \leqslant C^{|\alpha|+1} \cdot L_{|\alpha|}, \qquad \forall \alpha \in \mathbf{Z}_+^n,$$

with a constant C depending on V and f. In order to make $\mathscr{C}^L(U)$ stable under differential operators of finite order we assume that $L_{k+1} \leqslant C^k \cdot L_k$ for some constant C and all $k \geqslant 0$. We also assume that $L_k \geqslant k!$, $k \geqslant 0$, thus ensuring that $\mathscr{C}^L(U)$ contains the real analytic functions. It will not cause confusion to let the letter L also denote the increasing convex function on the positive numbers defined by $L(t) = \sup\{t^k/L_k : k \geqslant 0\}$; note that $L(t) \geqslant 1$. We shall assume that

$$\int_0^\infty \frac{\log L(t)}{1+t^2}\, dt = \infty. \tag{2.3}$$

By the Denjoy-Carleman theorem this is a necessary and sufficient condition for $\mathscr{C}^L(U)$ to contain no non-trivial element with compact support. $\mathscr{C}^L(U)$ is then called a *quasi-analytic* class. The corresponding sheaf obtained as U varies through the open sets of $\mathbb{R}^n$ is also called quasi-analytic.

3. Continuation of solutions

Let $\mathscr{B}$ be the sheaf of hyperfunctions on $\mathbb{R}^n$. When $\mathscr{F}$ is a subsheaf of $\mathscr{B}$ and U is an open set in $\mathbb{R}^n$ we let, as is customary, $\mathscr{F}_M(U)$ denote the set of solutions in $\mathscr{F}(U)^s$ of the system $p(D)u = 0$, where $p(D)$ is the differential operator associated with the matrix p, $D = (D_1, \ldots, D_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $i = \sqrt{-1}$; M stands here, as above, for the $\mathscr{P}$-module $\mathscr{P}^s/p'\mathscr{P}^t$.

Let $K \subset U \subset \mathbb{R}^n$, K compact, U open, both sets assumed convex. We may form the quotient space $\mathscr{F}_M(U \setminus K)/\mathscr{F}_M(U)$, where, by a slight abuse of notation, $\mathscr{F}_M(U)$ stands for the image of the restriction mapping $\mathscr{F}_M(U) \to \mathscr{F}_M(U \setminus K)$. Obviously, an element u of $\mathscr{F}_M(U \setminus K)$ can be extended to an element of $\mathscr{F}_M(U)$ if and only if u is equal to zero in the quotient space $\mathscr{F}_M(U \setminus K)/\mathscr{F}_M(U)$. More generally, let $\mathscr{F} \subset \mathscr{G}$ be subsheaves of $\mathscr{B}$ and consider the map

$$\frac{\mathscr{F}_M(U \setminus K)}{\mathscr{F}_M(U)} \to \frac{\mathscr{G}_M(U \setminus K)}{\mathscr{G}_M(U)}$$

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induced by the inclusion $\mathscr{F} \subset \mathscr{G}$. It is obvious that $u \in \mathscr{F}_M(U \setminus K)$ can be extended to an element of $\mathscr{G}_M(U)$ if and only if the image of $u + \mathscr{F}_M(U)$ under this map is zero.

THEOREM 1. *Let $\mathscr{C}^L$ be a quasi-analytic sheaf on $\mathbb{R}^n$. Let $K \subset U \subset \mathbb{R}^n$, K compact and convex, U open. In order that the image of the natural map*

$$\frac{\mathscr{C}_M^L(U \setminus K)}{\mathscr{C}_M^L(U)} \xrightarrow{\alpha} \frac{\mathscr{B}_M(U \setminus K)}{\mathscr{B}_M(U)}$$

be zero, it is necessary and sufficient that $\operatorname{Ext}^1(M, \mathscr{P})$ *has no elliptic component in its primary decomposition. In order that $\mathscr{C}_M^L(U \setminus K)/\mathscr{C}_M^L(U) = 0$, it is necessary and sufficient that, in addition to the above condition on* $\operatorname{Ext}^1(M, \mathscr{P})$, *$M$ is determined.*

