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The Verdier Hypercovering Theorem

J. F. Jardine

Abstract. This note gives a simple cocycle-theoretic proof of the Verdier hypercovering theorem. This theorem approximates morphisms $[X, Y]$ in the homotopy category of simplicial sheaves or presheaves by simplicial homotopy classes of maps, in the case where Y is locally fibrant. The statement proved in this paper is a generalization of the standard Verdier hypercovering result in that it is pointed (in a very broad sense) and there is no requirement for the source object X to be locally fibrant.

1 Introduction

The Verdier hypercovering theorem is a traditional and widely used method of approximating the morphisms $[X, Y]$ between two objects in homotopy categories of simplicial sheaves and presheaves by simplicial homotopy classes of maps.

In its standard form, the theorem says that the comparison function

$$\varinjlim_{[p]\colon Z\to X} \pi(Z, Y) \to [X, Y], \tag{1.1}$$

which is defined by taking the element

$$X \xleftarrow{[p]} Z \xrightarrow{[f]} Y$$

to the morphism $f \cdot p^{-1}$, is a bijection, provided that X and Y are locally fibrant in the sense that all of their stalks are Kan complexes.

Here, $\pi(Z, Y)$ denotes simplicial homotopy classes of maps and the colimit is indexed over homotopy classes represented by hypercovers $p\colon Z \to X$. A hypercover is a map which is a trivial fibration of simplicial sets in all stalks and is therefore invertible in the homotopy category.

The theorem is stated in the form displayed above (for simplicial sheaves) in [7], and the proof given there is a calculus of fractions argument which is adapted from Brown's thesis [3]. A more recent version of the classical proof for simplicial presheaves appears in [4].

The Verdier hypercovering theorem had multiple applications during the early development of simplicial sheaf homotopy theory, such as the identification of sheaf cohomology with homotopy classes of maps, which appeared in [7,9] for abelian and non-abelian cohomology, respectively. Variants of the theorem have become part of the basic tool kit for all work in this type of homotopy theory; see [6], for example.

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It turns out that the traditional proof of the theorem, based as it is on calculus of fractions techniques, is needlessly fussy. Cocycle category techniques from [10] can be used to give a much simpler and more conceptual proof of this result (Theorem 3), as well as proofs of several useful variations (Corollary 6). The purpose of this note is to present these proofs.

The statement of Theorem 3 is a generalization of the standard result in that it holds in pointed categories (suitably defined) and there is no requirement for the source object X to be locally fibrant in order for the map (1.1) to be a bijection. Only the target object Y needs to be locally fibrant, and this is a characteristic of all of the statements proved here. This weakening of the hypotheses of the hypercovering theorem first appeared without proof in [11].

The proofs of the variants of the Verdier hypercovering theorem which are displayed in this paper are quite simple. The ideas presented here represent a continuation of some of the themes of the paper [10], which contains several proofs of foundational results of simplicial sheaf and presheaf homotopy theory in which traditional appeals to hypercovers are replaced by simple arguments involving cocycles. Thus, it is no surprise that the Verdier hypercovering theorem itself is susceptible to the same analysis.

2 Hypercovers and Cocycles

Suppose that $\mathcal{C}$ is a small Grothendieck site. Write $s\mathbf{Pre}$ and $s\mathbf{Shv}$ for the categories of simplicial presheaves and simplicial sheaves on $\mathcal{C}$, respectively.

Recall [8] that the categories of simplicial presheaves and simplicial sheaves have local model structures for which the cofibrations are monomorphisms and the weak equivalences are defined stalkwise. In both cases the fibrations (called either *global* or *injective* fibrations) are defined by a right lifting property with respect to trivial cofibrations. The associated sheaf map $\eta \colon X \to \tilde{X}$ is a local weak equivalence, and the associated sheaf functor and its right adjoint define a Quillen equivalence between the local model structures for simplicial sheaves and simplicial presheaves on the site $\mathcal{C}$.

The discussion that follows will be confined to simplicial presheaves. It has an exact analog for simplicial sheaves.

