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Nagoya Math. J.
Vol. 129 (1993), 23–42

QUANTUM WHITE NOISES

WHITE NOISE APPROACH TO QUANTUM STOCHASTIC CALCULUS*

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I. Introduction

Let $H = L^2(R)$ be the Hilbert space of all complex-valued square integrable functions defined on R, $\Phi = \Gamma(H)$ be the Boson Fock space over H. For each $h \in H$, denote by $\varepsilon(h)$ the corresponding exponential vector:

$$\varepsilon(h) = 1 \oplus h \oplus h^{\otimes 2}/2! \oplus \cdots h^{\otimes n}/n! \oplus \cdots,$$

in particular $\varepsilon(0)$ is the Fock vacuum. It is well known (cf. [9,15]) that the family $\mathbf{E} = \{\varepsilon(h);\, h \in H\}$ is linearly independent and total in Φ. In developing their quantum stochastic calculus, Hudson and Parthasarathy [9] used the set $\mathbf{E}$ as "testing vectors": all operators on Φ were firstly defined on $\mathbf{E}$ and then extended to their proper domains. Instead of operator valued processes, they essentially dealt with vector valued ones and, therefore, obtained a quantum (i.e. noncommutative) version of Itô's product formula which was only based on the commutation rules of a free Boson field and Lebesgue integration. The three fundamental integrators are annihilation, creation and number processes which played the role of "quantum noises" in quantum stochastic evolutions. They are noncommutative extensions of classical Brownian motion and Poisson process.

On the other hand, the white noise approach initiated by T. Hida [5] has been proved highly effective to the classical stochastic integration theory. One natural question is: what one can do with it in quantum stochastic calculus?

In the present paper, we define the "quantum white noise" as a generalized quantum process in terms of Hida's derivative (or "causal calculus"). Since it could be rigorously treated in the framework of Hida's distributions over white noise space rather than in Fock space Φ, we establish some kind of chaos decomposition for operators which is a considerable extension of those decompositions for

Received September 2, 1991.
* Work supported by the National Natural Science Foundation of China.

vectors (or functionals). Moreover, we investigate quantum stochastic measures of the "normal form", define quantum stochastic integrals with respect to them and extend Hudson and Parthasarathy's formulae to the more general case.

II. Quantum white noises

In general description of quantum stochastic evolutions (cf. [1,7,9,15]), there usually is another Hilbert space K called "initial space" and a dense linear manifold D in K. A quantum stochastic process is a family of operators $X=(X(t),\ t\in R)$ on $K\otimes\Phi$ as well as its adjoint process $X^*=(X^*(t),\ t\in R)$ with domains containing $D\otimes\mathbf{E}$. However, in our presentation here, we shall neglect the initial space K and restrict ourselves to real Hilbert space H just for notational simplification. The general case will be treated by taking tensor product and complexification.

We briefly recall some notions and notations in white noise analysis. Let $A=1+t^2-d^2/dt^2$ be the harmonic oscillator in H and $\Gamma(A)$ be its second quantization. The Schwartz spaces of rapidly decreasing C^∞-functions on R and tempered distributions will be denoted by E and E^* respectively. Let μ be the white noise measure on E^* and $(L^2)=L^2(E^*,\mu)$. It is well known that (L^2) is isometrically isomorphic to the Boson Fock space $\Phi=\Gamma(H)$. This isomorphism gives the Fock space a "white noise interpretation" (cf. [7,15] for different probabilistic interpretations of Fock space) and, therefore, $\Gamma(A)$ could be considered as a densely defined selfadjoint operator in (L^2). For $k=0,1,2,\ldots,$ we put $E_k=\mathrm{Dom}(A^k)$ and $(E_k)=\mathrm{Dom}(\Gamma(A^k))$. Then, E_k (resp. (E_k)) is a Hilbert space with norm $|\xi|_k=|A^k\xi|_0$ (resp. $\|\phi\|_k=\|\Gamma(A^k)\phi\|_0$) where $|\cdot|_0$ (resp. $\|\cdot\|_0$) is the norm in H (resp. (L^2)). Denote by E_{-k} (resp. (E_{-k})) its dual space with dual norm $|\cdot|_{-k}$ (resp. $\|\cdot\|_{-k}$). Then, the projective limit $E=\bigcap_k E_k$ (resp. $(E)=\bigcap_k (E_k)$) is a nuclear Fréchet space and the inductive limit $E^*=\bigcup_k E_{-k}$ (resp. $(E)^*=\bigcup_k (E_{-k})$) is its topological dual space. Thus we have two Gel'fand triplets:

$$E\subset H\subset E^*$$

and

$$(E)\subset(L^2)\subset(E)^*$$

with dual pairings $\langle\cdot,\cdot\rangle$ $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ respectively. The elements in (E) are called Hida's testing functionals and elements in (E)* are referred to as Hida's generalized functionals (or distributions).

Note that the exponential functionals

$$\varepsilon(\xi) = \exp\{\langle \cdot\,, \xi\rangle - |\,\xi\,|_0^2/2\}, \quad \xi \in E \tag{2.1}$$

belong to (E) and for every $k \in N_0$, as $\xi \to \xi_0$ in E,

$$\|\,\varepsilon(\xi) - \varepsilon(\xi_0)\|_k^2 = \exp|\,\xi\,|_k^2 + \exp|\,\xi_0\,|_k^2 - 2\exp\,(\xi, \xi_0)_k \to 0 \tag{2.2}$$

hence the map $\xi \mapsto \varepsilon(\xi)$ is continuous from E to (E). Under the isomorphism between (L^2) and Φ, the exponential functionals just correspond to those exponential vectors (and accordingly we denote them by the same symbols). It follows that the family $\mathbf{E}_0 = \{\varepsilon(\xi), \xi \in E\}$ is linearly independent and total in (E). We shall use this much more restricted family as "testing vectors" and propose the following

DEFINITION 2.1. A generalized quantum process (abbr. GQP) is a pair of densely defined, mutually adjoint families of linear operators $(X(t), X^*(t)\,; t \in R)$ from (E) into $(E)^*$ with domains containing $\mathbf{E}_0$.

