

SYMMETRY CLASSES OF TENSORS ASSOCIATED TO NONABELIAN GROUPS OF ORDER pq

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(Received 15 October 2015; accepted 29 December 2015; first published online 16 March 2016)

Abstract

Necessary and sufficient conditions for the existence of an orthogonal $*$ -basis of symmetry classes of tensors associated to nonabelian groups of order pq are provided by using vanishing sums of roots of unity.

2010 Mathematics subject classification: primary 20C30; secondary 15A69.

Keywords and phrases: symmetry classes of tensors, orthogonal basis, nonabelian groups of order pq , vanishing sums of roots of unity.

1. Introduction

The study of symmetry classes of tensors is motivated by many branches of pure and applied mathematics: combinatorial theory, matrix theory, operator theory, group representation theory, differential geometry, partial differential equations, quantum mechanics and other areas (see, for example, [11] and the references cited below). In particular, finding examples of (higher) symmetry classes of tensors that possess an orthogonal basis of decomposable symmetrised tensors (orthogonal $*$ -basis or o-basis, for short) is of considerable interest. This topic arose from the question by Wang and Gong in [14], and the existence of an orthogonal $*$ -basis of symmetry classes of tensors has been studied for several classes of groups: for example, dihedral groups in [8], dicyclic groups in [2], semi-dihedral groups in [9], some subgroups of full symmetric groups and some types of p -groups in [6].

Nonabelian groups of order pq have applications in group theory and graph theory (see, for example, [4]). Aspects of the symmetry classes of tensors associated to these groups have been considered. In particular, Poursalavati computed some dimensions of the symmetry classes of tensors associated with certain Frobenius groups in [13]. However, he did not investigate the condition for the existence of an o-basis. To do so, we need to handle the complicated values in the character table. We carry this through with the help of a result of Lam and Leung [10] on vanishing sums of roots of unity.

The research was supported by Naresuan University under the project R2559C049.

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2. Preliminaries

Let V be an n -dimensional complex inner product space and G a permutation group on m elements. Let Γ_n^m be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_m)$, with $1 \leq \alpha_i \leq n$. Define the action of G on Γ_n^m by

$$\alpha\sigma = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(m)}).$$

Let $O(\alpha) = \{\alpha\sigma \mid \sigma \in G\}$ be the orbit of α . We write $\alpha \sim \beta$ if α and β belong to the same orbit in Γ_n^m . Let Δ be a system of distinct representatives of the orbits and let G_α be the stabiliser subgroup of α , that is, $G_\alpha = \{\sigma \in G \mid \alpha\sigma = \alpha\}$. Let χ be any irreducible character of G .

For any $\sigma \in G$, define the operator $P_\sigma : V^{\otimes m} \rightarrow V^{\otimes m}$ on the m -fold tensor space $V^{\otimes m} := \bigotimes_1^m V$ by

$$P_\sigma(v_1 \otimes \dots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(m)}).$$

The symmetry class of tensors associated with G and χ is the image in $V^{\otimes m}$ of the symmetry operator

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma,$$

and it is denoted by $V_\chi^n(G)$. We say that the tensor $T(G, \chi)(v_1 \otimes \dots \otimes v_m)$ is a decomposable symmetrised tensor, and we denote it by $v_1 * \dots * v_m$. The dimension of $V_\chi^n(G)$ is given by

$$\dim(V_\chi^n(G)) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle factorisation of σ (see [12]).

The inner product on V induces an inner product on $V_\chi(G)$ which satisfies

$$\langle v_1 * \dots * v_m, u_1 * \dots * u_m \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m \langle v_i, u_{\sigma(i)} \rangle.$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V . It is well known that $\{e_\alpha^\otimes \mid \alpha \in \Gamma_n^m\}$ forms an orthogonal basis for $V^{\otimes m}$ associated to the induced inner product. Here, e_α^\otimes denotes the m -fold tensor $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \dots \otimes e_{\alpha_m}$ and we also write $e_\alpha^* = e_{\alpha_1} * \dots * e_{\alpha_m}$. We have

$$\langle e_\alpha^*, e_\beta^* \rangle = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if } \alpha = \beta h. \end{cases} \tag{2.1}$$

In particular, for $\sigma_1, \sigma_2 \in G$ and $\alpha \in \Gamma_n^m$,

$$\langle e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x). \tag{2.2}$$

Thus,

$$\|e_\alpha^*\|^2 = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma).$$

Define

$$\Omega = \left\{ \alpha \in \Gamma_n^m \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\},$$

and put $\bar{\Delta} = \Delta \cap \Omega$. Then $e_\alpha^* \neq 0$ if and only if $\alpha \in \Omega$.

