CONSTRAINTS ON STELLAR EVOLUTION FROM PULSATIONS

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Consideration of the many types of intrinsic variable stars, that is, those that pulsate, reveals that perhaps a dozen classes can indicate some constraints that affect the results of stellar evolution calculations, or some interpretations of observations. Many of these constraints are not very strong or may not even be well defined yet. In this review we discuss only the case for six classes: classical Cepheids with their measured Wesselink radii, the observed surface effective temperatures of the known eleven double-mode Cepheids, the pulsation periods and measured surface effective temperatures of three R CrB variables, the δ Scuti variable VZ Cnc with a very large ratio of its two observed periods, the nonradial oscillations of our sun, and the period ratios of the newly discovered double-mode RR Lyrae variables. Unfortunately, the present state of knowledge about the exact compositions; mass loss and its dependence on the mass, radius, luminosity, and composition; and internal mixing processes, as well as sometimes the more basic parameters such as luminosities and surface effective temperatures prevent us from applying strong constraints for every case where currently the possibility exists.

Use of the pulsation existence as evidenced by observed periods is often of limited value because the actual mode of the pulsation is uncertain. For example, there are the questions whether some classical Cepheids are radial fundamental or first overtone pulsators, or whether the β Cephei variables are radial or nonradial pulsators. This problem is sometimes very embarrassing because the dispute over whether the Mira variables are radial fundamental or radial first overtone pulsators means that we cannot identify periods that are a factor of three different. For most cases, however, the principal mode of pulsation is well known, although any other modes present can be identified only with less certainty.

Perhaps the greatest successes for the constraints of pulsation theory are for the classical Cepheids, the RR Lyrae variables, and the population II Cepheids (BL Her and W Vir variables). Masses derived by some pulsation-based methods agree well with evolution theory, but there are

A. Maeder and A. Renzini (eds.), Observational Tests of the Stellar Evolution Theory, 421–440.

problems that still exist. These problems might eventually be bona-fide constraints to be settled by changes in our current ideas of the internal composition structure. It seems that, except for much needed accurate surface effective temperatures for Mira variables and for the extremely hot dwarf stars and more periods for double-mode RR Lyrae variables in the field and in globular clusters, the next advances in these pulsation-based constraints will come from theoretical calculations of internal mixing and element separation and from nonlinear pulsation calculations.

Cepheid masses have been shown by Cox (1979) to be consistent with evolution calculations using the new larger distance scale of Hanson (1977) and the cooler effective temperatures of Pel (1978), Flower (1977) and Bell and Parsons (1972). We ignore for this discussion the problems for the double-mode and bump Cepheids that can indicate internal helium enhancements or magnetic fields. For periods longer than 10 days, there has been a remaining problem that the Wesselink masses used with the period-mean density relation produce masses too small by typically a factor of 0.6-0.7. We here propose as Burki (1983) recently has done, that these Wesselink radii indicate mass loss rates that can be used as a constraint on evolution calculations which include mass loss.

The measured Wesselink radius has been discussed by Burki, who shows that the presence of a companion star can greatly affect the derived radius. Therefore, it is better that only truly single Cepheids be considered. Solution of three equations, the period-mean density relation, the definition of the surface effective temperature, and a fit for the pulsation constant Q as a function of mass, radius, luminosity, and effective temperature, are solved for three unknowns, Q, luminosity and mass when the period, effective temperature and Wesselink radius are given. The Cepheids RS Pup, l Car, U Car, and SV Vul (possibly a binary) are shown to have a mass as small as 0.3 the so-called theoretical mass. Burki shows that the Maeder (1981) evolution tracks with moderate to large mass loss (cases B and C) bracket the current masses for these four Cepheids. Figure 1 gives these Burki masses versus luminosity for these Cepheids together with the case A (no mass loss), B and C relations for the initial masses of 9 and 15 $M_\odot$. It appears that the mass loss rate almost as high as given for case C is indicated for these four variables.

Actually both RS Pup and SV Vul have measured luminosities and effective temperatures, allowing the calculation of their pulsation masses. RS Pup shows little mass loss in Figure 1, and indeed the Wesselink and pulsation masses agree at 10 $M_\odot$ within about 2 $M_\odot$. For SV VuL, however, it appears that evolution, theoretical and pulsation masses yield about 12 $M_\odot$, but the Wesselink value is less than 4. It may be that indeed SV Vul is a binary star with a blue companion which wrongly reduces its Wesselink radius by an appreciable amount and therefore reduces the Wesselink mass to a very low, incorrect value. We do not have luminosity data for the two other Carina stars to

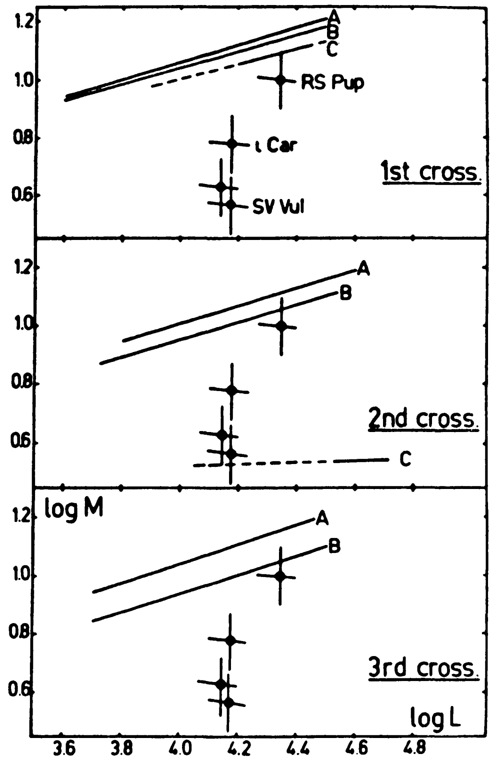

Figure 1. The luminosity-mass relation for four Cepheids with Wesselink masses and luminosities from the work of Burki. Also plotted are the Maeder theoretical relations during the first, second, and third crossings of the Cepheid instability strip for no mass loss (case A), moderate mass loss (case B) and high mass loss (case C).

dispute their Wesselink masses, and so factors of two mass loss for long period Cepheids may be correct. This result constitutes a constraint on the mass loss rates for yellow giants of original mass about 12 $M_{\odot}$.