By definition, all X_t and X_t^* are closable. Taking their closures if necessary, we may assume that they are closed operators. If the domain of X_t is the whole space (E), then, by closed graph theorem, it is a continuous linear operator from (E) into $(E)^*$.

The S-transform of functional $F \in (E)^*$ is defined by

$$(SF)\,(\xi) = \langle\!\langle F, \varepsilon(\xi)\rangle\!\rangle, \quad \xi \in E \tag{2.3}$$

and, as a functional on E, is characterized by the following two conditions (cf. [18]):

(i) it has a ray entire extension on E; i.e. for any $\xi, \eta \in E$ the function

$$R \ni \lambda \to (SF)\,(\eta + \lambda\xi)$$

admits an entire extension to C;

(ii) there exists $k \in N_0$ and constants $C_1, C_2 \geq 0$ such that

$$|\,(SF)\,(z\xi)\,| \leq C_1 \exp\{C_2\,|\,z\,|^2\,|\,\xi\,|_k^2\} \tag{2.4}$$

for all $\xi \in E$ and $z \in C$.

Any functional on E satisfies these two conditions is referred to as a U-functional.

As an important example of GQP, we investigate Hida's differential operator ∂_t $(t \in R)$ which is defined as

(2.5) $$\partial_t \phi = S^{-1}\left\{\frac{\delta}{\delta\xi(t)}(S\phi)(\xi)\right\}, \quad \phi \in (E)$$

where $\delta/\delta\xi(t)$ stands for Fréchet functional derivative (cf. [5,12]). This operator could also be interpreted as Gâteaux derivative in the direction δ_t, the Dirac delta function at t. More specifically, let $\phi \in (E)$ and $x, y \in E^*$, the Gâteaux derivative of ϕ at x in the direction y is defined as

(2.6) $$D_y\phi(x) = \frac{d}{ds}\phi(x+sy)\mid_{s=0}$$
$$= \lim_{s\to 0}\frac{1}{s}\{\Phi(x+sy) - \phi(x)\}.$$

It is known that (cf. [6,19]) for all $y \in E^*$, D_y is a continuous linear operator on (E) and if $y \in E$, it can be extended to a continuous linear operator on $(E)^*$. Accordingly, for all $y \in E^*$, the dual operator D_y^* is a continuous linear operator on $(E)^*$ and if $y \in E$, its restriction is a continuous linear operator on (E). For the special choice $y = \delta_t$, we have

(2.7) $$\partial_t = D_{\delta_t}$$

and for $\xi \in E$,

(2.8) $$\partial_t \varepsilon(\xi) = \xi(t)\varepsilon(\xi).$$

DEFINITION 2.2. The GQP $(\partial_t, \partial_t^*; t \in R)$ is referred to as quantum white noise process (abbr. QWN).

Remark. Sometimes we call ∂_t the annihilation operator and its dual ∂_t^* the creation operator. Note that for any $s, t \in R$, the product $\partial_s^* \partial_t$ is a well defined operator from (E) into $(E)^*$ but it is not the case when they are in inverse order. However, since for $\xi, \eta \in E$,

(2.9) $$D_\xi = \int_R \xi(t)\, \partial_t\, dt$$

and

(2.10) $$D_\eta^* = \int_R \eta(s)\, \partial_s^*\, ds,$$

we can interpret $\partial_t \partial_s^*$ as an operator valued distribution. By the well known canonical commutation relation:

(2.11) $$[D_\xi, D_\eta^*] = (\xi, \eta)\, I,$$

we have

(2.12) $$[\partial_t, \partial_s^*] = \delta(t-s)\, I.$$

III. Quantum multiple Wiener integrals, chaos decomposition for operators

It is remarkable that Hida, Obata & Saitô [6] have developed a general theory of operators which could be represented as integrals of ∂_t^* and ∂_t. In this connection see also Krée [10] and Meyer [16]. So far as we know, this idea trace back to Berezin [2] who proved that any bounded operator B in Φ has a representation in the following "normal form":

(3.1)
$$B = \sum_{l,m} \int_{R^{l+m}} \beta_{l,m}(s_1,\dots,s_l\,;\,t_1,\dots,t_m)\, \partial_{s_1}^* \cdots \partial_{s_l}^*\, \partial_{t_1} \cdots \partial_{t_m}\, ds_1 \cdots ds_l\, dt_1 \cdots dt_m$$

where $\beta_{l,m}$ are, generally speaking, generalized functions and are supposed to be symmetric in $s_1,\dots,s_l$ and $t_1,\dots,t_m$ separately. Note that in every summand, the creation operators (∂_s^*) stand to the left of the annihilation operators (∂_t), that is, in the Wick ordering. In the same spirit. Maassen [14] has developed a method to describe quantum processes by integral kernels. Here we use the results obtained in [6] to extend the integral representation (3.1) to any continuous linear operator K from (E) into $(E)^*$.

We note that, for any $\phi, \psi \in (E)$ and $l, m \in N_0$, the function $\eta_{\phi,\psi}$ on R^{l+m} defined by

(3.2) $$(s_1,\dots,s_l\,;\,t_1,\dots,t_m) \mapsto \langle\!\langle \partial_{s_1}^* \cdots \partial_{s_l}^*\, \partial_{t_1} \cdots \partial_{t_m} \phi, \psi \rangle\!\rangle$$

belongs to $E^{\otimes(l+m)}$ (for the proof we refer to [6]). Therefore, for any $\kappa \in [E^{\otimes(l+m)}]^*$, there exists a unique continuous operator

$$\Xi_{l,m}(\kappa) : (E) \to (E)^*$$

such that

(3.3) $$\langle\!\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\!\rangle = \langle \kappa, \eta_{\phi,\psi} \rangle.$$

The operator $\Xi_{l,m}(\kappa)$ has a formal integral expression:

(3.4)
$$\Xi_{l,m}(\kappa) = \int_{R^{l+m}} \kappa(s_1,\dots,s_l\,;\,t_1,\dots,t_m)\, \partial_{s_1}^* \cdots \partial_{s_l}^*\, \partial_{t_1} \cdots \partial_{t_m}\, ds_1 \cdots ds_l\, dt_1 \cdots dt_m.$$

Moreover, $\Xi_{l,m}(\kappa)$ is continuous from (E) into itself if and only if $\kappa \in E^{\otimes l} \otimes [E^{\otimes m}]^*$.