REMARK 1. Consider the natural mappings

$$\frac{\mathscr{C}_M^L(U \setminus K)}{\mathscr{C}_M^L(U)} \xrightarrow{\alpha'} \frac{\mathscr{D}'_M(U \setminus K)}{\widetilde{\mathscr{D}'_M(U)}} \xrightarrow{\alpha''} \frac{\mathscr{B}_M(U \setminus K)}{\mathscr{B}_M(U)}$$

where $\widetilde{\mathscr{D}'_M(U)} = \{u \in \mathscr{D}'_M(U \setminus K)\colon U \supset\supset {}^{\forall} V \supset\supset K,\ {}^{\exists}u_1 \in \mathscr{D}'_M(U)$ such that $u_1 = u$ on $U \setminus V\}$. By [10, Corollary 1.4], the map α'' is injective. Hence the image of $\alpha = \alpha'' \circ \alpha'$ is 0 if and only if the image of α' is 0. The same reasoning also applies to the mappings

$$\frac{\mathscr{C}_M^L(U \setminus K)}{\mathscr{C}_M^L(U)} \to \frac{\mathscr{C}_M^\infty(U \setminus K)}{\widetilde{\mathscr{C}_M^\infty(U)}} \to \frac{\mathscr{B}_M(U \setminus K)}{\mathscr{B}_M(U)}$$

with $\widetilde{\mathscr{C}_M^\infty(U)}$ defined like $\widetilde{\mathscr{D}'_M(U)}$. Thus, by replacing $\mathscr{B}_M(U \setminus K)/\mathscr{B}_M(U)$ by $\mathscr{D}'_M(U \setminus K)/\widetilde{\mathscr{D}'_M(U)}$ or by $\mathscr{C}_M^\infty(U \setminus K)/\widetilde{\mathscr{C}_M^\infty(U)}$, one obtains two other, equivalent versions of Theorem 1.

REMARK 2. The module $\operatorname{Ext}^1(M, \mathscr{P})$ in Theorem 1 may be replaced by the module $\mathscr{P}^t/p\mathscr{P}^s$. Indeed, (cf. [18, Chapter VIII, §14, 3°]) both modules have the same family of associated irreducible varieties except for the component of dimension n (which, of course, is not elliptic). The above condition on $\operatorname{Ext}^1(M, \mathscr{P})$ thus means that each of the varieties $N_\lambda(\mathscr{P}^t/p\mathscr{P}^s)$, $\lambda = 1, \ldots, l$, has a real point at infinity.

REMARK 3. Theorem 1 fails if $\mathscr{C}^L$ is not quasi-analytic. Following Remark 3 at the end of [9], consider $p \in \mathscr{P}$ and K with non-empty interior. Let $f \in \mathscr{C}_0^L(\mathring{K})$ be such that $p(D)u = f$ has no solution with compact support. It is well known that one can solve this equation in $\mathscr{C}^L(U)$; the solution is then a non-zero element of $\mathscr{C}_M^L(U \setminus K)/\mathscr{C}_M^L(U)$.

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Proof of Theorem 1. The second part of Theorem 1 follows from the first part and the fact that α is injective if and only if M is determined. This fact is an analogue of the second part of Corollary 1.4 in [10] and is proved analogously; we only remark that at a certain stage of the proof of necessity one has to use a family of cutoff functions instead of a single one, as it is done, for example, in [4].

The necessity in the first part of Theorem 1 follows via Lech's theorem [15] exactly as it is indicated in [10] for the real analytic case—so we may turn to the proof of the sufficiency.

Let $u \in \mathscr{C}^L_M(U \setminus K)$. Define the *Grušin transform* of u, denoted by $\tilde{d}.u$, to be the family $\{\tilde{d}^\lambda.u\}$ of vector-valued analytic functions on the algebraic varieties $N_\lambda = N_\lambda(\mathscr{P}^t/p\mathscr{P}^s)$, $\lambda = 0, 1, \ldots, l$, obtained as follows:

$$\tilde{d}^\lambda.u = \partial^\lambda(\zeta, D_\zeta)\left(p(D)((1-\chi)u)\right)\hat{}\,|_{N_\lambda}, \qquad \lambda = 0, 1, \ldots, l,$$

where $\chi \in \mathscr{C}_0^\infty(U)$ is equal to one in a neighbourhood of K and $\partial^\lambda(\zeta, D_\zeta)$ are the operators of Proposition 2. In other words, $\tilde{d}.u$ is the composition of the Noetherian Operator d with the Fourier transform, applied to $p(D)((1-\chi)u)$, and then restricted to N_λ. Note that this definition does not depend on the choice of χ:

$$\begin{aligned}\partial^\lambda\left(p(D)((1-\chi_1)u)\right)\hat{}\,\Big|_{N_\lambda} &- \partial^\lambda\left(p(D)((1-\chi_2)u)\right)\hat{}\,\Big|_{N_\lambda}\\ &= \partial^\lambda p(\zeta)((\chi_2-\chi_1)u)\hat{}\,|_{N_\lambda} = 0\end{aligned}$$

by the defining property of ∂^λ, $\lambda = 0, 1, \ldots, l$.