Actually the observed SV Vul period decrease reported by Fernie (1979) also indicates blueward evolution that occurs slowly compared to the first and possibly the third crossings of the instability strip when the original mass is about 15 $M_{\odot}$. Since evolution tracks indicate very rapid blueward evolution when the mass loss rate is as high as for case C, it is very unlikely that such a short-lived Cepheid would be observable with a rapidly decreasing period. The large mass loss of case C seems excluded by observations of the moderate long period Cepheid period decreases.

Since the large oxygen depletion reported by Luck and Lambert (1981) no longer seems to be correct, the large overshooting at 9 $M_{\odot}$ considered by Becker and Cox (1982) in the main sequence stages is not necessary. The result that the blue loops are at a much higher luminosity than thought before is unaffected, however, and the 15 or more day Cepheid evolution masses are reduced as much as 17% as reported by Becker and Cox and by Matraka, Wasserman and Weigert (1982).

For shorter period Cepheids, there is another problem. The two periods of the double-mode Cepheids and the bumps for the 5 - 15 day bump Cepheids suggest masses considerably less than the evolutionary, theoretical, or pulsation masses. The factor is 0.6 for the bump Cepheids, while it is about 0.3 for those double-mode Cepheids that directly display two periods. The question for the last 10 years has been what is the cause of such large theoretical P_1/P_0 period ratios and such low derived masses? Are they really correct or is something else wrong? The current proposed answers are that the surface layers have an unconventional structure which produces a few percent increase of the fundamental radial mode period relative to a similar increase of the first overtone mode period. This structure can be established by an actual mass loss or an enhanced helium content, a magnetic field, or an opacity increase in the outer 10^{-4} of the mass. The first and last of these suggestions now seems to be ruled out by other considerations that we will not discuss here.

Other discussion of the cause of this anomalously small period ratio, as indeed the cause of the double-mode pulsation, we leave to another talk. Here we merely look at the position of these variables on evolutionary tracks to see what constraints exist on evolutionary theory. Balona and Stobie (1979) made careful measurements of the intrinsic colors of eight of the 11 double-mode Cepheids. Conversion of these colors by a formula given by Bell and Parsons (1972), shows that all these variables have the same effective temperature of 5940 K within a small scatter of about 130 K. More recently, Barrell (1981) using the shape of the hydrogen α line, has verified both the narrow range of the temperature and its mean value for all 11 variables. Our present interest is to see if we can learn what this means for evolution tracks.

The cause of blue loops that produce Cepheids is the existence of the μ-gradient region left behind by a shrinking convective core in main sequence evolution. Too much mass loss can disrupt this structural feature and produce only a return to near the main sequence. Also for only rather low or rather high ratios of the mixing length to pressure scale height in the surface convection zone can these blue loops bring the tracks into the Cepheid variable instability strip.

Recently Huang and Weigert (1983) have investigated this last point as well as how much overshooting is allowed from the central hydrogen burning convective core to homogenize the bottom part of the μ-gradient shell. They find that, at 5 $M_\odot$, overshooting of no more than half of a pressure scale height above the convective core is allowed or no blue loops and no Cepheids will be predicted. Figure 2 shows the blue loops for the two cases with the core convection overshooting 0.5 and 1.0

https://doi.org/10.1017/S0074180900031259 Published online by Cambridge University Press

pressure scale heights. The effects of the outer ratio of convective mixing length to pressure scale height are also indicated for both core overshooting cases. This limit of only about 0.5 scale height core overshooting is reasonable because there is really a very strong μ-barrier against the outward motion of the heavy helium-rich core material into the lighter hydrogen-rich layers. For normal compositions then, there are these reasonable constraints: some, but not much, core overshooting, Maeder and Mermilloid (1981), and a convective envelope ratio of mixing length to pressure scale height not much smaller than 1.5, just as appropriate for our sun.

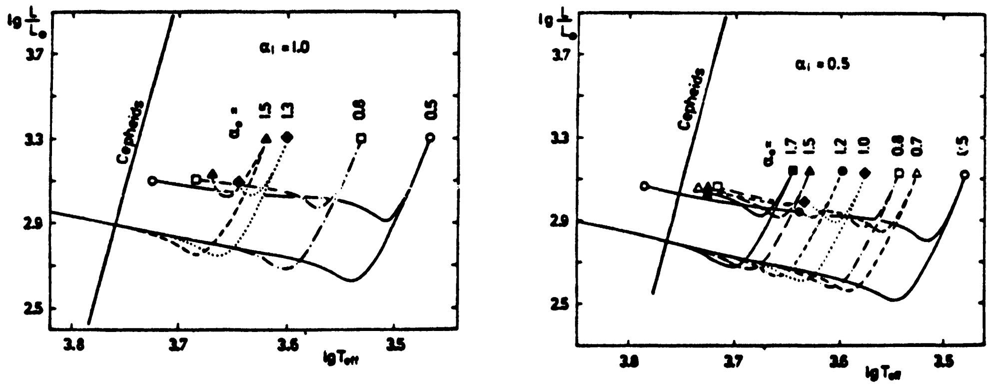

Figure 2. Theoretical Hertzsprung-Russell diagrams are given for the core overshooting of 1.0 and 0.5 pressure scale heights. For each case the ratio of the mixing length to pressure scale height ranges from 0.5 to 1.7 for the outer convective shell.