Recall that the S-transform of functional $F \in (E)^*$ is

$$(SF)(\xi) = \langle\langle F, \varepsilon(\xi)\rangle\rangle, \quad \xi \in E. \tag{3.5}$$

For any closed linear operator $K : (E) \to (E)^*$ with domain containing $\mathbf{E}_0$, we shall frequently consider its symbol:

$$\hat{K}(\xi, \eta) \equiv \langle\langle K\varepsilon(\xi), \varepsilon(\eta)\rangle\rangle, \quad \xi, \eta \in E. \tag{3.6}$$

This terminology was introduced by Berezin [3,4], Krêe & Rączka [11] in the context of operators on Fock space. Since $\mathbf{E}_0$ is total in (E), it follows that the operator K is uniquely determined by its symbol. Moreover, we have the following

PROPOSITION 3.1 *For any continuous linear operator $K : (E) \to (E)^*$, $\hat{K}$ is separately continuous on $E \times E$ and for any $\xi, \eta \in E$, $t \in R$*

$$\widehat{K\partial_t}(\xi, \eta) = \xi(t)\hat{K}(\xi, \eta)\,; \tag{3.7}$$

$$\widehat{\partial_t^* K}(\xi, \eta) = \eta(t)\hat{K}(\xi, \eta). \tag{3.8}$$

If K is continuous from (E) (resp. $(E)^*$) into itself, then $\hat{K}$ is Fréchet differentiable and

$$\widehat{\partial_t K}(\xi, \eta) = \frac{\delta}{\delta\eta(t)}\hat{K}(\xi, \eta)\,; \tag{3.9}$$

$$(\text{resp. } \widehat{K\partial_t^*}(\xi, \eta) = \frac{\delta}{\delta\xi(t)}\hat{K}(\xi, \eta)). \tag{3.10}$$

Proof. The first assertion is obvious. Since

$$\hat{K}(\xi, \eta) = (SK\varepsilon(\xi))(\eta),$$

it follows from equation (2.5) that

$$\begin{aligned}\frac{\delta}{\delta\eta(t)}\hat{K}(\xi, \eta) &= (S\partial_t K\varepsilon(\xi))(\eta)\\ &= \langle\langle \partial_t K\varepsilon(\xi), \varepsilon(\eta)\rangle\rangle.\end{aligned}$$

Similarly, in view of

$$\hat{K}(\xi, \eta) = (SK^*\varepsilon(\eta))(\xi),$$

we have

$$\frac{\delta}{\delta\xi(t)}\hat{K}(\xi, \eta) = (S\partial_t K^*\varepsilon(\eta))(\xi) \\ = \langle\langle \varepsilon(\xi), \partial_t K^*\varepsilon(\eta)\rangle\rangle \\ = \langle\langle K\partial_t^*\varepsilon(\xi), \varepsilon(\eta)\rangle\rangle.$$

This proves second part of the proposition. Q.E.D.

Let P_0 be the projection onto the vacuum (or 0-th chaos). It is easy to see that $\widehat{P_0} \equiv 1$ and, by the preceding proposition,

$$\widehat{\partial_{s_1}^* \cdots \partial_{s_l}^* P_0 \partial_{t_1} \cdots \partial_{t_m}}(\xi, \eta) = \xi(t_1)\cdots\xi(t_m)\eta(s_1)\cdots\eta(s_l). \tag{3.11}$$

According to Schwartz kernels theorem, for any $\kappa_{l,m} \in [E^{\hat{\otimes} l} \otimes E^{\hat{\otimes} m}]^*$ (resp. $E^{\hat{\otimes} l} \otimes [E^{\hat{\otimes} m}]^*$, where $\otimes^\wedge$ stands for symmetric tensor product), there is a unique continuous linear operator $K_l^m : E^{\hat{\otimes} m} \to [E^{\hat{\otimes} l}]^*$ (resp. $E^{\hat{\otimes} m} \to E^{\hat{\otimes} l}$), such that

$$\langle K_l^m \phi, \psi\rangle = \langle \kappa_{l,m}, \phi \otimes \psi\rangle \quad \forall\ \phi \in E^{\hat{\otimes} m},\ \psi \in E^{\hat{\otimes} l}. \tag{3.12}$$

Hence, it is reasonable to propose the following

Definition 3.2. For any continuous linear operator $K_l^m : [E^{\hat{\otimes} m} \to [E^{\hat{\otimes} l}]^*$, the quantum multiple Wiener integral (abbr. QWI) of K_l^m is a continuous linear operator $I_l^m(K_l^m) : (E) \to (E)^*$ denoted by

$$I_l^m(K_l^m) = \int_{R^{l+m}} \kappa_{l,m}(s_1, \dots, s_l\,;\, t_1, \dots, t_m) \cdot \\ \cdot\, \partial_{s_1}^* \cdots \partial_{s_l}^* P_0 \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m \tag{3.13}$$

whose symbol is

$$\widehat{I_l^m(K_l^m)}(\xi, \eta) = \langle K_l^m \xi^{\hat{\otimes} m}, \eta^{\hat{\otimes} l}\rangle. \tag{3.14}$$

Remark. By an argument similar to that in [6], we can prove that if K_l^m is continuous from $E^{\hat{\otimes} m}$ into $E^{\hat{\otimes} l}$, then $I_l^m(K_l^m)$ is continuous from (E) into itself.

