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Helioseismology: the Sun as a strongly-constrained, weakly-coupled plasma

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Abstract

Accurate measurements of observed frequencies of solar oscillations are providing a wealth of data on the properties of the solar interior. The frequencies depend on the solar structure, and on the properties of the plasma in the Sun. Except in the very outer layers, the stratification of the convection zone is almost adiabatic. There, the sound-speed profile is governed principally by the specific entropy, the (homogenous) chemical composition and the equation of state. It is therefore essentially independent of the uncertainties in the radiative opacities. The sensitivity of the observed frequencies is such that it enables to distinguish rather subtle features of the equation of state. An example is the signature of the heavy elements in the equation of state. This opens the possibility to use the Sun as a laboratory for thermodynamic properties.

Les fréquences observées des oscillations solaires constituent une base de données extrêmement riche qui nous permet d'étudier les propriétés de l'intérieur du soleil. Les fréquences dépendent de la structure solaire et des propriétés locales du plasma (surtout de la vitesse du son). Sauf dans les couches très exterieures, la structure de la zone convective du soleil est essentiellement adiabatique. Le profil de la vitesse du son est donc donné par l'entropie spécifique, la composition chimique (homogène) et l'équation d'état. L'opacité radiative ne joue pas de rôle. Grace à la grande précision des fréquences observées on arrive à distinguer des phénomènes assez sub-

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tiles dans l'équation d'état, comme la signature faible des élements lourds. Le soleil est devenu un laboratoire de physique des plasmas stellaires.

17.1 Introduction

Solar acoustic oscillations have opened a new window into the Sun. By their nature they link the local sound speed in the interior with the observed oscillation frequencies. The spatial resolution of the solar disk allows the identification of a large number of individual oscillation modes, which are classified in terms of spherical harmonics. Modes in a large range of angular degrees, between $l = 0$ and a few thousand, are observed. The frequencies of these modes are centered around 3 mHz, which corresponds to periods around 5 minutes. They have been determined with high precision: typical relative errors are of the order of 10^{-4}. The modes are confined to a cavity, which extends, broadly speaking, from the surface of the Sun, where the waves lose their material support, to the inner turning point which lies deeper the lower the angular degree l is. Radial modes, with $l = 0$, have no inner turning point and their cavity is the entire Sun.

The observed solar oscillation modes are standing acoustic waves; hence the quantity most obviously probed is sound speed. Since the oscillations are largely adiabatic (except very near the surface), the frequencies are determined predominantly by the local adiabatic sound speed, which is a thermodynamic quantity. In addition, the frequencies depend on the density distribution in the Sun. Therefore, these *helioseismic* frequencies can be used as a diagnosis of the plasma of the solar interior. A high-quality thermodynamic potential is needed for the pressure-density relation (*i.e.* the equation of state, which is essential for determining the hydrostatic equilibrium between pressure gradient and gravity) and for thermodynamic quantities (mainly adiabatic sound speed).

Introductions to helioseismology are, for example, the reviews by Deubner & Gough (1984), Christensen-Dalsgaard, Gough & Toomre (1985), Bahcall & Ulrich (1988), Christensen-Dalsgaard (1988), Libbrecht (1988), Vorontsov & Zharkov (1989), Gough & Toomre (1991), Libbrecht & Woodard (1991), Christensen-Dalsgaard & Berthomieu (1991), Gough (1992), and Turck-Chièze *et al.* (1993). The reviews by Christensen-Dalsgaard (1991) and Christensen-Dalsgaard & Däppen (1992) specifically address the helioseismic determination of the equation of state.

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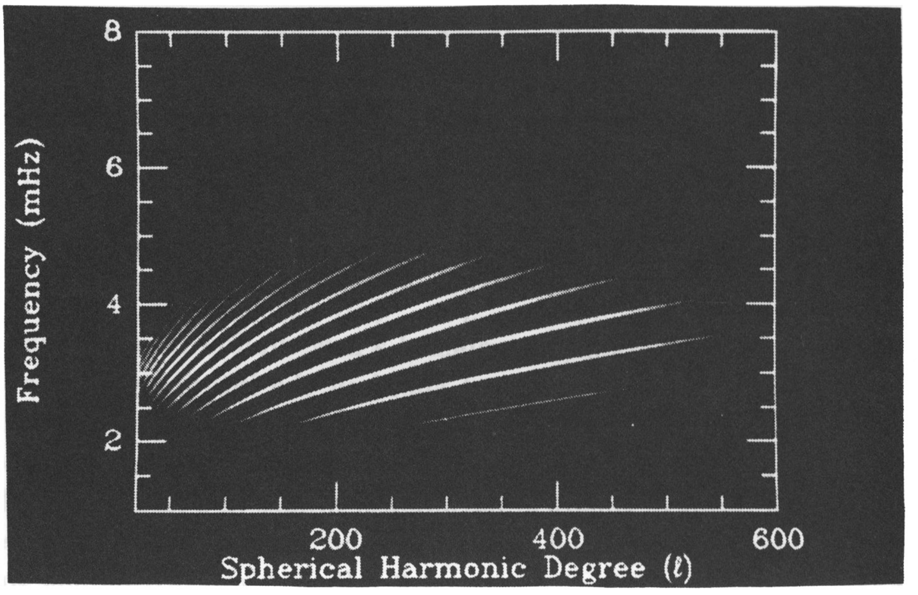

Fig. 17.1 Observed p-mode frequencies obtained from a 20-day sequence obtained at the 60-Foot Solar Tower of Mount Wilson Observatory.

17.2 Helioseismology: observations

After the discovery of the solar five-minute oscillations by Leighton *et al.* (1962), it took 15 years before they were recognized as global oscillations. Figure 1 shows a typical display of the helioseismic data. While early data looked extremely noisy, the observational progress made since has been tremendous, resulting in very clean data. In a typical representation of helioseismic data, the frequencies of all observed oscillation modes are plotted against their angular degree l. In general, for a given angular degree one observes more than one frequency. They belong to modes of different numbers n of radial nodes. If one plots the observed frequencies, those belonging to modes with the *same* number of radial nodes can be connected with smooth lines; this is true for *any* vibrating gas sphere. Figure 1 shows such a $\nu - l$ diagram obtained from current observations. Since modes with the same radial order n lie on the same ridge, one can therefore identify the radial order n with the different ridges of the diagram. Such an identification is possible up to an unkown global constant n_0. Duvall (1982) found a technique to resolve this remaining ambiguity and to identify the radial order uniquely.

In the mid-seventies, the ridges in the $\nu - l$ diagram began to emerge from the noise (Deubner, 1975); once they were seen, they definitely established the solar nature of the five-minute oscillations as a superposition of global oscillation modes, a suggestion made earlier by Ulrich (1969, 1970).

If the Sun were spherically symmetric, then each mode frequency ν_{nl} would be $2l+1$ times degenerate. The solar rotation (like any other non-spherical perturbation, such as, *e.g.*, magnetic fields) breaks this symmetry, thus each frequency is split into a multiplet. The splitting is small, since it is of the order of the angular frequency of the solar rotation, which has a period of a little less than a month. Therefore the rotational splitting is too small to be visible in a plot of absolute frequencies such as Figure 1. However, thanks to observational series of weeks and months, the splittings can be well observed for a wide range of l (see, *e.g.*, Harvey, 1988; Rhodes *et al.*, 1990).

Why did it take some 15 years before the oscillations were properly identified? The reason is that the oscillation velocities are tiny, less than $1\frac{\text{m}}{\text{s}}$. And yet, such velocities are observed using the Doppler effect of light. From each wing of a given spectral line, a narrow piece is cut out and sent through an interferometer into a comparator. The intensity difference of the two parts then becomes a measure of the Doppler line shift, and thus radial velocity. Since the solar disk can be well resolved, such measurements can nowadays be made typically for 1024×1024 pixels simultaneously, and this at a rate of a few times per minute.