Let A be a fixed choice of simplicial presheaf. The slice category $A/s\mathbf{Pre}$ has all morphisms $x \colon A \to X$ as objects and all diagrams

$$\begin{array}{ccc} & & X \\ & \nearrow^{x} & \\ A & & \downarrow{\scriptstyle f} \\ & \searrow_{y} & \\ & & Y \end{array}$$

as morphisms.

The intuition, in applications, is that $x \colon A \to X$ is a *base point* of X (geometric points of schemes are good examples to keep in mind) even though A could be nontrivial homotopically. The object A could also be empty, and $\varnothing/s\mathbf{Pre}$ is isomorphic to the category of simplicial presheaves.

Observe that $A/s\mathbf{Pre}$ is complete and cocomplete, with all limits formed in the category of simplicial presheaves.

By a standard formalism, the category $A/s\mathbf{Pre}$ inherits a local model structure from $s\mathbf{Pre}$, in that a morphism $f\colon x \to y$ as above is a local weak equivalence (respectively cofibration, fibration) if and only if the underlying map $f\colon X \to Y$ is a local weak equivalence (respectively cofibration, fibration) of simplicial presheaves.

In contrast with the model structure for simplicial presheaves, not all objects of the slice category are cofibrant: the identity morphsm $1\colon A \to A$ is initial, and so an object $x\colon A \to X$ is cofibrant if an only if the map x is a cofibration of simplicial presheaves.

The unique map $A \to *$ taking values in the terminal simplicial presheaf $*$ is the terminal object of $A/s\mathbf{Pre}$, and it follows that an object $x\colon A \to X$ is fibrant if and only if X is an injective fibrant simplicial presheaf.

There are various ways to say what a local fibration of simplicial presheaves should be. It is perhaps still most compelling to declare that a map $p\colon X \to Y$ is a local fibration if it has the *local right lifting property* with respect to all inclusions $\Lambda^n_k \subset \Delta^n$ of horns in simplices. This means that, given any commutative diagram of simplicial set maps

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\alpha} & X(U) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\beta]{} & Y(U) \end{array}$$

where U is an object of $\mathcal{C}$, the lifting problem can be solved after refinement along a covering family $U_i \to U$ in the sense that the dotted arrow liftings exist in all diagrams

$$\begin{array}{ccccc} \Lambda^n_k & \xrightarrow{\alpha} & X(U) & \longrightarrow & X(U_i) \\ \downarrow & & & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\beta]{} & Y(U) & \longrightarrow & Y(U_i) \end{array}$$

Equivalently, the simplicial presheaf map $p\colon X \to Y$ is a local fibration if and only if all presheaf maps

$$\hom(\Delta^n, X) \to \hom(\Lambda^n_k, X) \times_{\hom(\Lambda^n_k, Y)} \hom(\Delta^n, Y)$$

are local epimorphisms. Here, for example, $\hom(\Lambda^n_k, X)$ is the presheaf with sections

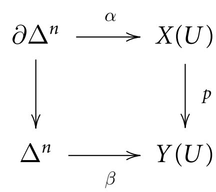
$$\hom(\Lambda^n_k, X)(U) = \hom(\Lambda^n_k, X(U))$$

given by the simplicial set maps $\Lambda^n_k \to X(U)$ for objects U of $\mathcal{C}$.

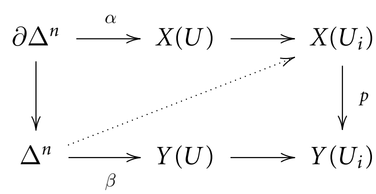
A map $q\colon Z \to W$ is a sectionwise fibration (or pointwise fibration) if all simplicial set maps $q\colon Z(U) \to W(U)$, $U \in \mathcal{C}$ are Kan fibrations. Every sectionwise fibration is a local fibration, so that, for example, every presheaf of Kan complexes is locally fibrant. Every injective fibration is a sectionwise fibration, so that every injective fibration is a local fibration. The converse is wildly false.

It is a basic result of local homotopy theory [8, Theorem 1.12] that a map $p\colon X \to Y$ is a local weak equivalence and a local fibration (or a local trivial fibration) if and only if it has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

This local lifting property means, by analogy with the definition of local fibration given above, that for any commutative diagram of simplicial set maps



there is a covering family $U_i \to U$ for which the dotted arrow liftings exist in all diagrams



Equivalently, all presheaf maps

$$\hom(\Delta^n, X) \to \hom(\partial\Delta^n, X) \times_{\hom(\partial\Delta^n, Y)} \hom(\Delta^n, Y)$$

should be local epimorphisms.