The crucial point of the proof is to show the triviality of $\tilde{d}.u$. If the family $\{N_\lambda\}$ contains a component of dimension n, say N_0, then by [18, Chapter IV, §4, 3°, Proposition 1] we may take $\partial^0(\zeta, D_\zeta) = p_1(\zeta)$, where p_1 is the matrix defined by (1.1): clearly

$$\tilde{d}^0.u = p_1 \cdot \left(p(D)((1-\chi)u)\right)\hat{} = \left(p_1(D)p(D)((1-\chi)u)\right)\hat{} = 0.$$

That $\tilde{d}^\lambda.u = 0$ also for the remaining values of λ, will be shown in two steps. The first step (Lemma 1) will establish estimates on the growth of $\tilde{d}^\lambda.u$, $\lambda > 0$; the estimates will depend on the regularity of u outside K. The second step (Lemma 2) will show that, in case each N_λ has a real point at infinity, the obtained estimates can only be satisfied by $\tilde{d}.u = 0$. Having obtained this, the proof is easily concluded. By Proposition 2 $\tilde{d}.u = 0$ implies that $p(D)((1-\chi)u) = p(D)v$ for some $v \in \mathscr{E}'[K']^s$, where K' is the convex hull of supp χ and χ is any function allowed in the definition of the Grušin transform of u. Then $(1-\chi)u - v$ is in $\mathscr{D}'_M(U)$ and coincides with u on $U \setminus K'$. By the freedom of choice of χ the set K' is an arbitrary convex compact neighbourhood of K, hence $u = 0$ considered as an element of $\mathscr{D}'_M(U\setminus K)/\widetilde{\mathscr{D}'_m(U)}$. The mapping α' of Remark 1 has thus image zero. By the same remark, this is also true about the mapping α. The proof is thus complete except for the two crucial lemmas.

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LEMMA 1. *Let K' be a convex compact subset of U containing K in its interior. Then there exists a constant C such that*

$$(3.1)\qquad |\tilde{d}^{\lambda}.u(\zeta)| \leqslant C \cdot \exp\left(H_{K'}(\operatorname{Im}\zeta) - \log L\left(\frac{|\zeta|}{C}\right)\right), \qquad 1 \leqslant \lambda \leqslant l;$$

here $H_{K'}$ is the supporting function of K': $H_{K'}(x) = \sup\{\langle x, y\rangle : y \in K'\}$.

PROOF. Let $(\chi_k)_{k\geqslant 0}$ be a sequence of test functions with support in K', equal to one in a fixed open neighbourhood V of K, and such that

$$|D^{\alpha}\chi_k(x)| \leqslant C^{|\alpha|+1} \cdot k^{|\alpha|}, \qquad x \in R^n, \qquad |\alpha| \leqslant k,$$

with some constant C independent of k. (Such test functions are sometimes referred to as the Ehrenpreis cutoff functions. For their construction see for example [4].) By the Leibniz' rule

$$|D^{\alpha}((1-\chi_k)v)(x)| \leqslant C_E^{k+1} \cdot L_k, \qquad x \in E, \qquad |\alpha| \leqslant k,$$

for any compact set E on which v is a function of class $\mathscr{C}^L$ (recall that $L_k \geqslant k! \geqslant (k/e)^k$); cf. [4, Lemma 2]. Moreover,

$$(3.2)\qquad |D^{\alpha}q(x, D)((1-\chi_{k+m})v)(x)| \leqslant C_{E,q}^{k+1} \cdot L_k, \qquad x \in E, \qquad |\alpha| \leqslant k,$$

whenever $q(x, D)$ is a differential operator of order m with polynomial coefficients. Now, for any multi-index α and for any test function χ with support in K' and equal to one on the neighbourhood V of K, $\zeta^{\alpha} \cdot \partial^{\lambda}(\zeta, D_{\zeta})(p(D)((1-\chi)u))\hat{}$ is the Fourier transform of the $\mathscr{C}^{\infty}$-function $D^{\alpha}\partial^{\lambda}(D, -x)\,p(D)((1-\chi)u)$ with support in $K' \setminus V$. Hence