Unfortunately, the length of the blue loops depends also on the composition. For low Z, the tracks can go much bluer, but such low Z values do not seem reasonable for the recently born 5 to 15 $M_\odot$ Cepheids. Figure 3 shows the track of a 6.0 $M_\odot$ short period Cepheid with a period of about five days at the blue loop tip. The upper blue loop is traversed very rapidly after the end of core helium burning. Notice that the mass is very reasonable and that the Z=0.02 value of the composition is normal. For lower luminosities and periods at the blue loop tips, the luminosity needs to be as low as $\log(L/L_\odot) = 3.0$ which occurs with a 5 $M_\odot$ model with a Z of 0.017. The very large number of the double-mode Cepheids (maybe one in four between periods of 2 and 5 days and almost one in two between 2 and 3 days) points to evolution tracks with a slight mass dependent Z, older stars having a lower Z, to give the long-lived blue loop tips (double-mode regime?) at the observed effective temperature for all these double-mode stars. To get the even hotter Cepheid SU Cas at an effective temperature of over 6300 K, one needs an even lower Z value, or perhaps more reasonably an assignment of this 1.95 day period Cepheid as a first crossing star. These tracks come from the library of tracks of Becker and Mathews, and the assumption here is for no core overshooting. If there

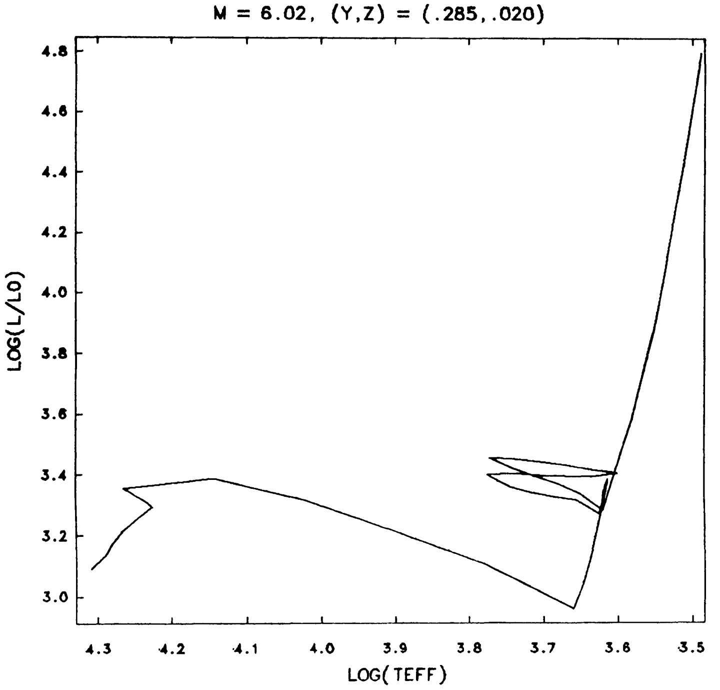

Figure 3. The evolution track for a 6 $M_\odot$ model with the Y and Z compositions, respectively, 0.285 and 0.020. The blue loops have their tips just at the effective temperature observed for the double-mode Cepheids.

is a little overshooting at 5 $M_\odot$, then the Z value needs to be even lower than 0.017, but not excessively so.

There is an unsolved problem here, though. Paczynski (1970), Robertson (1971), and Robertson and Faulkner (1972) have considered the possible overshooting of the core convection on the blue loops in the core helium burning stage. This overshooting extends the blue loops and may upset our argument about the Z value for the composition. One nice feature of their work is that with this overshooting during the blue loops, an even larger fraction of time is spent at the blue tips. With all the proper effects included, we may find that double-mode Cepheids naturally occur at the blue loop tips for a conventional composition. The final results of Robertson (1973) need further discussion, especially since the large opacity increases over the Cox and Stewart (1965) values no longer seem justified.

Turning now to another class of variable stars, Saio and Wheeler (1983) have considered the constraint of pulsation theory on the mass of three well known R CrB stars. These hydrogen-deficient stars presumably have lost their hydrogen envelope by a high mass loss rate either from a rather massive star, say 10 - 15 $M_\odot$, or from an older, less massive, star evolving through the asymptotic giant branch region.

Figure 4 shows the blue edge for the fundamental radial pulsation mode on the Hertzsprung-Russell diagram for 1, 2, and 3 $M_\odot$. The cause of the sharp turn of the blue edge towards higher effective temperatures at a critical luminosity is that the envelope density there becomes very low. At this mass-dependent critical luminosity the radiatively damping region of the star loses its effectiveness, while the cyclical ionization of helium even nearer to the surface still causes pulsation driving.

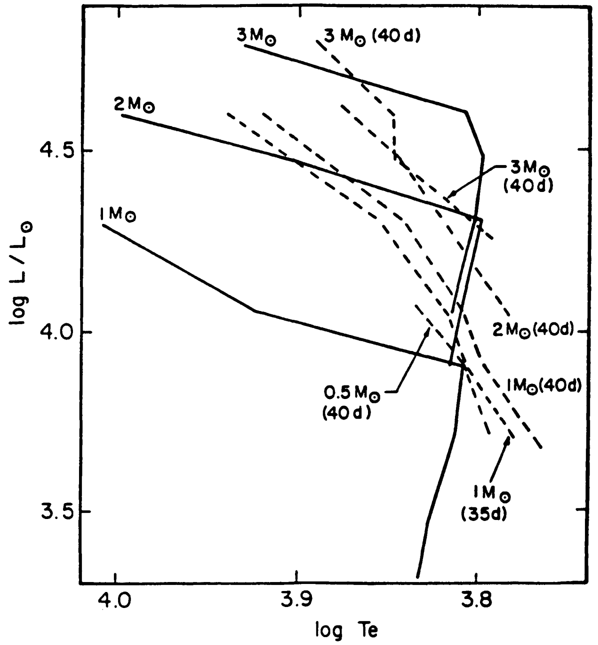

Figure 4. The Hertzsprung-Russell diagram in the region of the R CrB variables. Blue edges are marked as solid lines for 1, 2, and 3 $M_\odot$. Also given are lines of constant period for masses and periods as marked.

The figure also gives dashed lines for the constant period of 40 days and one case for 35 days. For the 3 $M_\odot$ case, we can see the sudden increase in luminosity near $\log(L/L_\odot)$ of 4.5 due to mode bumping. This bumping occurs when stars become very centrally concentrated at high luminosity and periods decrease compared to those for less concentrated models. This decrease of a so-called longer period strange mode period makes it close to that for the fundamental which is less decreased in its period. Therefore, the potential collision of the two periods is solved by the star when the fundamental period is bumped down in period. This means that to retain the 40-day period, the radius and luminosity need to be increased suddenly.