We have the following fundamental result:

Theorem 3.3. *Every continuous linear operator* $K : (E) \to (E)^*$ (*resp.* $(E) \to (E)$) *has a unique decomposition:*

$$K = \sum_{l,m} (l!m!)^{-1} I_l^m(K_l^m) \tag{3.15}$$

where $K_l^m : E^{\hat{\otimes} m} \to [E^{\hat{\otimes} l}]^*$ (resp. $E^{\hat{\otimes} m} \to E^{\hat{\otimes} l}$) *are continuous linear operators and the series converges weakly on the set* $\mathbf{E}_0$, *that is,*

$$\hat{K}(\xi, \eta) = \sum_{l,m} (l!m!)^{-1} \langle K_l^m \xi^{\otimes m}, \eta^{\otimes l} \rangle \quad \forall\ \xi, \eta \in E. \tag{3.16}$$

Proof. In view of the chaos decompositions of (E) and $(E)^*$ established in [8] (cf. also [13] for a different dual pair), for any $\phi \in (E)$ and any $F \in (E)^*$, we have unique decompositions:

$$\phi = \sum_n (n!)^{-1} \delta^n \phi_n, \tag{3.17}$$

$$F = \sum_n (n!)^{-1} \delta^n F_n \tag{3.18}$$

where $\phi_n \in E^{\hat{\otimes} n}$, $F_n \in E^{*\hat{\otimes} n}$ and δ^n is an isometric isomorphism of $E_k^{\hat{\otimes} n}$ into (E_k) for any $k \in Z$ (hence, it is a linear homeomorphism of $E^{\hat{\otimes} n}$ into (E) and is extended to that of $E^{*\hat{\otimes} n}$ into $(E)^*$). Moreover,

$$\langle\!\langle F, \phi \rangle\!\rangle = \sum_n (n!)^{-1} \langle F_n, \phi_n \rangle. \tag{3.19}$$

It is easy to see that series (3.17) (resp. (3.18)) converges in (E) (resp. $(E)^*$) and the projections J_n's are continuous. Hence, the direct sum decompositions for (E) and $(E)^*$ are topological. Consequently, we have a strongly convergent series

$$K = \sum_{l,m} K_{l,m} \tag{3.20}$$

where $K_{l,m} = J_l K J_m^{-1}$ is a continuous linear operator from m-th chaos of (E), $J_m(E)$, into the l-th chaos of $(E)^*$, $J_l(E)^*$. Letting

$$K_l^m = (\delta^l)^{-1} K_{l,m} \delta^m \tag{3.21}$$

for each l and m, we see that

$$\begin{aligned}
\hat{K}(\xi, \eta) &= \langle\!\langle K\varepsilon(\xi), \varepsilon(\eta) \rangle\!\rangle \\
&= \sum_{l,m} (l!m!)^{-1} \langle\!\langle K_{l,m} \delta^m \xi^{\otimes m}, \delta^l \eta^{\otimes l} \rangle\!\rangle \\
&= \sum_{l,m} (l!m!)^{-1} \langle\!\langle \delta^l K_l^m \xi^{\otimes m}, \delta^l \eta^{\otimes l} \rangle\!\rangle \\
&= \sum_{l,m} (l!m!)^{-1} \langle K_l^m \xi^{\otimes m}, \eta^{\otimes l} \rangle \\
&= \sum_{l,m} (l!m!)^{-1} \widehat{I_l^m (K_l^m)}\, (\xi, \eta)
\end{aligned}$$

as desired. Q.E.D.

Note that the operator $I_l^m (K_l^m)$ acts only on the m-th chaos of (E) and takes

https://doi.org/10.1017/S002776300000430X Published online by Cambridge University Press

value only in the l-th chaos of $(E)^*$. Since P_0 has representation

$$P_0 = \sum_n (-1)^n (n!)^{-1} \int_{R^n} \partial_{s_1}^* \cdots \partial_{s_n}^* \partial_{s_1} \cdots \partial_{s_n} ds_1 \cdots ds_n, \tag{3.22}$$

by inserting equation (3.22) into equation (3.15), we obtain a similar decomposition in terms $\Xi_{l,m}(\tilde{\kappa}_{l,m})$ where

$$\begin{aligned} &\tilde{\kappa}_{l,m}(s_1, \ldots, s_l\,;\, t_1, \ldots, t_m) \\ &= \sum_{k=0}^{l\wedge m} \frac{(-1)^k}{k!} \frac{l!m!}{(l-k)!(m-k)!} \kappa_{l-k,m-k}(s_1, \ldots, s_{l-k}\,;\, t_1, \ldots, t_{m-k}) \cdot \\ &\qquad \cdot \delta(s_{l-k+1}, t_{m-k+1}) \cdots \delta(s_l, t_m) \end{aligned} \tag{3.23}$$

and $\kappa_{l,m}$ is related to K_l^m by equation (3.12). Conversely, we have

$$\begin{aligned} &\kappa_{l,m}(s_1, \ldots, s_l\,;\, t_1, \ldots, t_m) \\ &= \sum_{k=0}^{l\wedge m} \frac{l!m!}{k!(l-k)!(m-k)!} \tilde{\kappa}_{l-k,m-k}(s_1, \ldots, s_{l-k}\,;\, t_1, \ldots, t_{m-k}) \cdot \\ &\qquad \cdot \delta(s_{l-k+1}, t_{m-k+1}) \cdots \delta(s_l, t_m). \end{aligned} \tag{3.24}$$

THEOREM 3.4. *Every continuous linear operator* $K : (E) \to (E)^*$ (*resp.* $(E) \to (E)$) *has a unique decomposition:*

$$K = \sum_{l,m} (l!m!)^{-1} \Xi_{l,m}(\tilde{\kappa}_{l,m}) \tag{3.25}$$

where $\Xi_{l,m}$*'s are defined by equation* (3.3) *and* $\tilde{\kappa}_{l,m}$*'s are determined by equation* (3.23). *The series converges weakly on the set* $\mathbf{E}_0$, *that is,*

$$\hat{K}(\xi, \eta) = \sum_{l,m} (l!m!)^{-1} \langle \tilde{\kappa}_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \exp \langle \xi, \eta \rangle. \tag{3.26}$$

Proof. In view of equation (3.3), we have

$$\widehat{\Xi_{l,m}(\tilde{\kappa}_{l,m})}(\xi, \eta) = \langle \tilde{\kappa}_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \exp\langle \xi, \eta \rangle.$$

Summing up for all l and m, we have

$$\begin{aligned} &\sum_{l,m} (l!m!)^{-1} \langle \tilde{\kappa}_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \exp \langle \xi, \eta \rangle \\ &= \sum_{l,m} \sum_{k=0}^{l\wedge m} \frac{(-1)^k}{k!(l-k)!(m-k)!} \langle \kappa_{l-k,m-k}, \eta^{\otimes (l-k)} \otimes \xi^{\otimes (m-k)} \rangle \langle \xi, \eta \rangle^k \exp \langle \xi, \eta \rangle \\ &= \sum_{k=0}^{\infty} (-1)^k (k!)^{-1} \langle \xi, \eta \rangle^k \sum_{\lambda,\mu} (\lambda!\mu!)^{-1} \langle \kappa_{\lambda,\mu}, \eta^{\otimes\lambda} \otimes \xi^{\otimes\mu} \rangle \exp \langle \xi, \eta \rangle \end{aligned}$$

$$= \sum_{\lambda,\mu} (\lambda!\mu!)^{-1} \langle \kappa_{\lambda,\mu}, \eta^{\otimes\lambda} \otimes \xi^{\otimes\mu} \rangle$$

which converges to $\hat{K}(\xi, \eta)$ according to equation (3.16). Q.E.D.