Using the Doppler effect of light, velocities of the order of $1\frac{\text{m}}{\text{s}}$ are only marginally detectable. One might therefore wonder why one can obtain so clean a picture as in Figure 1. This question is even more in order if one considers the seemingly chaotic motion on the Sun, granulation, super-granulation, flares, rotation, and so on. The answer lies in the extreme regularity and the surprisingly long life time of the modes, which allow the observers to follow an individual oscillation mode for days and weeks. Therefore, the strict periodicity of the signal is exploited, so that in the end the frequencies can be determined very accurately against all initial odds.

The data like those of Figure 1 allow a high precision analysis of the structure of the solar interior. Tabulated frequencies are given in the article by Libbrecht *et al.*, 1990. The relative precision, with which each of the observed mode frequencies ν_{nl} is determined, now attains 10^{-4}, which is at least one order of magnitude better than the uncertainties of any current theoretical predictions. The reason for this inadequacy of the theoretical models is that they are not (yet) sufficiently sophisticated, because the usual simplifying assumptions on convection, opacity, equation of state, nuclear physics, internal rotation, and other physical ingredients are not good enough to explain all the details encountered in the seismological data.

17.3 Helioseismology: theory

Broadly speaking, theoretical inferences from the observed helioseismic frequencies can be made in two ways. In the *forward* approach, we build a solar model and compute its normal modes. Then the "best" model is the one that satisfies all observational constraints. Should there be more than one "best" model, an aesthetic principle such as Occam's razor is invoked to select the simplest of them. In the *inverse* approach, we try to make as few theoretical assumptions as possible to infer the physical state of the solar interior directly from the oscillation frequencies.

17.3.1 The general equations for evolution and oscillations

For tutorial purposes I will go somewhat off the beaten track and discuss the evolution and oscillations of the star at the hand of the *same* set of hydrodynamic equations. Of course, the time scales of evolution and oscillations are so much different that in *practical* calculations one always separates the two parts. Making here as many simplifications as I dare, I refer the interested reader to the superb book by Unno *et al.* (1989). The *solar* case is extensively dealt with, for instance, in the reviews by Christensen-Dalsgaard and Berthomieu (1990) and Turck-Chièze *et al.* (1993). I neglect viscosity, and assume that any treatment of turbulent motion, or convective heat transfer, is done in terms of a mean-field approach. This means that state variables are averaged over time-scales of turbulent motion. Such an approach is justified except in a thin layer beneath the solar photosphere. Under the assumptions of inviscid motion and mean-field variables, the resulting 9 equations are

$$\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\frac{1}{\rho}\nabla p - \nabla\phi \tag{1}$$

$$\left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{v})\right] = 0 \tag{2}$$

$$\left[\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s\right] = -\frac{1}{T}\epsilon_{nuc} - \frac{1}{\rho T}\mathrm{div}\mathbf{F} \tag{3}$$

$$\Delta\phi = 4\pi G\rho \tag{4}$$

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$$\mathbf{F} = \mathbf{K}(\nabla T; T, \rho, \mathbf{X}) \tag{5}$$

$$\epsilon_{nuc} = \epsilon_{nuc}(T, \rho, \mathbf{X}) \tag{6}$$

$$\kappa = \kappa(T, \rho, \mathbf{X}) \tag{7}$$

$$p = p(T, \rho, \mathbf{X}) \tag{8}$$

$$s = s(T, \rho, \mathbf{X}) \tag{9}$$

Here, $\mathbf{v}$ is the (Eulerian) velocity field, p and ρ are pressure and density, respectively, ϕ is the (self-) gravitational potential, s is specific entropy (per unit mass), ϵ_{nuc} is the nuclear energy generation rate, $\mathbf{F}$ is the energy flux (in the mean-field sense) through the star, the vector $\mathbf{X} = (X_1, X_2, ...X_n)$ symbolizes the chemical composition, with X_i being the mass fractions of element i, and κ is the opacity. Note the square brackets in Eqs. (1-3). They are there for the discussion of the separate issues of evolution and oscillation (see below).

Taking into account the vector nature of Eq. (1) and (5), Eqs. (1-9) are 13 equations for the 13 hydrodynamic fields $\mathbf{v}, p, \rho, \phi, T, s, \mathbf{F}, \epsilon_{nuc}, \kappa$. Eqs. (1-4) are partial differential equations, Eq. (5-9) are "material" equations, and it is no surprise that they are the hard part of the overall problem. The toughest among them is the expression of the "conductivity" for energy (Eq. 5), because it is the result of wholly different *physical processes* according to the physical conditions, given by $T, \rho, \nabla T$. Energy transport by radiation, convection and electron conduction are the most familiar ones. As long as stellar matter is optically thick (which it is except near the stellar surface), Eq. (5) can be simplified with the help of the diffusion approximation

$$\mathbf{F} = -\frac{16\sigma T^4}{3\kappa\rho}\nabla T \tag{10}$$

but when matter is optically thin, strictly speaking even the *form* of Eq. (5) is inappropriate, because then radiative transport becomes intrinsically non-local, and radiation hydrodynamic will have to be brought into the game (see the book by Mihalas and Mihalas, 1984).

Eq. (5) also needs some "switch" to change to convective form of energy transport when the local conditions do not warrant a stable radiative stratification. For the equilibrium model, one usually assumes a stability criterion *à la* Schwarzschild or Ledoux plus some mixing-length formalism (see *e.g.* Cox & Giuli, 1968; Gough & Weiss, 1976; Unno *et al.*, 1989). For the oscillation part, the interplay between convective and oscillatory motion can become very complicated. Compared with the question of energy transport (Eq. 5), the rest of the material equations (6-9) look relatively harmless. The most difficult among them is opacity (7), which appears in the diffusion approximation (10).

17.3.2 Evolution

Formally speaking, the problem of stellar evolution is the one of Eqs. (1-9) without the parts in the big square brackets, that is without the inertia term of Eq. (1) and the thermal term of Eq. (3). Just to illustrate with a familiar equation, note that in the approximation of a spherically symmetric configuration Eqs. (1) and (4) allow elimination of ϕ and become the well-known equation of stellar structure (see, *e.g.*, Schwarzschild, 1958).

$$\frac{dp}{dr} = -\frac{GM_r\rho}{r^2}. \tag{11}$$

Here, as usual, M_r denotes the mass of the sphere of radius r and G the constant of gravitation. Similarly, in the spherical approximation, Eq. (3) becomes (with $L_r = 4\pi r^2 F$)

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon_{nuc}, \tag{12}$$

and Eq. (5) [in the form of the special case of Eq. (10)] becomes the equally familiar equation of the radiative temperature gradient

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{64\pi\sigma r^2 T^3} \tag{13}$$

With the dynamical terms of Eqs. (1-3) thus stripped away, the system of equations becomes formally time-independent. Evolution only happens through the transmutation of the chemical elements and the associated change of elemental abundances, which are reflected in the material properties (5-9). However, by all means one should avoid the impression that this simplification to a sequence of slowly changing static models will make the problem of stellar evolution easy. In fact, the evolutionary part is the hard part, not only because of its nonlinear nature (in contrast to the oscillation problem, here no linearization is available). The complexity of stellar evolution is due both to the rich variety of physical phenomena contained in the material equations (5-9) and to the varying chemical composition in the star.