The observed periods and surface effective temperatures for R CrB and RY Sgr are 44 and 39 days and 7000 K and 7100 K. This first effective temperature is uncertain by only 250 K, whereas the second has its uncertainty as 600 K. XX Cam has similar properties, but it does not pulsate, probably because its luminosity is just a bit lower than for the other two stars and it never enters the pulsation instability strip.

Mass and luminosity limits can be obtained for the two variable stars by noting where the line of the observed period intersects the nearly horizontal blue edge at possible effective temperatures of log T_e between 3.829 and 3.860 for R CrB and 3.813 and 3.886 for RY Sgr. Above about 3 $M_\odot$ the variable stars would not be predicted to pulsate because the line of constant period of 40 days is always in the stable, less luminous, region for all possible effective temperatures. Since RY Sgr can be a little bluer than R CrB, its mass and luminosity may possibly be a bit higher. For lower masses, the lines of constant period always are in the unstable region. In these cases, the luminosity limits always come from the observed effective temperature limits with, of course, a larger range for RY Sgr.

Figure 5 presents the relation between the mass and luminosity limits. At low mass, if it all is in a helium core, there is an upper limit to the luminosity that can be produced according to Paczynski (1971), and that is the dash-dot line. The Eddington limit for the luminosity at any given mass is the upper solid line. Recently Cottrell and Lambert (1982) have given for R CrB a value for the gravity that can be combined with the measured effective temperature to constrain the mass-luminosity relation, and that is the heavy dotted line. We see that the two R CrB variables that pulsate have masses between about 0.8 and 3.1 $M_\odot$ and luminosities between $\log(L/L_\odot)$ of 4.1 to 4.8. Further work on nonlinear pulsation calculations is now being done by Saio.

We now turn to some constraints from pulsation theory for δ Scuti variables. Here we consider the slow settling of helium below the bottom of the hydrogen-helium ionization convection zones. A 2 $M_\odot$ star evolves slowly enough that this draining of helium can operate even in the presence of mixing processes such as those from rotation and turbulent diffusion. During this stage the convection zones remain homogeneous due to the strong motions of convection, but the helium content steadily decreases. When the helium mass fraction decreases to about 0.08, then the deeper helium convection zone disappears. At this low helium content the opacity is now low enough so that all the internally generated luminosity can be carried by radiation alone. However, the outer hydrogen convection zone is, if anything, slightly enlarged because the high opacity of the hydrogen-helium mixture is even now increased with less dilution by helium which does not contribute so much to the opacity at 10,000 K.

New evolution tracks have been computed by Andreasen, Hejlesen, and Petersen (1983) allowing the helium to settle to depths of about 300,000 K and 1,000,000 K. As one might expect, the tracks are not greatly changed on the Hertzsprung-Russell diagram. However, these

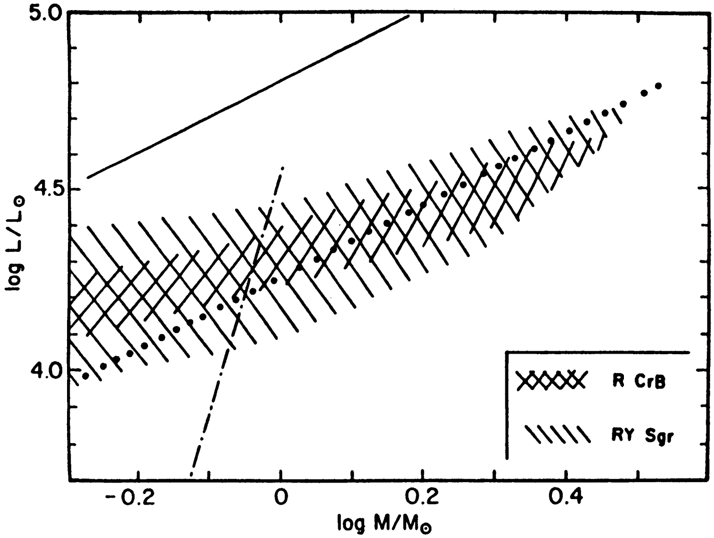

Figure 5. The allowed regions of mass and luminosities for R CrB and RY Sgr from pulsation theory are given as hatched areas. The maximum luminosity as a function of mass is also drawn for low masses. The solid line is the Eddington limit as a linear function of mass. The dotted line is the mass-luminosity relation derived from the observation of the effective temperature and gravity of R CrB.

authors have found that as the stars expand in their normal evolution and their pulsation periods increase, the period ratio of the first radial overtone to the fundamental increases to much larger values than ever considered before. Their period ratio reaches 0.79 compared to just over 0.76 for 1.7 to 2.0 $M_\odot$ when the helium is depleted down to about 300,000 K. Deeper helium depletion returns the period ratio back to or even slightly lower values.

This behavior of the overtone to fundamental period ratio is just as expected from the studies by Cox, King, and Hodson (1979) for decreasing the period ratios for double-mode Cepheids. In that case, surface enrichment was needed to reduce the ratio by about 0.01 to 0.05 for masses, respectively of 6 and 4 $M_\odot$. As discussed above, the physical effect to change the period ratio is a change in the apparent concentration of the star as seen differently by the two modes. For the double-mode Cepheids the enrichment seems best for depths of 250,000 K, not too much deeper than we will see is appropriate for VZ Cnc our δ Scuti variable.

With McNamara from New Mexico State University, I began wondering if the large-period ratio observed for the 0.178 day δ Scuti variable VZ Cnc might indicate this helium settling. This star has been reobserved

https://doi.org/10.1017/S0074180900031259 Published online by Cambridge University Press

recently by McNamara to compare with data obtained by Fitch (1955). Both observers get the ratio to be 0.8006, but the modern data give clearly a change in the amplitude ratio of the two modes with the overtone now 0.326/0.276 larger relative to the then existing fundamental amplitude. The accuracy of the periods themselves is not great enough to detect the few parts in a million period increase expected in the 30-year time interval, and therefore, a check that the expected redward evolution is really occurring is not yet possible. Anyway, it appears that the overtone is getting appreciably stronger rather than weaker as the star evolves to the red and to a more pure fundamental mode domain.