The canonical isomorphisms

$$\hom(\Delta^n, X) \cong X_n, \quad \hom(\partial\Delta^n, X) \cong \operatorname{cosk}_{n-1} X_n$$

imply that a map $p\colon X \to Y$ of simplicial presheaves is a local trivial fibration if and only if all induced maps $X_n \to \operatorname{cosk}_{n-1} X_n \times_{\operatorname{cosk}_{n-1} Y_n} Y_n$, $n \geq 0$, are local epimorphisms.

In particular, the canonical map $X \to *$ is a local trivial fibration if the maps $X_0 \to *$ and $X_n \to \operatorname{cosk}_{n-1}(X)_n$, $n > 0$, are local epimorphisms.

In the case where U is a simplicial sheaf for the étale site on a scheme S which is represented by a simplicial S- scheme U, then the simplicial sheaf map $U \to *$ is a local trivial fibration if and only if it is a hypercover in the sense of [1]. Based on this example, it has become the custom to use the term hypercover to refer to an arbitrary local trivial fibration of simplicial presheaves, and I shall continue to follow that practice here.

Say that a map $f\colon x \to y$ in the slice category $A/s\mathbf{Pre}$ is a *hypercover* if the underlying simplicial presheaf map $f\colon X \to Y$ is a hypercover (or a local trivial fibration). More generally, $f\colon x \to y$ is a local fibration if the map $f\colon X \to Y$ is a local fibration of simplicial presheaves. In particular, $x\colon A \to X$ is locally fibrant if X is locally fibrant.

The theory of cocycle categories of [10] applies without change to the model category $A/s\mathbf{Pre}$. Explicitly, a cocycle (g, f) from x to y is a diagram

$$x \xleftarrow[\simeq]{g} z \xrightarrow{f} y$$

in the slice category, or equivalently a commutative diagram of simplicial presheaf maps

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle x}\swarrow & \downarrow{\scriptstyle z} & \searrow{\scriptstyle y} & \\ X & \xleftarrow[g]{\simeq} & Z & \xrightarrow[f]{} & Y \end{array}$$

for which the map g is a weak equivalence. These cocycles are the objects of a category $H(x, y)$ which has morphisms $\theta\colon (g, f) \to (g, f')$ given by the commutative diagrams

$$\begin{array}{ccccc} & {\scriptstyle g}\swarrow & z & \searrow{\scriptstyle f} & \\ x & & \downarrow{\scriptstyle\theta} & & y \\ & {\scriptstyle g'}\nwarrow & z' & \nearrow{\scriptstyle f'} & \end{array} .$$

The category $H(x, y)$ is the category of cocycles from x to y.

Write $\pi_0 D$ for the set of path components of a category D, or equivalently, for the set of path components of the nerve BD.

The model structure on $A/s\mathbf{Pre}$ is right proper, and weak equivalences in this structure are closed under finite products, because these properties both hold for the category of simplicial presheaves. Thus, [10, Theorem 1] implies the following.

Lemma 1 *The function $\phi\colon \pi_0 H(x, y) \to [x, y]$, which is defined by $(g, f) \mapsto f \cdot g^{-1}$ for a cocycle (g, f) in the slice category $A/s\mathbf{Pre}$, is a bijection.*

Remark 2 There is no requirement that the target object y should be fibrant in the statement of Lemma 1. If y is fibrant, then the map ϕ of Lemma 1 is known to be a bijection by a theorem which holds for arbitrary simplicial model categories [2, Remark 2.7], [5].

The real utility of a statement like Lemma 1 in simplicial presheaf homotopy theory is for cases involving non-fibrant targets; see also [10]. The present context is the

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injective model structure for simplicial presheaves, the maps of interest have locally fibrant targets, and we know that locally fibrant objects are not injective fibrant in general.