17.3.3 Oscillations

Having solved the hard problem of stellar evolution, we arrive at an equilibrium model (often assumed to be spherically symmetric). Again, the reader is referred to the book by Unno *et al.* (1989). Let $p_0, \rho_0, T_0, \phi_0, s_0, \mathbf{F}_0$ be the variables and $\mathbf{X}(r)$ the profile of the chemical composition that define this equilibrium model at some point t_0 in the star's evolution. Introduce the Eulerian displacement variables of the kind $p' = p - p_0$, $\rho' = \rho - \rho_0$, and so forth, and insert these difference variables into the original equations (1-9). We then obtain equations that look essentially the same, though now they are written for the displacement variables. As long as no linearization or other simplification is made, the new equations are of course equivalent to the old ones.

However, the purpose of going to displacements is to make approximations. The two most important are those of a *static* and *spherically stratified* equilibrium. The assumption of a static equilibrium precisely reflects the vast gulf between oscillation and evolution time-scale, an assumption certainly valid until the very violent final phases of the star's life. The other assumption, that of a spherical equilibrium state, is a working hypothesis, not bad if rotation and magnetic fields do not distort the equilibrium state too much. According to Noether's (1918) theorem, the two assumed underlying symmetries (time translations, rotations) lead to conserved quantities, but they show up nicely only in a linear theory. Let us thus *linearize* the whole system (1-9), that is, neglect all second and higher-order terms in the displacement quantities. In the material equations (5-9) the complicated functional behavior is greatly simplified by linear expressions that involve the equilibrium quantities $p_0, \epsilon_{nuc,0}, \kappa_0$ and their partial derivatives evaluated at the equilibrium values. Under these assumptions (static and spherical equilibrium and linearization), there are special solutions that are products of a radial amplitude with an exponential time dependence and a (possibly vector-) spherical harmonic function. If we consider the example of pressure we write

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$$p' = p'_{lm}(r)e^{i\omega_l t}Y_l^m(\theta,\phi). \tag{14}$$

There are analogous expressions for all other variables. The general solution is a superposition of such particular solutions.

If we assume that thermal heat losses are negligible small during the fast oscillations, then we deal with *adiabatic* oscillations. Drastic simplifications become possible. The whole equation (3) disappears, because the left-hand side vanishes identically, showing that the equilibrium condition is exactly preserved (this statement is true even in the nonlinear case. In the absence of equation (3), temperature, energy flux and opacity do not participate in the oscillation equation, though they are of course important in the equilibrium part. Thus equations (5-7) are also gone. Thermodynamics becomes ultra simple, especially in the linear case, where the co-moving Lagrangian pressure and density fluctuations (δp and $\delta\rho$) are simply related through the equilibrium adiabatic gradient $\Gamma_1 = (\partial \ln p/\partial \ln \rho)_s$

$$\delta p = \delta\rho\Gamma_1 \frac{p}{\rho} \ . \tag{15}$$

In the *nonlinear* adiabatic case, this simple relation would have to be replaced by the function $p(\rho)$ that follows from the integral of motion s, *i.e.* from the implicit equation $s = s(p, \rho, \mathbf{X}) = \text{const}$.

A further simplification of the adiabatic problem is that the equation of continuity permits expressing the tangential component of the displacement field in terms of the pressure fluctuation (for details see Unno *et al.*, 1989). We thus arrive at the famous adiabatic eigenvalue problem of stellar oscillations, which plays a central role in helioseismology. The result is, for each angular degree l, an eigenvalue problem, which consists of coupled equations for the radial amplitudes of the displacement vector ξ_r, the fluctuation of pressure p' and of the gravitational field ϕ, and of the usual boundary conditions at the center and the surface of the star. For *radial* oscillations or in the so-called *Cowling approximation* for nonradial oscillations (where one neglects changes of the gravitational field during the oscillatory motion), the equations become especially simple. Their formal type is (again, see Unno *et al.*, 1989, for details)

$$\frac{dp'}{dr} = Ap' + B\xi' + C$$

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$$\frac{d\xi'}{dr} = Dp' + E\xi' + F \tag{16}$$

The Coefficients A, B, C, D, E, F are not constant but functions of the radius. They contain the properties of the equilibrium model and, most importantly, the *eigenvalue* ω_l. As mentioned, boundary conditions complement the equations (16). Thus the problem of adiabatic stellar oscillations is, for each l, completely analogous to an inhomogeneous vibrating string. For each l there is a set of solutions with different radial nodes n and frequencies ω_{nl}.

It should be clear by now that the solution of the eigenvalue problem is much easier than finding the equilibrium model through stellar evolution. Of course one can make things more complicated here as well. By considering *nonadiabatic* motion, the energy equation (3) is coming back, and with it temperature, which forces us to bring in ϵ_{nuc}, κ, and convection. Again, I refer to the book by Unno *et al.* (1989). Near the solar surface there are nonadiabatic effects that have to be treated properly before the theoretical data will match the observations shown in Figure 2.

17.3.4 Inverse analysis

If one writes Eq. (16) very formally, the oscillation frequencies ω_{nl} can be written as functionals

$$\omega_{nl} = \mathcal{F}_{nl}\left[\rho(r),\ p(r), \ldots\right] \tag{17}$$

of the structure of the Sun. So far we have discussed how to obtain the frequencies, given the structure. With the ability to do so, one can compare observed frequencies with computations based on different models, and in this way obtain some information about the solar structure. However, it is evidently desirable to attempt to invert the process, to obtain more extensive information about the properties of the solar interior from the observed frequencies. Such *inverse analyses* are, in a certain sense, implicit in any type of scientific measurement, since a raw measurement rarely supplies the quantity that one is interested in. However, in the present case the relation between the desired properties of the Sun, *e.g.* $\rho(r)$, and the observed quantities is more complex, since each frequency is sensitive to the structure of a substantial part of the Sun; thus the inverse problem is correspondingly more difficult. Similar problems are encountered in other branches of science, such as geophysics and radiation theory, and there is a substantial literature dealing with them (*e.g.* Parker 1977; Deepak 1977; Craig & Brown 1986; Tarantola 1987).

An alternative method of inversion is based on *asymptotic* theory, where local propagation properties for acoustic waves are approximately examined in the spirit of a JWKB analysis. The need for such an approximate discussion comes from the fact that, although the numerical solution of the equations of adiabatic oscillations is relatively simple, it does not immediately provide an understanding of the properties of the oscillations. Such a direct understanding can come from the approximate asymptotic analysis It was shown by Gough (*cf.* Deubner & Gough 1984; Gough 1986) how to write down an approximate form of the oscillation equations, from which it is straightforward to obtain the asymptotic behavior of the solution. It turned out immediately that this asymptotic approach also opens the door for elegant asymptotic inversion methods. I refer the reader to the papers by Gough (1985), Thompson (1991), Gough & Thompson (1991), Brodsky & Vorontsov (1993), and Gough & Vorontsov (1993). The last two papers deal with a *nonlinear* asymptotic inversion. The power of such inversions for the equation of state is illustrated in the article by Vorontsov *et al.* (*these proceedings*).

17.4 Comparison of theory with observations

The most direct way to compare theory and observation is to compute the analogue of Fig. 1 with the forward techniques mentioned above, so that the difference between each observed and computed frequency can be taken. Figure 2 shows four such diagrams of frequency differences, each for a different theoretical model. Two equations of state and two different opacity tables were used in the models. The two equations of state were (i) the Eggleton, Faulkner & Flannery (1973) (EFF) equation of state and (ii) the CEFF equation of state, which is, as explained below, an EFF plus a Coulomb term (Christensen-Dalsgaard, 1991; Christensen-Dalsgaard & Däppen, 1992). The opacities used were the Cox and Tabor (CT) (1976) and Los Alamos Opacity Library (LAOL) tables. Since I merely want to illustrate the sensitivity of the helioseismic method, it doesn't matter that these opacities are not the most current ones. A recent calculation based on Livermore opacitites can be found in Berthomieu *et al.* (1993a).