Figure 6 gives the period ratio being discussed for five calculated depths of the helium depletion in a model for a 2 $M_\odot$ star with the observed surface effective temperature and a luminosity such that the observed period is matched to the radial fundamental. To reproduce the observed period ratio for VZ Cnc, a depletion profile is needed that has a surface helium mass fraction, Y, of either 0.00 or 0.08 down to almost 200,000 K. Deeper the Y increases to 0.08 until a depth is reached where the temperature is almost 300,000 K. Then we set Y=0.18. Finally Y=0.28, the normal value, is used for depths deeper than about 400,000 K.

The apparently correct interpretation for VZ Cnc is that its helium has settled to below 200,000 K. This makes the blue edge of the instability strip for such a star just at 7000 K, and the red edge is also probably at about 7000 K. Thus this star just barely has an instability strip, and it is driven by hydrogen γ and κ effects alone. Note that as the helium settles, the fundamental mode blue edge evolves to the red due to the depletion of helium. When a star is at or near this

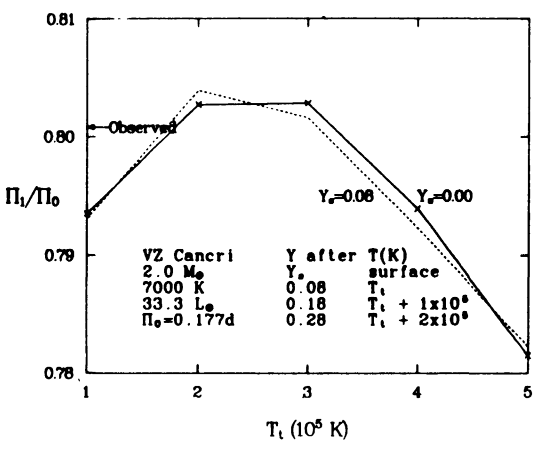

Figure 6. The period ratio versus transition temperature is plotted for five depths of settling of helium in a 2 $M_\odot$ model at 7000 K with the period of 0.177 day. Two surface helium content mass fractions have been considered. The observed period ratio is also indicated.

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blue edge there is a strong tendency to move over to an overtone pulsation as we see for double-mode Cepheids and double-mode RR Lyrae variables. It appears that this star has a redward evolution of its fundamental mode blue edge that is faster than the evolution of the star itself to the red, because the overtone mode is getting stronger. Helium depletion to depths such as we have found seem reasonable from the current results of diffusion theory.

One of the current interests in the constraints of pulsation theory on evolution theory is in the solar 5 minute oscillations. The low degree modes with $\ell=0$ to 5 give sensitivity to the central temperature and density structure, and that in turn bears on the persistent problem of the higher than observed predictions for the neutrino flux from the sun. The reason for the sensitivity of the periods of low degree oscillations is that the spacing between nodes near the center of the sun for, say the 10th overtone, is comparable to the horizontal distance between node lines when $\ell=1$ to 5. Thus a change in the central conditions results in a change in the nodal structure which reflects itself in changes of periods.

Figure 7 gives the internal structure of a solar model calculated by us at Los Alamos. The logarithms of the temperature, opacity, and density are plotted versus the external mass fraction. The central temperature for this model is about 15 million kelvin, and the central density is about 122 g/cm^3, both somewhat different from other recent models. These differences may be due to the equation of state of the solar material that we used. Some coulomb corrections for the pressure have been included, but maybe not in as elaborate way as Ulrich and his collaborators or Shibahashi and his collaborators have done. The very high opacity across most of the figure produces a convective shell down to a temperature of just over 2.5 million kelvin.

Figure 8 shows the hydrogen composition mass fraction profile for four different models. The basic profile is due to Christensen-Dalsgaard (1982) who has made an actual evolution calculation using standard methods. The solid stepped line shows our Los Alamos model which attempts to track the Christensen-Dalsgaard model as well as possible. The differences are needed due to slight differences between the equation of state and opacities. Note that we need a smaller amount of helium everywhere and it is not clear whether the sun produces that 10% smaller amount during its lifetime. The figure also shows two partially mixed models, one with a homogeneous shell between q=0.3 and 0.5 and another with the central 15% of the mass homogeneous as shown. The pulsation periods for low degree, high overtone modes will later be calculated, and the effects of these two mixed layers on the periods will be given.

Observations of these modes with frequencies between 900 and 4500 μHz are given in Figure 9. The plus signs, circles and triangles give data for many of the modes, and the dots are the latest Shibahashi, Noels, and Gabriel (preprint) theoretical results. The figure is really only

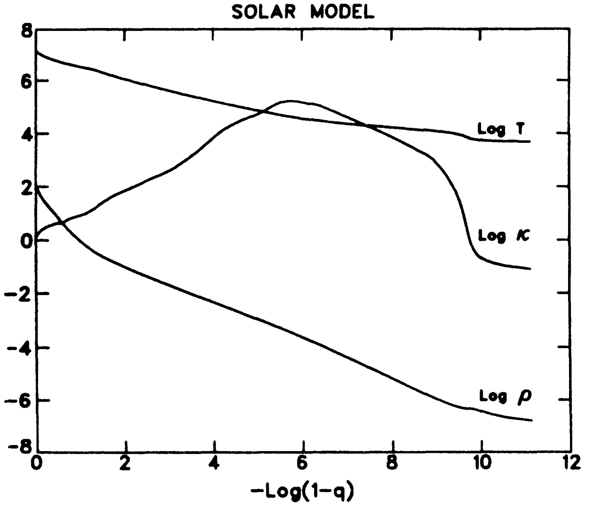

Figure 7. The logarithms of the temperature, opacity, and density are plotted for our solar model versus surface mass fraction. The units are: temperature (K), opacity (cm^2/g), density (g/cm^3).