Suppose that $f, g\colon x \to y$ are morphisms of the slice category $A/s\mathbf{Pre}$. A (naive) pointed homotopy from f to g is a commutative diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x\times\Delta^1}\downarrow & & \downarrow{\scriptstyle y} \\ X \times \Delta^1 & \xrightarrow[h]{} & Y \end{array}$$

such that h is a simplicial homotopy from f to g in the usual sense. Here, the projection map $\mathrm{pr}\colon A \times \Delta^1 \to A$ onto A defines the constant homotopy on A.

Equivalently, such a pointed homotopy is a map

$$h\colon (X \times \Delta^1) \bigcup_{A\times\Delta^1} A \to Y.$$

In the pushout diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ {\scriptstyle x\times\Delta^1}\downarrow & & \downarrow \\ X \times \Delta^1 & \xrightarrow[pr_*]{} & (X \times \Delta^1) \bigcup_{A\times\Delta^1} A \end{array}$$

the map pr_* is a weak equivalence if the map $x\colon A \to X$ is a cofibration, or if x is a cofibrant object of $A/s\mathbf{Pre}$. In that case, the pushout object is a cylinder for x in the slice category.

Every object $x\colon A \to X$ has a cofibrant model, meaning a diagram

$$\begin{array}{ccc} A & \xrightarrow{v} & Z \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

such that v is a cofibration and p is a weak equivalence. If the maps $f, g\colon x \to y$ are pointed homotopic and $p\colon v \to x$ is a cofibrant model of x, then the composites fp and gp are pointed homotopic and therefore represent the same map in the homotopy category since v is cofibrant. But then p is an isomorphism in that category, so that $f = g$ in the homotopy category.

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The objects of the category Triv$/x$ are the pointed homotopy classes of maps $[p]\colon z \to x$ which are represented by hypercovers $p\colon z \to x$. The morphisms of this category are commutative triangles of pointed homotopy classes of maps in the obvious sense.

There is a contravariant set-valued functor which takes an object $[p]\colon z \to x$ of Triv$/x$ to the set $\pi(z, y)$ of pointed homotopy classes of maps between z and y. There is a function

$$\phi_h\colon \varinjlim_{[p]\colon z\to x} \pi(z, y) \to [x, y]$$

which is defined by sending the diagram of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y$$

to the morphism $f \cdot p^{-1}$ in the homotopy category.

The colimit $\varinjlim_{[p]\colon z\to x} \pi(z, y)$ is the set of path components of a category $H_h(x, y)$ whose objects are the pictures of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y,$$

such that $p\colon z \to x$ is a hypercover, and whose morphisms are the commutative diagrams

(2.1)
$$\begin{array}{ccccc} & & z & & \\ & \swarrow^{[p]} & \Big\downarrow{\scriptstyle [\theta]} & \searrow^{[f]} & \\ x & & & & y \\ & \nwarrow_{[p']} & & \nearrow_{[f']} & \\ & & z' & & \end{array}$$

in pointed homotopy classes of maps. The map ϕ_h therefore has the form

$$\phi_h\colon \pi_0 H_h(x, y) \to [x, y].$$

The following result is a generalized Verdier hypercovering theorem.

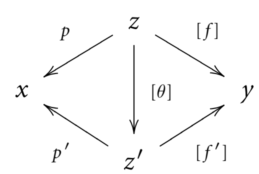
Theorem 3 *The function $\phi_h\colon \pi_0 H_h(x, y) \to [x, y]$ is a bijection if y is locally fibrant.*

Remark 4 Theorem 3 specializes to a generalization of the standard form of the Verdier hypercovering theorem [3, p. 425], [7] if $A = \varnothing$ for the unique map $x\colon \varnothing \to X$. The object X is not required to be locally fibrant.

Before proving Theorem 3, let us observe that there are multiple variations of the category $H_h(x, y)$:

(i) Write $H'_h(x, y)$ for the category whose objects are pictures $x \xleftarrow{p} z \xrightarrow{[f]} y$, where p is a hypercover and $[f]$ is a pointed homotopy class of maps. The morphisms of $H'_h(x, y)$ are diagrams

(2.2)

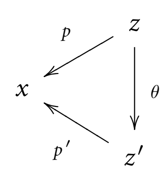


such that $[\theta]$ is a fibrewise pointed homotopy class of maps over x, and $[f'][\theta] = [f]$ as pointed homotopy classes. There is a functor $\omega\colon H'_h(x, y) \to H_h(x, y)$ which is defined by the assignment $(p, [f]) \mapsto ([p], [f])$, and which sends the morphism (2.2) to the morphism (2.1).