For $\alpha \in \bar{\Delta}$, $V_\alpha^* := \langle e_{\alpha\sigma}^* : \sigma \in G \rangle$ is called the orbital subspace of $V_\chi(G)$. By (2.1),

$$V_\chi(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^*$$

is an orthogonal direct sum. In [5], it is shown that

$$\dim(V_\alpha^*) = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma). \tag{2.3}$$

Thus, we deduce that if χ is a linear character, then $\dim V_\alpha^* = 1$ and, in this case, the set $\{e_\alpha^* \mid \alpha \in \bar{\Delta}\}$ is an orthogonal basis of $V_\chi(G)$. An orthogonal basis which consists of the decomposable symmetrised tensors e_α^* is called an *orthogonal *-basis* or *o-basis* for short. If χ is not linear, it is possible that $V_\chi(G)$ has no orthogonal *-basis.

The following facts will also be needed. The first follows from [7, Lemma 1.3].

PROPOSITION 2.1. *Let $\tau \in G$. If $B = \{e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^*, \dots, e_{\alpha\sigma_r}^*\}$ is an o-basis for V_α^* , then so is $\tau B := \{e_{\alpha\sigma_1\tau^{-1}}^*, e_{\alpha\sigma_2\tau^{-1}}^*, \dots, e_{\alpha\sigma_r\tau^{-1}}^*\}$.*

The second is the main result of Lam and Leung [10] on vanishing sums of roots of unity. For a given natural number m , if there exist m th roots of unity $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in \mathbb{C}$ such that $\epsilon_1 + \epsilon_2 + \dots + \epsilon_k = 0$, then the equation is said to be a vanishing sum of m th roots of unity of *weight* k . Let $W(m)$ be the set of weights k for which there exists a vanishing sum $\epsilon_1 + \epsilon_2 + \dots + \epsilon_k = 0$, where each ϵ_i is an m th root of unity.

THEOREM 2.2 [10]. *Let m be a positive integer and write its prime factorisation as $m = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$. The weight set $W(m)$ is exactly given by $\mathbb{N}_0 p_1 + \dots + \mathbb{N}_0 p_t := \{k_1 p_1 + \dots + k_t p_t \mid k_1, \dots, k_t \in \mathbb{N}_0\}$, where \mathbb{N}_0 is the set of all nonnegative integers.*

3. Nonabelian groups of order pq

Let q be prime and p a positive integer such that $p \mid q - 1$. It is well known that there is only one (up to isomorphism) nonabelian group G of order pq , namely, G is a semidirect product of a cyclic group $C_q = \langle a \rangle$ of order q and a cyclic group $C_p = \langle b \rangle$ of order p . That is, $G = C_q \rtimes_\phi C_p$, where $\phi : C_p \rightarrow \text{Aut}(C_q)$ is a homomorphism with $\text{ord}(\phi_b(a)) = p$. A presentation of G may be given by

$$G = \langle A, B \mid A^q = B^p = 1, BAB^{-1} = A^r \rangle,$$

where r is a primitive root of the congruence $z^p \equiv 1 \pmod{q}$ and $A = (a, 1), B = (1, b)$. In particular, if $p = 2$, then G is the dihedral group of order $2q$. An embedding of G into the symmetric group S_q is also well known. Explicitly, from [1], $A = (1\ 2 \cdots q)$ as an element of S_q , and B is the product of p disjoint cycles, where the cycle containing i sends i to $1 + (i - 1)r$.

Since $G = C_q \rtimes_\phi C_p$, we can view C_q as a C_p -set with action given by $\phi_b(a) = a^r$. This action is induced as an action of C_p on the set of irreducible representations $\text{Irr}(C_q) := C_q^\vee$ of C_q . Indeed,

$$b \cdot x = x\phi_b \quad \text{for each } x \in C_q^\vee.$$

Let O be an orbit of this action and $(C_p)_x$ the stabiliser of x in C_p . For each $x \in O$ and $U \in \text{Irr}((C_p)_x)$, it can be shown (see, for example, [3]) that

$$V_{(O,U)} = V_{O,x,U} = \text{Ind}_{(C_p)_x}^{C_p} U = \{f : C_p \longrightarrow U \mid f(hg) = hf(g), h \in (C_p)_x\},$$

is an irreducible representation of G and $V_{O,x,U} \cong V_{O,y,U}$ for any $x, y \in O$. Furthermore, if $\{O_1, O_2, \dots, O_k\}$ is the set of all disjoint orbits for the action of C_p on C_q^\vee , then

$$\{V_{O_i,U} \mid U \in \text{Irr}((C_p)_{x_i}), i = 1, 2, \dots, k\},$$

with $x_i \in O_i$, forms a complete set of irreducible representations of $G = C_q \rtimes_\phi C_p$ [3]. The character of $V = V_{(O,U)}$, such that $x \in O$, is given by the Mackey-type formula

$$\chi_V(a, g) = \begin{cases} \frac{1}{|(C_p)_x|} \sum_{b \in C_p} x\phi_b(a)\chi_U(g) & \text{if } g \in (C_p)_x, \\ 0 & \text{if } g \notin (C_p)_x. \end{cases} \tag{3.1}$$