$$(3.3)\qquad \begin{aligned} &|\zeta^{\alpha}\partial^{\lambda}(\zeta, D_{\zeta})(p(D)((1-\chi)u))\hat{}(\zeta)| \\ &\qquad \leqslant \|D^{\alpha}\partial^{\lambda}(D, -x)p(D)((1-\chi)u)\|_{L^1(K'\setminus V)} \cdot \exp(H_{K'}(\operatorname{Im}\zeta)) \end{aligned}$$

for all $\zeta \in C^n$. Each component of $\partial^{\lambda}(D, -x)p(D)((1-\chi)u)$ is a finite sum of terms of the form $q(x, D)((1-\chi)v)$, q being a differential operator with polynomial coefficients and $v \in \mathscr{C}^L(U \setminus K)$. Each of these terms, upon choosing $\chi = \chi_{k+m}$ with m determined by the degrees of the operators $\partial^{\lambda}(D, -x)$ and $p(D)$, satisfies (3.2). Hence, observing that $K' \setminus V \subset\subset U \setminus K$, we have

$$(3.4)\qquad \|D^{\alpha}\partial^{\lambda}(D, -x)p(D)((1-\chi_{k+m})u)\|_{L^1(K'\setminus V)} \leqslant C^{k+1} \cdot L_k,$$

$|\alpha| \leqslant k$, with a constant C independent of k. By (3.3) and (3.4) we may conclude that

$$(3.5)\qquad \frac{|\zeta|^k}{C^k \cdot L_k} \cdot |\partial^{\lambda}(\zeta, D_{\zeta})(p(D)(1-\chi_{k+m})u)\hat{}(\zeta)| \leqslant C \cdot \exp(H_{K'}(\operatorname{Im}\zeta)),$$

$\zeta \in C^n$, $k = 0, 1, \ldots$, possibly with a new constant C. Since all the functions $\partial^\lambda(\zeta, D_\zeta)(p(D)((1 - \chi_{k+m})u))\hat{}$, $k \geqslant 0$, coincide on N_λ, we finally obtain (3.1) by taking the least upper bound of all the left sides in (3.5).

LEMMA 2. *Let $A \in R$, $C > 0$. If an analytic function f on C^n satisfies the estimate*

$$|f(\zeta)| \leqslant C \cdot \exp\left(A\,|\operatorname{Im}\zeta| - \log L\left(\frac{|\zeta|}{C}\right)\right) \tag{3.6}$$

on an irreducible algebraic variety V with a real point at infinity then f vanishes identically on V.

PROOF. By the assumption there exists a real vector ζ ($\neq 0$) in the Zariski cone of V. Let $\zeta_0 \in V$. Apply Proposition 1 and consider the composed analytic function $g = f \circ \gamma$ on Δ. By (2.2) and (3.6) we may estimate the growth of g:

$$\log|g(s)| \leqslant C_1 + A_1|\operatorname{Im} s| + B_1|s|^\rho - \log L\left(\frac{|s|}{C_1}\right)$$

for some constants A_1, B_1, C_1, $\rho < 1$, and all s in the, somewhat shrunk if necessary, set Δ. In particular, g is of exponential type (recall that $L \geqslant 1$). Hence, whenever $\tau \in R$ and $R + i\tau \subset \Delta$, we must have

$$\int_{-\infty}^{\infty} \frac{\log^+|g(\sigma + i\tau)|}{1 + \sigma^2}\,d\sigma < \infty. \tag{3.7}$$

Also, by (2.3),

$$\int_{-\infty}^{\infty} \frac{|\log|g(\sigma + i\tau)||}{1 + \sigma^2}\,d\sigma = \infty. \tag{3.8}$$

But, by [2, Theorem 6.3.6] the only analytic function of exponential type on a half-plane containing $R + i\tau$ which satisfies (3.7) and (3.8) is the trivial function. Hence $g \equiv 0$. We conclude that $f = 0$ on the open set $\gamma(\Delta)$ of Γ, and therefore, by the uniqueness of analytic continuation, $f = 0$ on the connected analytic manifold of regular points of Γ. The set of regular points is dense in Γ, hence $f = 0$ on Γ; in particular $f(\zeta_0) = 0$. Since ζ_0 was an arbitrary point of V, $f = 0$ on V.

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