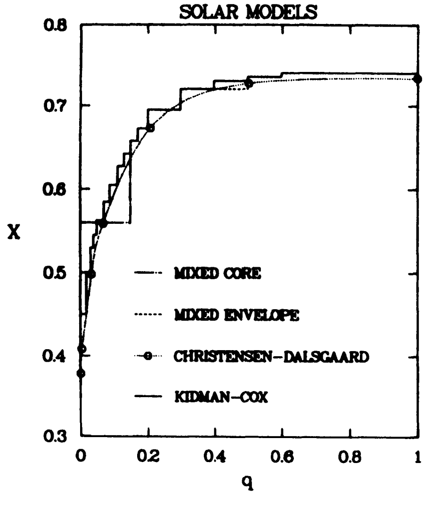

Figure 8. The hydrogen mass fraction, X, is plotted versus internal mass fraction for four models including one with a mixed region in the envelope between q=0.3 and 0.5 and one homogenized in the central 0.15 of the mass

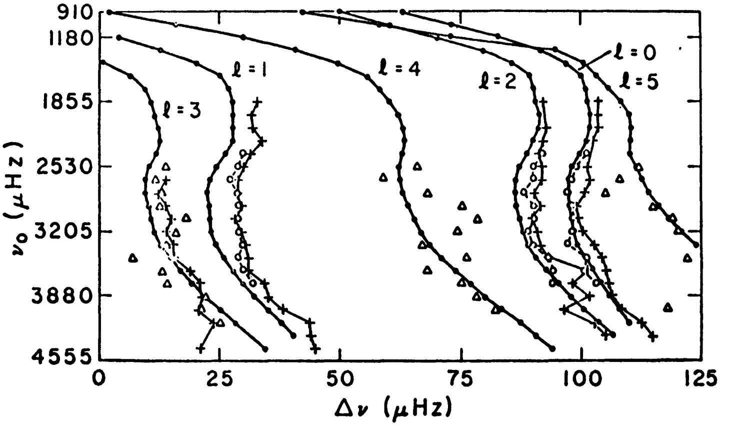

Figure 9. Frequencies of solar oscillations calculated by Shibahashi, Noels and Gabriel are plotted in 135 μH sections and placed above one another. The observed nonradial modes are also given as presented by Claverie et al. (1981) (open circles), Grec et al. (1980) (crosses), and Sherrer et al. (1982) (triangles).

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one-dimensional, but the line of increasing frequency is cut into segments of 135 μH and these lines are displayed one above the other. The match between theory and observation is good for these adiabatic periods. Our nonadiabatic periods are typically 0 to 7 μHz smaller in frequency. A thing to note about this figure is that the even ℓ values are close together and so are the odd ones if the l=5 curve had been plotted one raster line below. As shown by Vandakurov (1967) and recently by Tassoul (1980), the difference in frequency between two modes separated by +2 in ℓ and -1 in n (overtone) is almost zero. The splitting of these modes is a good probe of the internal solar structure and its possible mixing either now or in the past.

The table of frequencies give our values for our standard, mixed envelope, and mixed core models. We have been limited at the present time to only 330 mass shells, but we believe that maybe 100 more zones are required to get accurate frequencies even for these longer than average periods, and lower than average order. Thus we get frequencies too large by 1.0 to 1.5%. To approximately correct for this defect in order to assess the effects of the model structure on frequency splitting, we have corrected all of our frequencies by the ratio of the measured to standard frequencies. This should reduce our splitting prediction errors to much less than 1%.

The second table, with the separation of successive modes at a fixed ℓ value, averaged over all the four overtones, and with the above mentioned ℓ splitting averaged over four pairs shows the effect of the two mixed models. Our models still seem to have the separations slightly too large, though there is apparently no sensitivity to the possible internal mixing as expected. This separation is determined by the model structure in the outer 10% of the mass. We do see that homogenizing the envelope shell seems to slightly increase the ℓ splitting. An even larger increase, as also discussed recently by Ulrich and Rhodes (1983), is seen for the core mixed model. It seems that the ℓ splitting is sensitive to the core structure. Mixing increases the splitting, but our core spacial resolution is too coarse to match observed splittings. No mixing at all seems to be a more correct model for our sun.

As the final subject, we discuss the masses of the RR Lyrae variables. The discovery of the double-mode RR Lyrae variables in M15 and M3, as well as in the field (AQ Leo), in M68, and in the Draco galaxy, has allowed us to measure their masses. The value of 0.65 $M_\odot$ seems just right for the Oosterhoff group II cluster M15. Only two stars are available for the Oosterhoff group I cluster M3, and they seem to indicate 0.55 $M_\odot$. This mass is too low for current ideas of the 0.8 $M_\odot$ stars evolving with some unknown mass loss rate on the red giant branch and their helium-rich core mass when they experience the core flash. If the low mass for M3 is correct, there is a strong constraint for red giant evolution theory. At 0.55 $M_\odot$, stars are very blue and not in the pulsational instability strip unless their helium core mass is also quite low, 0.425 rather than the 0.475 $M_\odot$, as currently accepted as appropriate.

NONADIABATIC FREQUENCIES (μHz)

DEGREE	ORDER	STANDARD MODEL	MIXED ENVELOPE	MIXED CORE	MEASURED
0	10	1595	1595	1597	
	11	1732	1732	1734	
	12	1867	1868	1869	1824
	13	2002	2003	2004	1954
	14	2138	2139	2140	2094
1	10	1645	1645	1645	
	11	1782	1782	1783	1753
	12	1919	1919	1920	1888
	13	2056	2056	2057	2022
	14	2193	2193	2194	2157
2	10	1698	1699	1698	
	11	1835	1835	1835	1812
	12	1971	1972	1971	1947
	13	2110	2110	2110	2082
	14	2247	2248	2247	
3	10	1749	1749	1749	
	11	1887	1887	1887	
	12	2025	2025	2024	
	13	2163	2163	2162	
	14	2301	2301	2301	
4	10	1795	1794	1794	
	11	1934	1934	1934	1918
	12	2074	2074	2074	2052
	13	2213	2212	2212	
	14	2352	2352	2351	2323

Table 1. Frequencies of our solar model are given for the longer observed periods in the 5 minute region.