I remark in passing that in such comparisons of observed with computed data ("O–C diagrams"), it is useful if an appropriate *scale factor* is taken out (see, *e.g.*, Christensen-Dalsgaard 1988; Christensen-Dalsgaard

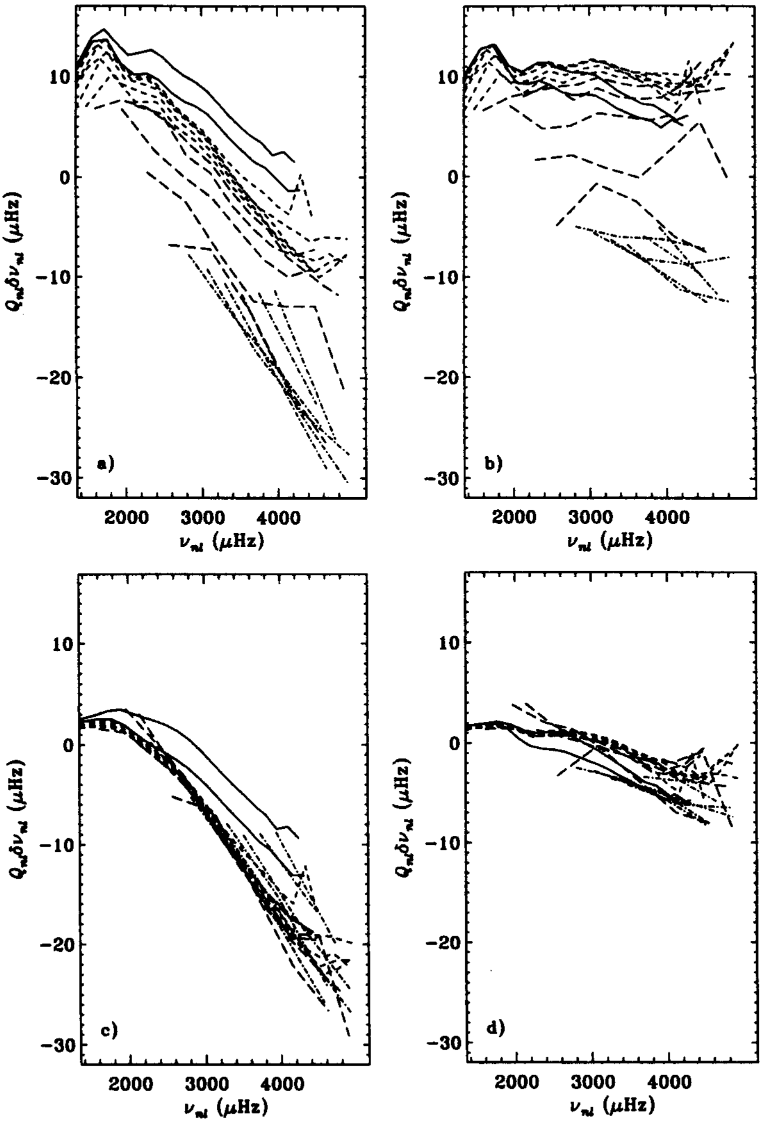

Fig. 17.2 Frequency differences, scaled by the factor Q_{nl} (see text), between observed frequencies in the compilation by Libbrecht *et al.* (1990) and four sets of computed frequencies, in the sense (observation) − (theory). The abscissa is cyclic frequency ν_{nl}. The points have been connected with lines according to the value of the degree l: $l = 20, 30$: ————; $l = 40, 50, 60, 80, 100$: ············; $l = 120, 150, 200, 300, 400$: — — — — —; and $l = 500, 600, 700, 800, 900, 1000$: —·—·—·—. The models are distinguished by their equation of state and opacity (a) EFF equation of state, CT opacity; (b) EFF equation of state, LAOL opacity; (c) CEFF equation of state, CT opacity; (d) CEFF equation of state, LAOL opacity (from Christensen-Dalsgaard & Däppen, 1992).

& Berthomieu 1991). This scale factor Q_{nl}, which is essentially the inertia - or kinetic energy - of the mode with quantum numbers nl (normalized to the same surface amplitude), contains the principal l and frequency dependence of the individual mode frequencies ν_{nl}.

The purpose of the illustration in Fig. 2 is to show the sensitivity of the helioseismic analysis with respect to changes in the physics of the model. A perfect model would yield a horizontal line corresponding to all $\delta\nu_{nl} = 0$. Note that the discrepancies between theory and observation are huge compared to the observational errors which are nowadays significantly below 1 μHz. Such a combination of quantity and quality of astrophysical data is truly exceptional.

17.5 The equation of state

As we have seen, the three basic material properties required in stellar models are the equation of state, opacity, and the nuclear-energy generation rate. At this meeting, the focus is on the equation of state. I shall use the term equation of state in a slightly broader sense than usual, so that it encompasses not only pressure as a function of temperature and density, but also all thermodynamic quantities. These quantities must be consistent with each other, that is, their appropriate Maxwell relations have to be satisfied. Such *formal* consistency is always achieved if the equation of state and the thermodynamic quantities stem from a single thermodynamic potential. In trivial models (*e.g.* in a plasma assumed to be fully ionized everywhere) it is possible to write down a consistent equation of state and thermodynamic quantities independently. However, in more realistic cases, modeling a *thermodynamic potential* is the only practical way to obtain the equation of state and thermodynamic quantities.

A quick glance at Fig. 2 reveals that solar observations are indeed very sensitive to details of the equation of state. One might go further and conclude that the Sun prefers the CEFF to the EFF equation of state. However, such conclusions are fraught with danger, although probably not in this clear-cut case. The reason why one has to be prudent is that there are too many uncertainties in the solar model, coming, *e.g.*, from convection or opacity, so that one has to be alert to the possibility that by changing the equation of state one could trigger changes in the other physical parameters. An illustration for this is found at each railroad crossing in France, where a sign warns: "un train peut en cacher un autre" (which, applied to our situation, means: proceed with caution, watch out for a hidden train of thought). If, say, the opacity is bad, one can not rule out that a *worse* equation of state could cause an overall better agreement with observations. Only when simultaneous progress with the other physical quantities is made (that is, if someone is watching the other track, to use the train metaphor), we will learn how to disentangle the different effects. However, for a sensitivity analysis, Figure 2 is already sufficient. The transitions from panels *a* to *c* and *b* to *d*, respectively, are obtained by putting some additional nonideal effects (the Coulomb pressure) into the equation of state *with everything else unchanged*. The response of the Sun, as seen through the "eyes" of helioseismology, is huge.

I will not elaborate how the equation of state is modeled. Several authors of these proceedings do it (Rogers, Alastuey, Saumon, and Chabrier). My message is different: I intend to show why there is still a long way to go before rigorous theories (for instance that presented by Alastuey, *these proceedings*) can be used in solar and stellar models. I will begin with requirements for any solar or stellar equation of state. I insist that *formal* aspects (such as consistency and smoothness) play a crucial role. As a consequence, I would like to raise sympathy for the many home-grown formalisms that stellar modelers have been constantly developing. Then I will discuss the nonideal plasma effects that have to be included in realistic solar equations of state. Finally I will present a few selected results from equation of state comparisons. In the absence of a perfect equation of state, the comparisons can give us at least important information about the amount of the current uncertainty in the equation of state. Also, it will tell us at which temperatures and densities the uncertainty is most noticeable and to what degree solar observations can discriminate between various models.