Consider the particular case when $m = 0$:

$$I_l(\kappa_l)1 = \int_{R^l} \kappa_l(s_1, \ldots, s_l)\, \partial_{s_1}^* \cdots \partial_{s_l}^* \, ds_1 \cdots ds_l \, 1$$

is just the (generalized) multiple Wiener integral of κ_l. In fact, the S-transform of $I_l(\kappa_l)1$ is

$$(SI_l(\kappa_l)1)(\xi) = \langle\!\langle I_l(\kappa_l)\varepsilon(0), \varepsilon(\xi) \rangle\!\rangle = \widehat{I_l(\kappa_l)}(0, \xi) = \langle \kappa_l, \xi^{\otimes l} \rangle. \tag{3.27}$$

Thus we have

Theorem 3.5. *If* $\phi = \sum_m (m!)^{-1} I_m(\phi_m)1 \in (E)$ *and*

$$K = \sum_{l,m} (l!m!)^{-1} I_l^m(K_l^m) : (E) \to (E)^*,$$

then

$$K\phi = \sum_l (l!)^{-1} I_l(\sum_m (m!)^{-1} K_l^m \phi_m)1. \tag{3.28}$$

If, moreover, $K : (E) \to (E)$ *and*

$$\Lambda = \sum_{n,l} (n!l!)^{-1} I_n^l(\Lambda_n^l) : (E) \to (E)^*.$$

then

$$\Lambda K = \sum_{n,m} (n!m!)^{-1} I_n^m(\sum_l (l!)^{-1} \Lambda_n^l K_l^m).$$

New we give some examples:

Example 1. Translation and Gâteaux differentiation of $\phi \in (E)$ in the direction $y \in E^*$ (cf. [19]):

$$\tau_y \phi(x) = \phi(x + y) ; \tag{3.30}$$

$$D_y\phi = \lim_{\varepsilon \downarrow 0} (\tau_{\varepsilon y} \phi - \phi) / \varepsilon. \tag{3.31}$$

Since

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$$\hat{\tau}_y(\xi, \eta) = \exp\langle y, \xi\rangle \exp\langle \xi, \eta\rangle$$

and

$$\hat{D}_y(\xi, \eta) = \langle y, \xi\rangle \exp\langle \xi, \eta\rangle,$$

it follows that

$$D_y = \int_R y(t)\, \partial_t\, dt \tag{3.32}$$

and that

$$\tau_y = \sum_n (n!)^{-1}\Big(\int_R y(t)\, \partial_t\, dt\Big)^n = \exp D_y. \tag{3.33}$$

EXAMPLE 2. Scaling transformation and second quantization of the multiplication by $\lambda \in R$ (cf. [19]):

$$Z_\lambda \phi(x) = \phi(\lambda x), \quad \phi \in (E)\,; \tag{3.34}$$

$$\Gamma(\lambda)\, \varepsilon\, (\xi) = \varepsilon(\lambda \xi), \quad \xi \in E. \tag{3.35}$$

Since

$$\hat{Z}_\lambda(\xi, \eta) = \exp\{\lambda\langle \xi, \eta\rangle\}\exp\{(\lambda^2 - 1)\,|\,\xi\,|^2/2\}$$

and

$$\widehat{\Gamma(\lambda)}\, (\xi, \eta) = \exp\{\lambda\langle \xi, \eta\rangle\},$$

it follows that

$$Z_\lambda = \sum_{l,m} (l!m!)^{-1} \int_{R^{l+m}} \lambda^l((\lambda^2 - 1)/2)^m \partial_{s_1}^* \cdots \partial_{s_l}^* P_0 \partial_{s_1} \cdots \partial_{s_l} \partial_{t_1}^2 \cdots \partial_{t_m}^2 \cdot \tag{3.36}$$
$$\cdot\, ds_1 \cdots ds_l\, dt_1 \cdots dt_m$$

and that

$$\Gamma(\lambda) = \Sigma_n\, (n!)^{-1} \int_{R^n} \lambda^n\, \partial_{t_1}^* \cdots \partial_{t_n}^* P_0 \partial_{t_1} \cdots \partial_{t_n}\, dt_1 \cdots dt_n. \tag{3.37}$$

EXAMPLE 3. Renormalization operator (cf. Yan [20]):

$$R = Z_{1/\sqrt{2}}\, \Gamma(\sqrt{2})\,; \tag{3.38}$$
$$R^{-1} = Z(1/\sqrt{2}) Z_{\sqrt{2}}.$$

By formulae (3.29), (3.36) and (3.37), we have

$$(3.39)\quad R = \sum_{l,m} (l!m!)^{-1} \int_{R^{l+m}} (-1/2)^m \partial_{s_1}^* \cdots \partial_{s_l}^* P_0\, \partial_{s_1} \cdots \partial_{s_l} \cdot \partial_{t_1}^2 \cdots \partial_{t_m}^2\, ds_1 \cdots ds_l\, dt_1 \cdots dt_m$$

and

$$(3.40)\quad R^{-1} = \sum_{l,m} (l!m!)^{-1} \int_{R^{l+m}} (1/2)^m \partial_{s_1}^* \cdots \partial_{s_l}^* P_0\, \partial_{s_1} \cdots \partial_{s_l} \cdot \partial_{t_1}^2 \cdots \partial_{t_m}^2\, ds_1 \cdots ds_l\, dt_1 \cdots dt_m.$$