PROPOSITION 3.1. *For the nonabelian group G of order pq , where p is a positive integer and q is a prime such that $p \mid q - 1$, there are $(q - 1)/p$ irreducible characters of degree p and p irreducible characters of degree one.*

PROOF. By the above discussion, it is sufficient to find the orbits and stabilisers for the action of C_p on C_q^\vee . Since p is the smallest positive integer such that $r^p \equiv 1 \pmod{q}$,

$$[x] = \{b^t \cdot x \mid t = 0, 1, 2, \dots, p - 1\}$$

contains exactly p elements for each $x \in C_q^\vee - \{1\}$. Thus, there are $(q - 1)/p$ orbits of size p and one orbit of size 1. Let $\{1, x_1, x_2, \dots, x_{(q-1)/p}\}$ be the set of representatives of these orbits. By the orbit-stabiliser theorem, $(C_p)_{x_i} = \{1\}$ for each $i = 1, 2, \dots, (q - 1)/p$ and $(C_p)_1 = C_p$. Now, $\text{Irr}((C_p)_{x_i}) = \{\widetilde{0}\}$ and $\text{Irr}((C_p)_1) = \{\widetilde{0}, \widetilde{1}, \widetilde{2}, \dots, \widetilde{p - 1}\}$, where $\widetilde{k}(t) = e^{2\pi kti/p}$ for each $k, t = 0, 1, 2, \dots, p - 1$. Hence, $V_{([x_i], \widetilde{0})}$ and $V_{([1], \widetilde{k})}$, for each $i = 1, 2, \dots, (q - 1)/p$ and $k = 0, 1, \dots, p - 1$, form a complete list of irreducible representations of $G = C_q \rtimes_\phi C_p$. By the Mackey-type formula (3.1), we compute $\dim(V_{([x_i], \widetilde{0})}) = p$ and $\dim(V_{([1], \widetilde{k})}) = 1$, which completes the proof. \square

4. Symmetry classes of tensors associated to G and nonlinear characters

Denote by $\text{Irr}^p(G)$ the set of degree- p irreducible characters of $G = C_q \rtimes_{\phi} C_p$. In the proof of the Proposition 3.1, we have seen that

$$\text{Irr}^p(G) = \{\chi_{V_{(i, \bar{0})}} \mid i = 1, 2, \dots, (q - 1)/p\}.$$

Suppose that V is a finite-dimensional inner product space, $\chi_V \in \text{Irr}^p(G)$ and $\alpha \in \Gamma_{\dim(V)}^q$. Let $e_{\alpha}^* := T(G, \chi_V)(e_{\alpha}^{\otimes})$.

PROPOSITION 4.1. *For $\alpha \in \Gamma_{\dim(V)}^q$, $e_{\alpha}^* = 0$ if and only if α is a constant sequence.*

PROOF. It is clear that $(a^s, 1) \in G_{\alpha}$ if and only if $a^s \in (C_q)_{\alpha}$. Since $(C_q)_{\alpha}$ is a subgroup of C_q and $|C_q| = q$ (which is prime),

$$(C_q)_{\alpha} = \begin{cases} \{1\} & \text{if } a \notin (C_q)_{\alpha}, \\ C_q & \text{if } a \in (C_q)_{\alpha} \end{cases} \tag{4.1}$$

$$= \begin{cases} \{1\} & \text{if } \alpha \text{ is not a constant sequence,} \\ C_q & \text{if } \alpha \text{ is a constant sequence.} \end{cases} \tag{4.2}$$

By (3.1) and the fact that $(C_p)_{x_i} = \{1\}$, for each $i = 1, 2, \dots, (q - 1)/p$,

$$\chi_{V_{(i, \bar{0})}}(a^s, b^l) = \begin{cases} 0 & \text{if } l \neq 0, \\ \sum_{j=0}^{p-1} x_i \phi_{bj}(a^s) & \text{if } l = 0. \end{cases} \tag{4.3}$$

Thus, for $\chi_V = \chi_{V_{(i, \bar{0})}}$,

$$\begin{aligned} \sum_{\sigma \in G_{\alpha}} \chi_V(\sigma) &= \sum_{(a^s, 1) \in G_{\alpha}} \sum_{j=0}^{p-1} x_i \phi_{bj}(a^s) \\ &= \sum_{a^s \in (C_q)_{\alpha}} \sum_{j=0}^{p-1} x_i \phi_{bj}(a^s) \\ &= \sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_{\alpha}} x_i \phi_{bj}(a^s). \end{aligned}$$