17.5.1 Requirements on an equation of state for stellar models

A stellar equation of state has to satisfy four conditions: (i) a large domain of applicability (in ρ, T), (ii) a high precision of its numerical realization, (iii) consistency between the thermodynamic quantities, and (iv) the possibility to take into account relatively complex mixtures with at least several of the more abundant chemical elements. More specifically, the first condition demands that the formalism can be used from the stellar surface (the photosphere), where T is typically a few 10^3 K and ρ some 10^{-7} g/cm^3, to the center of a star where T is, again typically, about 10^7 K and ρ some 10^2 g/cm^3. The second condition demands that a given formalism can be cast in an algorithm that converges without ambiguity and with sufficient precision, so that all required thermodynamic derivatives (such as adiabatic gradients) can be computed. Note, that for this only *formal* precision is required: reality of the physical description is a different issue. The third condition, consistency, states that all thermodynamic quantities stem from a single thermodynamic potential. This condition is often violated in two- or more-zone formalisms, which contains a different physical theory in different parts of a star. An example is the *ad hoc* imposition of full ionization in the central region, in order to mimic a pressure-ionization device, in combination with a conventional Saha equation in the envelope of the star. Such a formalism leads to a discontinuous thermodynamic potential and a violation of thermodynamic identities.

Such violations of thermodynamic identities are inadmissible in calculations of stellar structure and oscillations. As we have seen, calculations of stellar oscillation frequencies often exploit thermodynamic quantities to transform one variable into another. Equation (15) shows such a transformation. There the adiabatic gradient Γ_1 is used to establish a connection between density and pressure changes, and it is an absolute necessity that the Γ_1 is consistent with the equation of state and other thermodynamic variables of the model. This example illustrates the necessity of formal consistency. Finally, the third and last condition, *i.e.* the possibility to describe rather realistic chemical compositions, is a bit less important for the equation of state itself. However, for opacity, heavy elements are very important, and a good equation of state plays an important role in any opacity calculation.

17.5.2 The role of the solar convection zone

Energy transport by radiation is treated adequately in the solar interior in the diffusion approximation; on the other hand, energy transport by convection is usually treated in a rather crude way, with an *a priori* unknown parameter, the so-called mixing length (see, *e.g.*, Cox and Giuli, 1968). Near the surface, convection is probably sufficiently vigorous to cause dynamic effects on the average hydrostatic equilibrium, yet such effects are often ignored. At the lower boundary of the convection zone, motion is normally supposed to stop at the point where convective instability ceases; there is no doubt, however, that motion extends into the convectively stable region through convective overshoot, although the extent of the overshoot is uncertain (see, *e.g.* Berthomieu *et al.*, 1993b).

Despite the complications it introduces, in a certain sense convection simplifies the structure of the outer parts of the Sun. Regardless of the uncertain details of convective energy transport, there is no doubt that except in a thin boundary layer near its top the convection zone is very nearly adiabatically stratified (*e.g.* Gough & Weiss 1976). One can show (Christensen-Dalsgaard, 1986) that the structure of the almost adiabatically stratified convection zone only depends on the equation of state, the composition and the constant value of the specific entropy, which in turn is essentially fixed by the value of the mixing-length parameter; particular, the convection zone structure is insensitive to the opacity. Another simplification of convection is that it makes the chemical composition homogeneous in the convection zone, although there is of course the possibility of gravitational settling (for a recent calculation, see Christensen-Dalsgaard *et al.*, 1993).

Beneath the convection zone, the stratification becomes highly dependent on radiative opacity. It is difficult to disentangle the helioseismic effects of equation of state and opacity, but if opacity can be nailed down relatively accurately, an equation of state diagnosis can also become possible. An examlpe of an equation of state issure is the possibility of partial recombination of He^+ ions in the solar center (see Christensen-Dalsgaard & Däppen, 1992).

17.6 Equation of state comparisons

The most direct way to test the equation of state would be laboratory experiments. However, so far they have not yet helped to check realistic stellar equations of state. For instance, attempts to use constraints from a high-precision optical emission spectrum (*e.g.* Wiese, Paquette & Kelleher, 1973) have failed, because line-broadening effects were overshadowing the subtle details of statistical mechanics. It is therefore no wonder that - despite their difference in statistical mechanics - several of the currently popular equations of state have been able to reproduce that optical experiment (Däppen, Anderson and Mihalas, 1987; Seaton, 1990; Iglesias & Rogers, 1992).

An alternative "experimental" approach is to use solar oscillation data. As the comparisons between observed and theoretical solar oscillation frequencies (Fig. 2) demonstrate, one can use the Sun to test the equation of state (for more details, see Christensen–Dalsgaard, Däppen & Lebreton, 1988; Christensen–Dalsgaard, 1991; Christensen–Dalsgaard & Däppen, 1992). Inversions of solar oscillation frequencies, such as those presented by Vorontsov *et al.* (*these proceedings*), have also demonstrated a high diagnostic potential for subtle effects, such as the location of the pressure-ionization region of helium and the influence of heavy elements in the equation of state. The disadvantage of a solar diagnosis is of course that we cannot vary the parameters: we have to accept solar conditions as they are. Only asteroseismology carries the promise to overcome this handicap (for a recent review, see *e.g.*, Christensen-Dalsgaard, 1993).

In the absence of a rigorous computation of the equation of state (to the needed accuracy), one can make comparisons between different models of the equation of state. Such comparisons will give us information about the overall uncertainty in the equation of state. But they also allow solar physicists to determine how uncertainties in the equation of state propagate into theoretically predicted oscillation frequencies. In this way, a "map" of the $T-\rho$ plane can be drawn, showing localized "interesting" regions, where nonideal effects of one or another kind are important.

I will briefly present the equations of state used in the comparisons. More details about them (and further references) can be found in the article by Christensen-Dalsgaard & Däppen (1992). I just recall that all currently used stellar equations of state can be classified in terms of the so-called "chemical picture" and the "physical picture" (Krasnikov, 1977). While in the more conventional chemical picture bound configurations (atoms, ions and molecules) are introduced and treated as new and independent species, only *fundamental* particles (electrons and nuclei) appear in the physical picture. In the chemical picture, reactions between the various species occur, and thus the thermodynamic equilibrium must be sought among the stoichiometrically allowed set of concentration variables by means of a maximum entropy (or minimum free-energy) principle. In contrast, the physical picture has the aesthetic advantage that there is no need for a minimax principle; the question of bound states is dealt with implicitly through the Hamiltonian describing the interaction between the fundamental particles. For exhaustive treatments of these issues, consult the three books by Ebeling, Kraeft & Kremp (1976), Kraeft *et al.* (1986), Ebeling *et al.* (1991).

17.6.1 EFF

Eggleton, Faulkner & Flannery (1973) developed a simple equation of state in the chemical picture (EFF) that is formally consistent and includes an *ad hoc* pressure ionization device that works at least qualitatively correctly. The device is not based on a physical model (*e.g.* a description of an atom and its surrounding particles), but is imposed by forcing the anticipated result, *i.e.*, full ionization at high densities. In addition, the EFF equation of state incorporates a correct treatment of the partially degenerate electrons according to Fermi-Dirac statistics. Bound systems (atoms and ions) are always assumed to be in their ground state; the ground-state energy is constant and equal to the free-particle value.