Let $I_n(\phi_n)1$ be the Wiener-Itô's multiple integral of ϕ_n. By formula (3.28), we obtain that

$$R^{-1} I_n(\phi_n)1 = \sum_{m=0}^{[n/2]} \frac{n!}{(n-2m)!m!2^m} \int_{R^{n-m}} \phi_n(s_1, \ldots, s_{n-2m};\, t_1,\, t_1, \ldots, t_m,\, t_m) \cdot \partial_{s_1}^* \cdots \partial_{s_{n-2m}}^*\, ds_1 \cdots ds_{n-2m}\, dt_1 \cdots dt_m 1$$

which implies the transformation formula from Wiener-Itô's multiple integrals into Wiener-Stratonovich's ones (cf. [20])

$$(3.41)\qquad X^{\hat{\otimes} n} = \sum_{m=0}^{[n/2]} \frac{n!}{(n-2m)!m!2^m} : x^{\hat{\otimes}(n-2m)} : \hat{\otimes}\, \mathrm{Tr}^{\otimes m}$$

where Tr is the trace operator defined by

$$(3.42)\qquad \langle \mathrm{Tr}, \omega \rangle = \int_R \langle \delta_t^{\otimes 2}, \omega \rangle dt = \int_R \omega(t, t) dt$$

for $\omega \in E \hat{\otimes} E$.

IV. Mutual quadratic variation of quantum stochastic measures

In the integral representation (3.4), for every $t \in R$, let the kernel $\kappa(t) \in [E^{\otimes(l+m)}]^*$ defined as follows:

$$(4.1)\qquad \langle \kappa(t), \omega \rangle = \int_{-\infty}^{t} \omega(s, \ldots, s) ds, \quad \forall\ \omega \in E^{\otimes(l+m)},$$

then

$$(4.2)\qquad W_{l,m}(t) \equiv \int_{-\infty}^{t} \partial_s^{*l}\, \partial_s^m\, ds, \quad t \in R$$

is a family of continuous linear operator from (E) into $(E)^*$. Thus,

$$W^*_{l,m}(t) \equiv \int_{-\infty}^{t} \partial_s^{*m} \partial_s^l \, ds, \quad t \in R \tag{4.3}$$

and $(W_{l,m}(t),\ W^*_{l,m}\ (t)\ ;\ t \in R)$ is a GQP.

Generally speaking, we can define GQP by any Wick polynomials of QWN. For example,

$$\begin{aligned} X_n(t) &\equiv \int_{-\infty}^{t} :(\partial_s^* + \partial_s)^n: ds \\ &= \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{t} \partial_s^{*j} \partial_s^{n-j} \, ds, \end{aligned} \tag{4.4}$$

or, formally

$$X_n(t) \equiv \int_{-\infty}^{t} H_n(x(s)\ ;\ 1/ds) ds \tag{4.5}$$

where H_n is the Hermite polynomial of degree n with variance parameter $1/ds$ and $x(s) = \partial_s^* + \partial_s$. The most interesting GQPs are representable by polynomials of at most second degree (quadratic processes), namely:

1° The annihilation and creation process:

$$\begin{aligned} A(t) &\equiv W_{0,1}(t) = \int_{-\infty}^{t} \partial_s \, ds \\ A^*(t) &\equiv W_{1,0}(t) = \int_{-\infty}^{t} \partial_s^* \, ds. \end{aligned} \tag{4.6}$$

Note that

$$Q(t) \equiv A^*(t) + A(t), \quad t \in R \tag{4.7}$$

is a quantum Brownian motion. It is reasonable to call the GQP

$$x(t) \equiv \partial_t^* + \partial_t, \quad t \in R \tag{4.8}$$

a quantum Gaussian white noise.

2° The number process:

$$N(t) \equiv W_{1,1}(t) = \int_{-\infty}^{t} \partial_s^* \partial_s \, ds. \tag{4.9}$$

It is remarkable that the process

$$N^\lambda(t) \equiv N(t) + \sqrt{\lambda}\, Q(t) + \lambda t \tag{4.10}$$

is a quantum Poisson process with parameter λ (cf. [9,15]). So the GQP

$$n^{\lambda}(t) \equiv \partial_t^* \partial_t + \sqrt{\lambda}\, x(t) + \lambda I \tag{4.11}$$

could be reasonably interpreted as a quantum Poisson white noise.

3° The Volterra Laplacian process:

$$\begin{aligned} V(t) &\equiv W_{0,2}(t) = \int_{-\infty}^{t} \partial_s^2\, ds \\ V^*(t) &\equiv W_{2,0}(t) = \int_{-\infty}^{t} \partial_s^{*2}\, ds. \end{aligned} \tag{4.12}$$

Differentiation of these processes yields operator valued measures on R which could be regarded as quantum stochastic measures (abbr. QM). For example,

$$dW_{l,m}(t) = \partial_t^{*l}\, \partial_t^m\, dt, \tag{4.13}$$

$$dX_n(t) = :x(t)^n:\, dt \tag{4.14}$$

etc. They will play the role of integrators for quantum stochastic integration.

Definition 4.1. Let $(X(t), X^*(t); t \in R)$ be a GQP which is weakly measurable in t on $\mathbf{E}_0$ (that is, for any $\xi, \eta \in E$, the map $t \mapsto \hat{X}_t(\xi, \eta)$ is Lebesgue measurable). The central (resp. right, left) quantum stochastic integral (abbr. QI(c), QI(r), QI(l) respectively) of $X(t)$ with respect to QM $dW_{l,m}(t)$ is a family of closed linear operators:

$$\int_{-\infty}^{t} \partial_s^{*l}\, X_s\, \partial_s^m\, ds : (E) \to (E)^*, \quad t \in R \tag{4.15}$$

$$\left(\text{resp.} \int_{-\infty}^{t} \partial_s^{*l}\, \partial_s^m\, X_s\, ds, \quad \int_{-\infty}^{t} X_s\, \partial_s^{*l}\, \partial_s^m\, ds\right)$$

with domains containing $\mathbf{E}_0$ whose symbols are

$$\int_{-\infty}^{t} \xi(s)^m\, \eta(s)^l \hat{X}_s(\xi, \eta)\, ds, \quad t \in R \tag{4.16}$$

$$\left(\text{resp.} \int_{-\infty}^{t} \eta(s)^l \frac{\delta^m}{\delta \eta(s)^m} \hat{X}_s(\xi, \eta)\, ds, \quad \int_{-\infty}^{t} \xi(s)^m \frac{\delta^l}{\delta \xi(s)^l} \hat{X}_s(\xi, \eta)\, ds\right)$$

provided these Lebesgue integrals exist and are U-functionals of η.