Note that $x_i \phi_{bj}(1) = 1$ for each j , because $x_i \phi_{bj} \in \text{Irr}(C_q)$. Thus, $e_{\alpha}^* = 0$ if and only if $\sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_{\alpha}} x_i \phi_{bj}(a^s) = 0$, which happens if and only if $(C_q)_{\alpha} \neq \{1\}$ (by (4.1) and the second orthogonality relation for irreducible characters). The proof is now completed by (4.2). □

To obtain the condition for the existence of an o-basis, it is necessary to calculate the dimension of the orbital subspace V_{α}^* . This is a direct consequence of Freese’s theorem, which we stated at (2.3).

PROPOSITION 4.2. For $\alpha \in \Gamma_{\dim(V)}^q$,

$$\dim(V_\alpha^*) = \begin{cases} 0 & \text{if } \alpha \text{ is a constant sequence,} \\ \frac{p^2}{|G_\alpha|} & \text{otherwise.} \end{cases}$$

PROOF. By Proposition 4.1, if α is constant, then $\dim(V_\alpha^*) = 0$. Suppose that α is not a constant sequence. Then $(C_q)_\alpha = \{1\}$. By (4.3) and Freese’s theorem, (2.3),

$$\begin{aligned} \dim(V_\alpha^*) &= \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \\ &= \frac{p}{|G_\alpha|} \sum_{\tilde{a} \in (C_q)_\alpha} \chi((\tilde{a}, 1)) \\ &= \frac{p^2}{|G_\alpha|}, \end{aligned}$$

which completes the proof. □

THEOREM 4.3. Suppose that p is a positive integer and q is a prime with $p \mid q - 1$, G is a nonabelian group of order pq and χ is an irreducible character of G .

- (1) If $\dim V = 1$ or χ is linear, then $V_\chi(G)$ always admits an o-basis.
- (2) If $\dim V > 1$ and χ is nonlinear, then $V_\chi(G)$ does not admit an o-basis.

PROOF. It is well known that if χ is linear, then $V_\chi(G)$ always admits an o-basis. If $\dim V = 1$, then $\dim(V^{\otimes m}) = 1$ for any positive integer m . Thus, $\dim(V_\chi(G)) \leq 1$ (for any irreducible character χ) and hence $V_\chi(G)$ admits an o-basis.

For the nonlinear case with $\dim V > 1$, it is enough to consider the condition on each orbital subspace V_α^* . Let $\alpha = (1, 2, 1, 1, \dots, 1)$. Since $\dim V > 1$, $\alpha \in \Gamma_{\dim V}^q$. By the embedding of G in S_q , where q is prime, it is easy to see that $G_\alpha = \{1\}$. Now, by Proposition 4.2, $\dim(V_\alpha^*) = p^2$. Assume that V_α^* has $B = \{e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^*, \dots, e_{\alpha\sigma_{p^2}}^*\}$ as an o-basis. Then, by the pigeonhole principle, there must exist $1 \leq l \leq p$ and $i_1, i_2, \dots, i_p \in \{1, 2, \dots, p^2\}$ such that $\{\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_p}\} \subseteq \{(a, b^l), (a^2, b^l), \dots, (a^q, b^l)\}$. Moreover, by Proposition 2.1, $B^\circ := \sigma_{i_l} B$ is an o-basis for V_α^* as well. Thus, B° contains

$$S = \{e_\alpha^*, e_{\alpha(a^{t_1}, 1)}^*, e_{\alpha(a^{t_2}, 1)}^*, \dots, e_{\alpha(a^{t_{p-1}}, 1)}^*\}$$

for some $t_1, t_2, \dots, t_{p-1} \in \{1, 2, \dots, q\}$. Since elements in S are pairwise orthogonal, by (2.2), for each $k = 1, 2, \dots, p - 1$,

$$0 = \langle e_\alpha^*, e_{\alpha(a^k, 1)}^* \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in (a^k, 1)G_\alpha} \chi(\sigma) = \frac{1}{q} \chi((a^k, 1)) = \frac{1}{q} \sum_{j=0}^{p-1} x\phi_{bj}(a^k).$$

Hence, $\sum_{j=0}^{p-1} x\phi_{bj}(a^k) = 0$, which is a vanishing sum of q th roots of unity (because $x\phi_{bj}$ is an irreducible character of C_q and q is prime). The weight of the sum, that is, the number of terms, is p . This contradicts Theorem 2.2, which asserts that the weight is in $\mathbb{N}_0 q$. □

Acknowledgement

The author is grateful to the anonymous referee for recommendations to improve the paper.

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