AVERAGE CORRECTED SEPARATIONS AND SPLITTINGS (μHz) (N=10-14)

L	STANDARD MODEL	MIXED ENVELOPE	MIXED CORE	MEASURED
$\overline{\Delta\nu}(0)$	134	134	134	135
$\overline{\Delta\nu}(1)$	135	135	135	135
$\overline{\Delta\nu}(2)$	135	135	135	135
$\overline{\Delta\nu}(3)$	136	136	136	
$\overline{\Delta\nu}(4)$	137	138	137	134
$\overline{\delta\nu}(0)$	31	32	33	10
$\overline{\delta\nu}(1)$	31	32	32	18
$\overline{\delta\nu}(2)$	36	38	37	30

Table 2. Separations and ℓ-splitting in units of μHz are given for ℓ values of 0 - 4.

Figure 10 shows how photometric data can reveal double-mode behavior. From a Sky and Telescope report we see on top the Sandage, Katem, and Sandage (1981) blue magnitude data for M15 variable V31. The period used to phase the observations made over three summers is 0.408231 day. The scatter in the data indicates that maybe the derived period is not accurate enough. Cox, Hodson, and Clancy (1981,1983) noticed that the scatter in this light curve and many others with periods near 0.4 days revealed the presence of two periods of the star. The separation of the two components is also in the figure with the periods of the overtone and fundamental modes indicated. The lower two panels show bolometric magnitudes rather the photographic blue magnitudes above. There are 17 RR Lyrae variables with periods between 0.389 and 0.429 day in

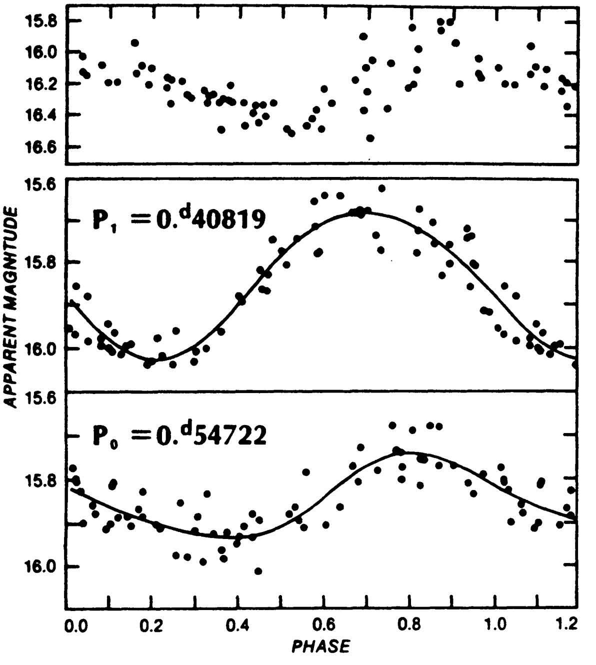
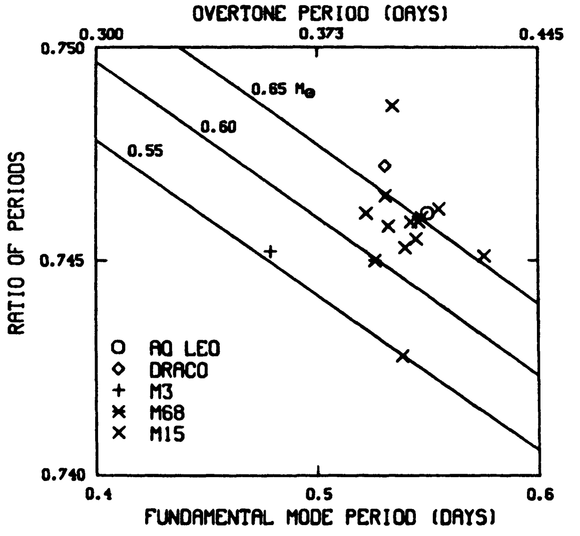


Figure 10. In the upper panel are given blue magnitudes for V31 as published by Sandage, Katem and Sandage. Below are phased bolometric magnitude data for the two separate modes with their periods indicated.

M15, and there are indications that all pulsate with two modes simultaneously. For the 12 cases where a good period ratio can be determined, the overtone is always the stronger mode.

From ideas first presented by Jorgensen and Petersen (1967), it is possible to use the period ratio to derive a mass for each star. This is physically possible because the two modes sample the outer structure differently due to their different oscillation amplitude variations. The best way to portray this mass dependence is to show in Figure 11 the period ratio versus period. This diagram depends only slightly on the model luminosity and effective temperature, and so it is usable because it has been calculated for these two parameters being close to those for real RR Lyrae variables. In addition to 12 M15 variables, we show also in the figure the period ratio for the field RR Lyrae variable AQ Leo, a representative point for nine Draco galaxy RR Lyrae variables, a point representing the only two known M3 variables, and a point for the single known M68 double-mode variable. It appears that the M15 stars have a mass of 0.65 $M_\odot$ as well as those nine (only one plotted) in Draco and for AQ Leo. The intermediate Oosterhoff group cluster M68 seems to have a mass for its single known double-mode variable near 0.60 $M_\odot$. According to Nemec at this meeting V75 in Draco also has a mass of 0.60 $M_\odot$. Finally the two M3 variables and V165 in Draco point to 0.55 $M_\odot$. Since M3 is representative of an Oosterhoff group I cluster, there is the possible interpretation that group I clusters have horizontal branch stars of 0.55 $M_\odot$ and the group II clusters, with certainly higher luminosity and lower metal composition stars, have masses of their stars of 0.65 $M_\odot$.

Figure 11. For all the listed double-mode RR Lyrae variables are plotted their period ratios versus their fundamental and overtone periods. The lines of constant mass are computed from linear nonadiabatic radial pulsation theory.



With a mass of 0.65 $M_\odot$, it is possible to make a theoretical Hertzsprung-Russell diagram with the RR Lyrae variable instability strip blue and red edges shown. Figure 12 also gives the lines of constant fundamental mode period and lines of constant period ratio. A standard composition called King Ia has been used to derive the periods and blue edges from linear pulsation theory. The red edge comes from the calculations of Deupree (1977) which show how dependent the red edge is upon the helium content. On this diagram, we have plotted the mean luminosities of the M15 RR Lyrae variables with F for the fundamental mode variables and O for the overtones. The double-mode variables are plotted as + signs. Here we use only the relative luminosities and periods as observed. The observed effective temperatures are not used. A distance modulus of 15.28 for M15 results in the luminosities that are plotted. Possible errors in the magnitudes and that from a change in mass by 0.05 $M_\odot$ from the assumed 0.65 $M_\odot$ are indicated. The interesting thing is that the the eight plotted double-mode variables all lie between the fundamental and overtone pulsators suggesting that they are the result of some kind of mode switching.