(ii) Write $H''_h(x, y)$ for the category whose objects are the pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y,$$

where p is a hypercover and $[f]$ is a pointed homotopy class (as before). The morphisms of $H''_h(X, Z)$ are commutative diagrams



such that $[f' \cdot \theta] = [f]$. There is a canonical functor $H''_h(x, y) \xrightarrow{\omega'} H'_h(x, y)$ that is the identity on objects and takes morphisms θ to their associated fibrewise pointed homotopy classes.

(iii) Let $H_{\mathrm{hyp}}(x, y)$ be the full subcategory of $H(x, y)$ whose objects are the cocycles

$$x \xleftarrow{p} z \xrightarrow{f} y,$$

with p a hypercover. There is a functor $\omega''\colon H_{\mathrm{hyp}}(x, y) \to H''_h(x, y)$ that takes a cocycle (p, f) to the object $(p, [f])$.

Lemma 5 *Suppose that y is locally fibrant. Then the inclusion functor*

$$i : H_{\mathrm{hyp}}(x, y) \subset H(x, y)$$

is a homotopy equivalence.

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Proof Objects of the cocycle category $H(x, y)$ can be identified with maps $(g, f) : z \to x \times y$ such that the morphism g is a weak equivalence and morphisms of $H(x, y)$ are commutative triangles in the obvious way. Maps of the form (g, f) have functorial factorizations

$$\begin{array}{ccc} z & \xrightarrow{j} & v \\ & \searrow^{(g,f)} & \downarrow{\scriptstyle (p,g')} \\ & & x \times y \end{array} \tag{2.3}$$

such that j is a sectionwise trivial cofibration and (p, g') is a sectionwise Kan fibration. It follows that (p, g') is a local fibration. The projection map pr is a local fibration since y is locally fibrant, so the map p is a local fibration. The map p is also a local weak equivalence, and hence a hypercover.

It follows that the assignment $(g, f) \mapsto (p, g')$ defines a functor

$$\psi' \colon H(x, y) \to H_{\text{hyp}}(x, y).$$

The weak equivalences j of the diagram (2.3) define homotopies $\psi' \cdot i \simeq 1$ and $i \cdot \psi' \simeq 1$. ■

Proof of Theorem 3 The composite

$$\pi_0 H(x, y) \xrightarrow{\psi'_*} \pi_0 H_{\text{hyp}}(x, y) \xrightarrow{\omega''_*} \pi_0 H''_h(x, y) \xrightarrow{\omega'_*} \pi_0 H'_h(x, y) \xrightarrow{\omega_*} \pi_0 H_h(x, y) \xrightarrow{\phi_h} [x, y] \tag{2.4}$$

is the bijection ϕ of Lemma 1. The function ψ'_* is a bijection by Lemma 5, and the functions ω''_*, ω'_*, and ω_* are surjective, as is the function ϕ_h. The functions which make up the string (2.4) are therefore all bijections. ■

The following is a corollary of the proof of Theorem 3 which deserves independent mention.

Corollary 6 *Suppose that the object $y \colon A \to Y$ of A/s**Pre** is locally fibrant. Then the induced functions*

$$\pi_0 H_{hyp}(x, y) \xrightarrow{\omega''_*} \pi_0 H''_h(x, y) \xrightarrow{\omega'_*} \pi_0 H'_h(x, y) \xrightarrow{\omega_*} \pi_0 H_h(x, y)$$

are bijections, and all of these sets are isomorphic to the set $[x, y]$ of morphisms $x \to y$ in the homotopy category $\mathrm{Ho}(s/$**Pre**$)$.

The bijections of the path component objects in the statement of Corollary 6 with the set $[x, y]$ all represent specific variants of the Verdier hypercovering theorem.

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Mathematics Department, University of Western Ontario, London, ON N6A 5B7
e-mail: jardine@uwo.ca