17.6.2 CEFF

To overcome the lack of a Coulomb term in the EFF equation of state, Jørgen Christensen-Dalsgaard and I have added a Coulomb configurational term in the Debye-Hückel approximation (taken from the MHD equation of state). Such an upgrade of the EFF equation of state was motivated by the fact that adding a Coulomb term to the EFF equation of state makes a significant contribution towards a more realistic equation of state (see below and the papers by Christensen-Dalsgaard, 1991; Christensen-Dalsgaard & Däppen, 1992). Of course the remaining disadvantages of the EFF equation of state still point to the need of more complete formalisms. However, the successful application of the CEFF equation of state to solar physics makes it very well suited as a reference equation of state.

17.6.3 MHD

The Mihalas-Hummer-Däppen (MHD) equation of state (Hummer & Mihalas, 1988; Mihalas *et al.*, 1988; Däppen *et al.*, 1988) is realized in the chemical picture with the free-energy minimization method. Occupation probabilities are introduced on the one hand to avoid the famous (or rather notorious) discontinuities that come along with simple cut-off recipes for internal partition functions. On the other hand they represent a result that should come from quantum mechanics, namely the fraction of atoms or ions for which a given state can exist (given the constraints of the surrounding particles). Only then, these "available" states are populated according to statistical mechanics. It is clear that such an approach is largely intuitive. However, its advantage is that complicated plasmas can be modeled, with *detailed* internal partition functions for a large number of atomic, ionic, and molecular species. All particles are allowed to interact with each other. Also, full thermodynamic consistency is assured by analytical expressions of the free energy and its first- and second-order derivatives. This not only allows an efficient Newton-Raphson minimization, but, in addition, the ensuing thermodynamic quantities are of analytical precision and can therefore be differentiated once more, this time numerically. Reliable third-order thermodynamic quantities are thus calculated. The MHD equation of state was realized for the international "Opacity Project" (see Seaton, 1987).

17.6.4 OPAL

The OPAL equation of state is realized in the physical picture. A detailed presentation is given by Rogers (*these proceedings*). In the physical picture, the concept of a perturbed atom in a plasma is not needed at all. Therefore, no assumptions about energy-level shifts or the convergence of internal partition functions have to be made. On the contrary, properties of energy levels and the partition functions come out from the formalism. The OPAL equation of state was developed by a group at Livermore as part of their opacity project (Rogers, 1986; Iglesias, Rogers & Wilson, 1987; Rogers, *these proceedings*). This equation of state does satisfy the requirements from stellar modelling that I mentioned above; however, a systematic application application of the OPAL equation of state to helioseimology is still awaiting.

17.6.5 Results from the comparisons

Early comparisons showed a striking agreement between the MHD and OPAL equation of state for conditions as found in the hydrogen-helium ionization zones of the Sun (Däppen, Lebreton & Rogers, 1990; Däppen, 1990). For convenience, a representative result from this early comparison is shown in Figure 3, which compares the MHD and OPAL results with that of the simple EFF formalism (which is essentially a consistent ground-state-only Saha equation of state under these conditions). The absolute curves of part *a* of Figure 3 are merely able to show the difference between MHD (or OPAL) and the simple EFF results. To see the difference between the MHD and OPAL results, one needs the magnified part *b*, which shows the *relative* differences between MHD and EFF, and between OPAL and EFF values, respectively. This relative plot now not only allows one to see the difference between MHD and OPAL results, but also their striking similarity.

Later, it turned out that this agreement was nearly accidental. The reason for this was found by varying the parameters of the MHD equation of state. It followed that on the chosen isochore, all thermodynamic quantities are dominated by the Coulomb pressure correction (Däppen, 1990; Christensen-Dalsgaard, 1991; Christensen–Dalsgaard & Däppen, 1992). The Coulomb correction overshadows the effect of the excited states (which are of course treated differently in the MHD and OPAL approach). Note that the Coulomb term acts directly and indirectly, at least in the language of the chemical picture, because it is not mainly the free-energy of the Debye-Hückel term itself, but rather also the Coulomb-induced shift in the ionization equilibrium, which is responsible for the deviation from the unperturbed EFF result.

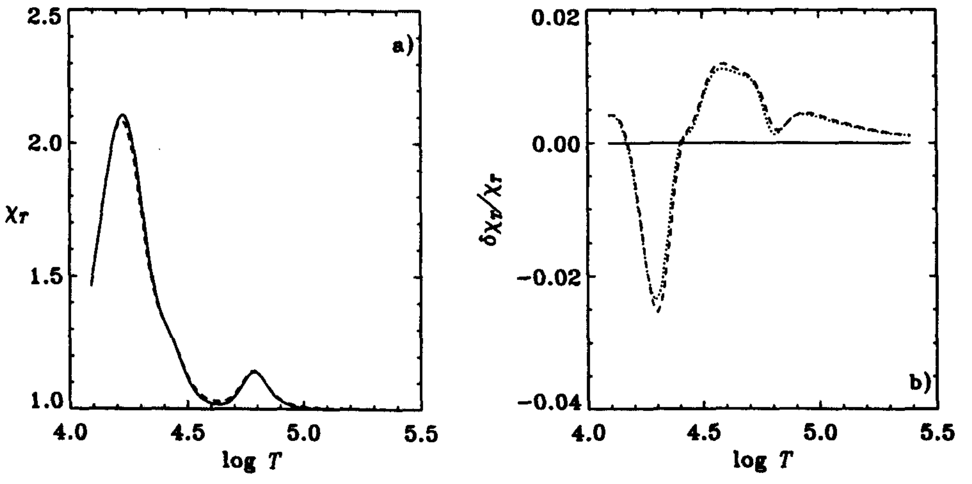

Fig. 17.3 Comparison of $\chi_T = (\partial \ln p/\partial \ln T)_\rho$ for $\rho = 10^{-5.5}$ g cm^{-3}. Absolute quantities (a) and relative differences (with respect to EFF) (b) are shown. See text for more details.

Of course, solar physicists were happy that two completely different formalisms delivered the same equation of state, but, by the same token, a first attempt to use the Sun as an equation-of-state test was also thwarted. This discovery suggested to upgrade the simple EFF equation of state with the help of the Coulomb interaction term. The resulting equation of state (called CEFF) has become a useful tool for solar physics (Christensen–Dalsgaard, 1991; Christensen–Dalsgaard & Däppen, 1992); at the same time, however, it became also clear that a helioseismic test of the important issue of chemical versus physical picture would be more difficult than first thought.

For reasons not yet fully understood it seems that in the chemical picture, the signature of internal partition functions, such as those employed in the MHD equation of state, is much less visible in the thermodynamic quantities than a naive estimation of the shift in the ionization equilibrium would predict. It is likely that there are accidental cancellations in the derivatives of the free energy. The cancellations of partition-function effects in the chemical picture seem to be greatest for the ionization zone of hydrogen and somewhat less for those of helium. A more recent comparison of MHD

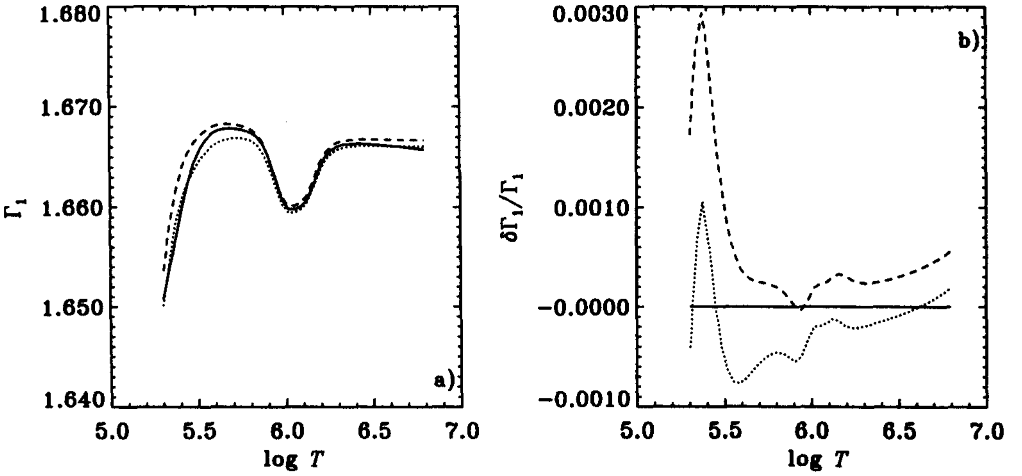

Fig. 17.4 Γ_1 for $\rho = 5.00 \times 10^{-3}$g cm^{-3} and a representative solar mixture of H, He, and O. Parts (a) and (b) as in Fig.3, but here with CEFF instead of EFF. See text for more details.

and OPAL values (Däppen, 1992) has examined selected cases of higher densities (where sizeable discrepancies appear) and a first case of a mixture involving a representative solar heavy element (oxygen). It appears that for the heavier elements, the internal partition functions finally lead to the intuitively expected consequences for the thermodynamic quantities.