From the mass of 0.55 $M_\odot$, we can construct another Hertzsprung-Russell diagram that can be used for plotting the M3 RR Lyrae variables. For this globular cluster we need a distance modulus of 15.00 to fit the variables into the theoretical instability strip. This is Figure 13 that again shows the separation of the fundamental and overtone pulsators. The two double-mode stars have not been observed on the photometric scale of Sandage (1981), and therefore, are not plotted. The appearance of overtone pulsators in the theoretical fundamental mode region is a problem for both M15 and M3. We would expect that overtone pulsation could occur slightly redward of the fundamental blue edge as the overtone there may be able to grow in a fundamental mode pulsator.

https://doi.org/10.1017/S0074180900031259 Published online by Cambridge University Press

But the overtones and the double-mode pulsators seem to be much too red to match the best currently available blue edges and the fundamental-overtone transition line.

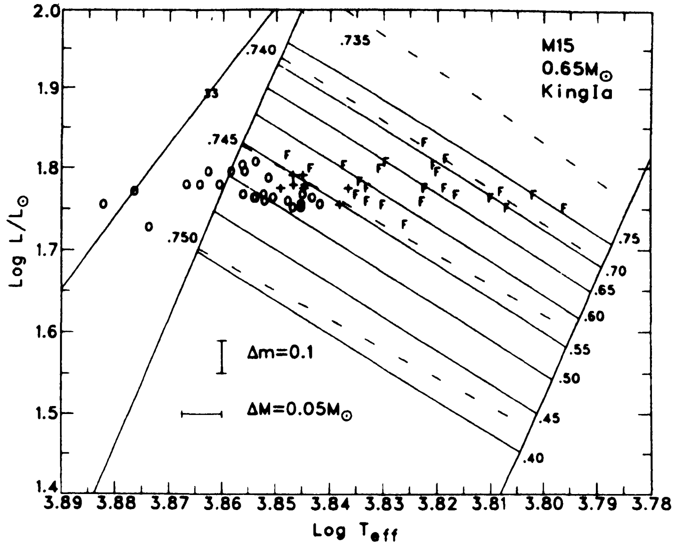


Figure 12. The RR Lyrae pulsation instability strip is plotted on the Hertzsprung-Russell diagram for a mass of 0.65 $M_\odot$ and the King Ia mixture composition. Mean luminosities over the pulsation cycle are plotted for the M15 RR Lyrae variables with the fundamental pulsators marked by F, the overtones by O, and the double-mode variables by +.

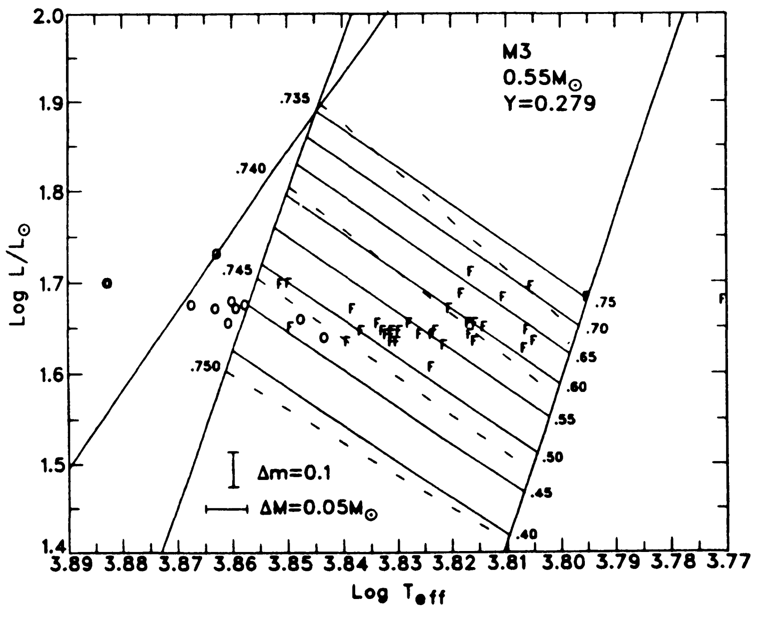


Figure 13. The RR Lyrae pulsation instability strip is plotted on the Hertzsprung-Russell diagram for a mass of 0.55 $M_\odot$ and the mixture composition with Y=0.279. Mean luminosities over the pulsation cycle are plotted for the M3 RR Lyrae variables with the fundamental pulsators marked by F, the overtones by O, and the double-mode variables by +.

The constraint on evolution theory is that maybe the mass of the RR Lyrae variables differs with Oosterhoff groups, being lower in the higher metal content Oosterhoff group I clusters. Actually, with a core mass of about 0.475 $M_{\odot}$, as currently preferred, a mass of 0.55 $M_{\odot}$ would be born on the horizontal branch and evolve at much bluer positions than the instability strip. The only way to get RR Lyrae variables then is to reduce the helium-rich core to values as low as 0.425 $M_{\odot}$ according to the calculations of Sweigart and Gross (1976). Does this mean that the higher Z clusters have more mass loss and the core helium flash occurs at an earlier time in its evolution when less helium has been produced in the core?

We conclude from this discussion that there are some constraints on evolution theory from the various results of pulsation theory. The solar problem is interesting in that it reinforces our standard ideas of evolution, leaving the neutrino problem unsolved. The two other potentially strong constraints are from the double-mode Cepheids and the double-mode RR Lyrae variables. Future work for the first of these will be in theoretical studies of blue loop parameters. For the latter case, more accurate observations of RR Lyrae period ratios will eventually define the horizontal branch star masses to produce very strong constraints for red giant and horizontal branch evolution calculations.

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