In this sense, $W_{l,m}(t)$ can be regarded either as QI(r) of ∂_t^m with respect to $dW_{l,0}(t)$ or as QI(l) of ∂_t^{*l} with respect to $dW_{0,m}(t)$.

Remark. Recently, N. Obata [17] obtained a criterion for functionals on $E \times E$ to be symbols of continuous linear operators from (E) to $(E)^*$. It is quite useful for derivation of conditions which ensure the existence and continuity of quantum stochastic integrals.

If, for any t, X_t and ∂_t commute, that is,

$$\xi(t)\, \hat{X}_t(\xi, \eta) = \frac{\delta}{\delta\eta(t)} \hat{X}_t(\xi, \eta) \tag{4.17}$$

for any $\xi, \eta \in E$, then QI(r) coincides with QI(c). Especially, we introduce the following

Definition 4.2. A GQP $(X(t), X^*(t)\,;\, t \in R)$ is said to be adapted (with respect to the filtration generated by QWN$(\partial_t, \partial_t^*)$) if for any $t \in R$, $X(t)$ commute with all ∂_u and ∂_u^* whenever $u \geqq t$. More precisely,

$$\begin{aligned} X_t \partial_u &= \partial_u X_t, \\ \partial_u^* X_t &= X_t\, \partial_u^*, \quad \forall\, u \geqq t, \end{aligned} \tag{4.18}$$

on $\mathbf{E}_0$, or equivalently,

$$\begin{aligned} \xi(u)\, \hat{X}_t(\xi, \eta) &= \frac{\delta}{\delta\eta(u)} \hat{X}_t(\xi, \eta), \\ \eta(u)\, \hat{X}_t(\xi, \eta) &= \frac{\delta}{\delta\xi(u)} \hat{X}_t(\xi, \eta), \quad \forall\, u \geqq t \end{aligned} \tag{4.19}$$

for all $\xi, \eta \in E$.

For example, for annihilation process (4.6),

$$\hat{A}_t(\xi\,;\,\eta) = \int_{-\infty}^{t} \xi(s)\,ds \exp\langle \xi, \eta\rangle$$

which obviously satisfies equation (4.19) and, therefore, is an adapted GQP.

Note that any operator $K : (E) \to (E)^*$ which commute with ∂_t^* and ∂_t for all t is a multiple of identity. This can be shown very easily by the following argument: In equation (4.19), substituting X_t by K, we see that the unique solution for functional equations (4.19) for all $u \in R$ is a multiple of the exponential functional $\exp\langle \xi, \eta\rangle$. So, roughly speaking, any adapted GQP acts like identity in the future.

To obtain the quantum Itô's formula, the essential step is to compute the so-called mutual quadratic variation of QMs which is determined by the "integration by part" formula.

DEFINITION 4.3. The mutual quadratic variation of QM $dX(t)$ and $dY(t)$ is the QM $dX(t) \cdot dY(t)$ defined by

$$\int_{-\infty}^{t} dX(s) \cdot dY(s) = X(t)Y(t) - \int_{-\infty}^{t} X(s)dY(s) - \int_{-\infty}^{t} (dX(s))Y(s) \tag{4.20}$$

provided the right hand side makes sense.

According to this definition, we have

THEOREM 4.4. *For any $l, m, n \in N_0$,*

$$1° \quad dW_{l,m}(t) \cdot dW_{0,n}(t) = 0\,; \tag{4.21}$$

$$2° \quad dW_{n,0}(t) \cdot dW_{l,m}(t) = 0. \tag{4.22}$$

Proof. Since for $\xi, \eta \in E$,

$$\begin{aligned}
&\widehat{W_{l,m}(t)\,W_{0,n}(t)}\,(\xi, \eta) \\
&\quad = \int_{-\infty}^{t} \xi(s)^m \,\eta(s)^l \, ds \int_{-\infty}^{t} \xi(r)^n \, dr \ \exp\langle \xi, \eta\rangle \\
&\quad = \Big[\int_{-\infty}^{t} \Big(\int_{-\infty}^{r} \xi(s)^m \,\eta(s)^l \, ds\Big)\, \xi(r)^n \, dr \, + \\
&\qquad + \int_{-\infty}^{t} \xi(s)^m \,\eta(s)^l \Big(\int_{-\infty}^{s} \xi(r)^n \, dr\Big)\, ds\Big] \exp\langle \xi, \eta\rangle,
\end{aligned}$$

it follows that the equation

$$W_{l,m}(t)\ W_{0,n}(t) = \int_{-\infty}^{t} W_{l,m}(r)\ \partial_r^n \, dr + \int_{-\infty}^{t} \partial_s^{*l}\, \partial_s^m \ W_{0,n}(s)\, ds$$

holds on $\mathbf{E}_0$. Thus we have proved (4.21). The equation (4.22) can be proved similarly. Q.E.D.