Figure 4 shows the result of this comparison with oxygen for the quantity Γ_1. Density was chosen as $\rho = 0.005$ g cm^{-3}, a value suggested by a helioseismic study of the solar helium abundance (Kosovichev *et al.*, 1992). Here, not only do the large MHD partition functions cause shifts in the ionization balance but these shifts also significantly propagate into the thermodynamic quantities. The effect is large enough so that it appears, despite the small relative number of the heavy elements in the mixture, to be within reach of helioseismology (for more details see Christensen-Dalsgaard & Däppen, 1992; Däppen *et al.* (1993)).

To examine the MHD ionization fractions, a single case was examined ($T = 2.10 \times 10^5 K, \rho = 5.00 \times 10^{-3}$g cm^{-3}), once with the full MHD equation of state, once with a "stripped-down" version of MHD, which does not contain any excited states (but is otherwise identical). The resulting ionization fractions of O^{3+}, O^{4+}, O^{5+} were, respectively, 0.314, 0.248, 0.364 for the stripped-down MHD (without excited states), and 0.304, 0.476, 0.182 for the full MHD. (The result for the stripped-down very closely reflects the ground-state weights of the ions). Not unexpectedly in view of the Planck-Larkin partition function (see Rogers, *these proceedings*), the OPAL equation of state predicts ionization fractions close to those of the stripped-down MHD equation of state (Rogers, *private communication*).

This comparison for the first time establishes a clear case of disagreement between the MHD and OPAL results. Clearly, the origin of the discrepancy in the ionization degrees is due to the treatment of the excited states. Of course, only some 2 percent of the matter in the Sun consist of elements heavier that H and He, and therefore the signature of the MHD-OPAL discrepancy in Γ_1 (Figure 4) is small (of the order of 10^{-3}). Nevertheless, as has been demonstrated by Christensen-Dalsgaard & Däppen (1992), even the resulting tiny sound-speed differences are within reach of a helioseismic diagnosis.

17.7 Conclusions

Even weakly-coupled plasmas can pose tough problems if high accuracy is demanded. Solar oscillations are an example of a case where the present observational material is much better than the theoretical models. The solar convection zone is especially well suited for a study of the equation of state. It was suggested in a number of early papers (*e.g.* Berthomieu *et al.*, 1980; Ulrich, 1982; Shibahashi *et al.*, 1983, 1984) that improvements in the equation of state can reduce discrepancies between theory and observations. Later, Christensen-Dalsgaard, Däppen & Lebreton (1988) showed that the MHD equation of state significantly reduced these discrepancies for a large range of oscillation modes. Since the MHD equation of state simultaneously incorporates several different types of non-ideal corrections, it did not become immediately clear which one of these corrections was mainly contributing to this success.

¿From selected comparisons of the MHD with the OPAL equation of state, it turned out, rather surprisingly, that the net effect of the hydrogen and helium bound states on thermodynamic quantities was to a large degree eclipsed beneath the influence of the Coulomb term, which was thus recognized as the dominant non-ideal correction in the hydrogen and helium ionization zones. This discovery led to an upgrade of the simple EFF equation of state through the inclusion of the Coulomb interaction term (CEFF).

However, for the heavier elements it appears that, in the chemical picture, the internal partition functions finally lead to the expected consequences for the thermodynamic quantities. The heavy elements can thus become the ideal testing ground for the effects of bound states in partially ionized plasmas. The small abundance of heavy elements in the Sun will make a diagnosis difficult and stretch the power of helioseismology to its limits, but as the study by Vorontsov *et al.* (*these proceedings*) shows, there are encouraging signs that the difficulties can be overcome.

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References

Alastuey, A., *these proceedings*

Bahcall, J.N. & Ulrich, R.K., *Rev. Mod. Phys.* **60**, 297-372 (1988)

Berthomieu, G., Cooper, A.J., Gough, D.O., Osaki, Y., Provost, J. & Rocca, A., in *Lecture Notes in Physics*, Vol. **125**: Nonradial and Nonlinear Stellar Pulsation, eds Hill, H.A. & Dziembowski, W., Springer, Berlin, 307-312 (1980)

Berthomieu, G., Provost, J., Morel, P., Lebreton, Y., *Astron. Astrophys.* **268**, 775-791 (1993a)

Berthomieu, G., Morel, P., Provost, J., Zahn, J.-P., in *Proc. IAU Symposium No 137: Inside the Stars*, eds. Werner W. Weiss and Annie Baglin, ASP Conference Series, Vol. 40, 60-62 (1993b)

Brodsky, M. & Vorontsov, S.V., *Astrophys. J.*, **409**, 455 (1993)

Chabrier, G., *these proceedings*

Christensen-Dalsgaard, J., in *Seismology of the Sun and the distant stars*, ed D.O. Gough, Reidel, Dordrecht), 23-53 (1986)

Christensen-Dalsgaard, J., in *Seismology of the Sun & Sun-like Stars*, eds Domingo, V. & Rolfe, E.J., ESA SP-286, Noordwijk, The Netherlands, 431-450 (1988)

Christensen-Dalsgaard, J., In *Lecture Notes in Physics*, Vol. **388**: Challenges to Theories of the Structure of Moderate-mass Stars, eds Gough, D.O. & Toomre, J., Springer, Heidelberg, 11-36 (1991)

Christensen-Dalsgaard, J., in *Proc. IAU Symposium No 137: Inside the Stars*, eds. Werner W. Weiss and Annie Baglin, ASP Conference Series, Vol. 40, 483-496 (1993).