THEOREM 4.5. *For any $l, m \in N$ and $t \in R$,*

$$1° \quad [A(t),\, W_{l,m}(t)] = lW_{l-1,m}(t)\,; \tag{4.23}$$

$$2° \quad [W_{l,m}(t),\, A^*(t)] = mW_{l,m-1}(t). \tag{4.24}$$

Accordingly,

$$3° \quad dA(t) \cdot dW_{l,m}(t) = l\,dW_{l-1,m}(t)\,; \tag{4.25}$$

$$4° \quad dW_{l,m}(t) \cdot dA^*(t) = m\,dW_{l,m-1}(t). \tag{4.26}$$

Proof. Let $t \in R$ be arbitrarily fixed. By the canonical commutation rule, for $\xi, \eta \in E$ and $f, g \in E$ with supports in $(-\infty, t)$, we have

$$\begin{aligned} &\langle\!\langle D_f D_g^{*l} D_g^m \varepsilon(\xi), \varepsilon(\eta)\rangle\!\rangle \\ &= \langle\!\langle D_g^{*l} D_g^m D_f \varepsilon(\xi), \varepsilon(\eta)\rangle\!\rangle + l \cdot (f, g) \langle\!\langle D_g^{*(l-1)} D_g^m \varepsilon(\xi), \varepsilon(\eta)\rangle\!\rangle \end{aligned}$$

which means that, in distribution sense,

$$\begin{aligned} &\widehat{A(t) W_{l,m}(t)} (\xi, \eta) \\ &= \widehat{W_{l,m}(t) A(t)} (\xi, \eta) + l \cdot \widehat{W_{l-1,m}(t)} (\xi, \eta). \end{aligned}$$

Since the right hand side is well defined, so is the left hand side hence the equation (4.23) follows. By definition (cf. equation (4.20)) we have

$$\begin{aligned} dA(t) \cdot dW_{l,m}(t) &= dW_{l,m}(t) \cdot dA(t) + l dW_{l-1,m}(t) \\ &= l dW_{l-1,m}(t). \end{aligned}$$

equations (4.24) and (4.26) can be proved similarly. Q.E.D.

COROLLARY 4.6. *For any* $n \in N$, $t \in R$,

(4.27) 1° $[A(t), X_n(t)] = nX_{n-1}(t)$;

(4.28) 2° $[X_n(t), A^*(t)] = nX_{n-1}(t)$;

(4.29) 3° $[Q(t), X_n(t)] = 0$.

Consequently,

(4.30) 4° $dA(t) \cdot dX_n(t) = ndX_{n-1}(t)$;

(4.31) 5° $dX_n(t) \cdot dA^*(t) = ndX_{n-1}(t)$;

(4.32) 6° $dQ(t) \cdot dX_n(t) = dX_n(t) \cdot dQ(t) = ndX_{n-1}(t)$.

Proof. It suffices to look at the following equalities:

$$\begin{aligned} [A(t), X_n(t)] &= [A(t), \sum_{j=0}^{n} \binom{n}{j} W_{j,n-j}(t)] \\ &= \sum_{j=0}^{n} \binom{n}{j} [A(t), W_{j,n-j}(t)] \\ &= \sum_{j=1}^{n} \binom{n}{j} j W_{j-1,n-j}(t) \\ &= n \sum_{K=0}^{n-1} \binom{n-1}{k} W_{k,n-1-k}(t) \\ &= nX_{n-1}(t) \end{aligned}$$

https://doi.org/10.1017/S002776300000430X Published online by Cambridge University Press

and

$$\begin{aligned}[X_n(t), A^*(t)] &= \sum_{j=0}^{n} \binom{n}{j} [W_{j,n-j}(t), A^*(t)] \\ &= \sum_{j=0}^{n-1} \binom{n}{j} (n-j)\, W_{j,n-1-j}(t) \\ &= n \sum_{j=0}^{n-1} \binom{n-1}{j} W_{j,n-1-j}(t) \\ &= nX_{n-1}(t).\end{aligned}$$

The remaining parts are obvious. Q.E.D.

In particular, for the three fundamental QMs $dA(t)$, $dA^*(t)$ and $dN(t)$, we have Hudson-Parthasarathy's formulae:

$$\begin{cases} dA(t) \cdot dA^*(t) = dt \\ dA(t) \cdot dN(t) = dA(t) \\ dN(t) \cdot dA^*(t) = dA^*(t) \\ dN(t) \cdot dN(t) = dN(t). \end{cases} \tag{4.33}$$

Other mutual quadratic variations (following the Wick ordering) all vanish. This approach gives the quantum Itô's product formula a very concise form:

$$\begin{cases} dt \cdot dt = 0\,; \\ \partial_t\, \partial_t^{*m} = m\partial_t^{*(m-1)}\, (dt)^{-1} \\ \partial_t^n\, \partial_t^* = n\partial_t^{n-1}\, (dt)^{-1} \end{cases} \tag{4.34}$$

For example, we can easily obtain that

$$\begin{aligned} & dQ(t) \cdot dQ(t) = \partial_t\, \partial_t^* (dt)^2 = dt\,; \\ & dN^\lambda(t) \cdot dN^\lambda(t) \\ & \quad = (\partial_t^*\, \partial_t\, \partial_t^*\, \partial_t + \sqrt{\lambda}\, \partial_t^*\, \partial_t\, \partial_t^* + \sqrt{\lambda}\, \partial_t\, \partial_t^*\, \partial_t + \lambda\, \partial_t \partial_t^*)\, (dt)^2 \\ & \quad = (\partial_t^*\, \partial_t + \sqrt{\lambda}\, \partial_t^* + \sqrt{\lambda}\, \partial_t + \lambda)\, dt \\ & \quad = dN^\lambda(t). \end{aligned}$$

Finally, for adapted integrands, we have the Itô's formula:

THEOREM 4.7. *If* $(X(t), X^*(t)\,;\, t \in R)$ *and* $(Y(t), Y^*(t)\,;\, t \in R)$ *are adapted* GQPs, *then, for* $l, m \in N$,

$$\begin{cases} X_t\, \partial_t\, dt \cdot Y_t \partial_t^{*l}\, \partial_t^m\, dt = lX_t Y_t\, \partial_t^{*(l-1)}\, \partial_t^m\, dt \\ Y_t\, \partial_t^{*l}\, \partial_t^m\, dt \cdot X_t\, \partial_t^*\, dt = mY_t X_t\, \partial_t^{*l}\, \partial_t^{m-1}\, dt \end{cases} \tag{4.35}$$

provided the involved QIs exist. Other mutual quadratic variations with Wick ordering all vanish.

Acknowledgement. The author is very grateful to Professors T. Hida, N, Obata, L. Streit, H. H. Kuo, A. Yu. Khrennikov, J. A. Yan and Y. J. Lee for their helpful comments. He also gratefully acknowledges the referee for his valuable suggestions.

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