Christensen-Dalsgaard, J. & Berthomieu, G., in *Solar Interior and Atmosphere*, eds Cox, A.N., Livingston, W.C. & Matthews, M., Space Science Series, University of Arizona Press, Tucson, 401-478 (1991)

Christensen-Dalsgaard, J. & Däppen, W., *Astron. Astrophys. Review* **4**, 267–361 (1992)

Christensen-Dalsgaard, J., Däppen, W. & Lebreton, Y., *Nature* **336**, 634-638 (1988)

Christensen-Dalsgaard, J., Gough, D.O. & Toomre, J., *Science* **229**, 923-931 (1985)

Christensen-Dalsgaard, J., Profitt, C.R. & Thompson, M.J., *Astrophys. J.* **403**, L75 (1993)

Cox, A.N. & Tabor, J.E.: 1976, *Astrophys. J. Suppl.* **31**, 271-312 (1985)

Cox, J.P., and Giuli, R.T., *Principles of Stellar Structure*, Gordon and Breach, New York, 1968

Craig, I.J.D. & Brown, J.C., *Inverse Problems in Astronomy: a Guide to Inversion Strategies for Remotely Sensed Data*, Adam Hilger, Bristol (1986)

Däppen, W., in *Lecture Notes in Physics*, Vol. **367**: *Progress of Seismology of the Sun and Stars*, eds. Osaki, Y. & Shibahashi, H., Springer, Berlin, 33-40 (1990)

Däppen, W., in *Astrophysical Opacities*, eds. C. Mendoza & C. Zeippen (*Revista Mexicana de Astronomía y Astrofísica*) 141-149 (1992)

Däppen, W., Anderson, L.S. & Mihalas, D., *Astrophys. J.* **319**, 195-206 (1987)

Däppen, W., Lebreton, Y. & Rogers, F., *Solar Physics* **128**, 35-47 (1990)

Däppen, W., Mihalas, D., Hummer, D.G. & Mihalas, B.W., *Astrophys. J.* **332**, 261-270 (1988)

Däppen, W., Gough, D.O., Kosovichev, A.G. & Rhodes, E.J., Jr., in *Proc. IAU Symposium No 137: Inside the Stars*, eds. Werner W. Weiss and Annie Baglin, ASP Conference Series, Vol. 40, 304-306 (1993).

Deepak, A., *Inversion Methods in Atmospheric Remote Sounding*, Academic Press, New York (1977)

Deubner, F.-L., *Astron. Astrophys.* **44**, 371 (1975)

Deubner, F.-L. & Gough, D.O., *Ann. Rev. Astron. Astrophys.* **22**, 593-619 (1984)

Duvall, T.L., *Nature* **300**, 242-243 (1982)

Ebeling, W., Kraeft, W.D. & Kremp, D., *Theory of Bound States and Ionization Equilibrium in Plasmas and Solids*, Akademie Verlag, Berlin, DDR (1976)

Ebeling, W., Förster, A., Fortov, V.E., Gryaznov, V.K. & Polishchuk, A.Ya., *Thermodynamic Properties of Hot Dense Plasmas*, Teubner, Stuttgart, Germany (1991)

Eggleton, P.P., Faulkner, J. & Flannery, B.P., *Astron. Astrophys.* **23**, 325-330 (1973)

Gough, D.O., *Solar Phys.* **100**, 65-99 (1985)

Gough, D.O., in *Hydrodynamic and Magnetohydrodynamic Problems in the Sun and Stars*, ed. Osaki, Y., University of Tokyo Press, 117-143 (1986)

Gough, D.O., in *Astrophysical Fluid Dynamics*, eds Zahn, J.-P. & Zinn-Justin, J., North-Holland, Amsterdam (1992)

Gough, D.O. & Thompson, M.J., in *Solar Interior and Atmosphere*, eds Cox, A.N., Livingston, W.C. & Matthews, M., Space Science Series, University of Arizona Press, 519-561 (1991)

Gough, D.O. & Toomre, J., *Ann. Rev. Astron. Astrophys.* **29**, 627-685 (1991)

Gough, D.O. & Vorontsov, S.V., *Mon. Not. R. astr. Soc.*, (1993) submitted

Gough, D.O. & Weiss, N.O., *Mon. Not. R. astr. Soc.* **176**, 589-607 (1976)

Harvey, J., in *Seismology of the Sun & Sun-like Stars*, eds Domingo, V. & Rolfe, E.J., ESA SP-286, Noordwijk, The Netherlands, 55-66 (1988)

Hummer, D.G. & Mihalas, D., *Astrophys. J.* **331**, 794-814 (1988)

Iglesias, C.A. & Rogers, F.J., in *Astrophysical Opacities*, eds. C. Mendoza & C. Zeippen (*Revista Mexicana de Astronomía y Astrofísica*) 161-170 (1992)

Iglesias, C.A., Rogers, F.J. & Wilson, B.G., *Astrophys. J.* **322**, L45 (1987)

Kosovichev, A.G., Christensen-Dalsgaard, J., Däppen, W., Dziembowski, W.A., Gough, D.O., & Thompson, M.J., *Mon. Not. R. astr. Soc.* **259**, 536-558 (1992)

Kraeft W.D., Kremp D., Ebeling W. & Röpke G., *Quantum Statistics of Charged Particle Systems*, Plenum, New York (1986)

Krasnikov Yu.G., *Zh. Eksper. teoret. Fiz.* **73**, 516 (1977) (English translation: *Soviet Phys. - JETP* **46**, 270-274; author's name mis-spelt as "Karsnikov")

Leighton, R.B., Noyes, R.W., Simon, G.W., *Astrophys. J.* **135**, 474 (1962)

Libbrecht, K.G., *Space Sci. Rev.* **47**, 275-301 (1988)

Libbrecht, K.G. & Woodard, M.F., *Science* **253**, 152-157 (1991)

Libbrecht, K.G., Woodard, M.F. & Kaufman, J.M., *Astrophys. J. Suppl.* **74**, 1129-1149 (1990)

Mihalas, D.M. & Mihalas, B.W., *Foundations of radiation hydrodynamics* (Oxford University Press) (1984)

Mihalas, D., Däppen W. & Hummer, D.G., *Astrophys. J.* **331**, 815-825 (1988)

Noether, E., *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.*, 235-257 (1918) [for an English translation see *Transport Theory and Stat. Phys.* **1** (1971), 186-207]

Parker, R.L., *Ann. Rev. Earth Planet. Sci.* **5**, 35-64 (1977)

Rhodes, E.J., Jr., Cacciani, A., Korzennik, S., Tomczyk, S., Ulrich, R.K., Woodard, M.F., *Astrophys. J.*, **351**, 687–700 (1990)

Rogers, F.J., *Astrophys. J.* **310**, 723-728 (1986)

Rogers, F.J., *these proceedings*

Saumon, D., *these proceedings*

Schwarzschild, M., *Structure and evolution of the stars* (Princeton University Press, Princeton, New Jersey) (1958)

Seaton, M., *J. Phys. B: Atom. Molec. Phys.* **20**, 6363-6378 (1987)

Seaton, M., *J. Phys. B: Atom. Molec. Phys.* **23**, 3255-3296 (1990)

Shibahashi, H., Noels, A. & Gabriel, M., *Astron. Astrophys.* **123**, 283-288 (1983)

Shibahashi, H., Noels, A. & Gabriel, M., *Mem. Soc. Astron. Ital.* **55**, 163-168 (1984)

Tarantola, A., *Inverse Problem Theory*, Elsevier, Amsterdam (1987)

Thompson, M.J., in *Lecture Notes in Physics* Vol. **388**: Challenges to Theories of the Structure of Moderate-mass Stars, eds Gough, D.O. & Toomre, J., Springer, Berlin, 61-80 (1991)

Turck-Chièze, S., Däppen, W., Fossat, E., Provost, J., Schatzman, E., Vignaud, D., *Physics Report* **230**, 57-235 (1992)

Ulrich, R.K., it Astrophys. J, **158**, 427 (1969)

Ulrich, R.K., it Astrophys. J, **162**, 993 (1970)

Ulrich, R.K., *Astrophys. J.* **258**, 404-413 (1982)

Unno, W., Osaki, Y., Ando, H. & Shibahashi, H., *Nonradial Oscillations of Stars, 2nd Edition*, University of Tokyo Press, Tokyo (1989)

Vorontsov, S.V., Baturin, V.A., Gough, D.O., Däppen, W., *these proceedings*

Vorontsov, S.V. & Zharkov, V.N., *Sov. Sci. Rev. E. Astrophys. Space Phys.* **7**, 1-103 (1989)
Wiese, W.L., Kelleher, D.E. & Paquette, D.R., *Phys. Rev.* **A6** 1132-1153 (1972)