# A REMARK ON THE GENERALIZED NUMERICAL RANGE OF A NORMAL MATRIX

*by* YIK-HOI AU-YEUNG and FUK-YUM SING

(Received 1 April, 1976)

**1. Introduction.** Let $A$ be an $n \times n$ complex normal matrix and let $\mathcal{W}(A) = \{\operatorname{diag} UAU^* : U$ is unitary) where $U^*$ is the conjugate transpose of $U$. It is known that $\mathcal{W}(A)$ may not be convex [1, 3] and it is convex when $A$ is Hermitian [1, 2]. In this note we show that $\mathcal{W}(A)$ is convex if and only if the eigenvalues of $A$ are collinear (i.e. there exist complex numbers $\alpha\ (\neq 0)$ and $\beta$ such that $\alpha A + \beta I$ is Hermitian).

Hence for most normal matrices $A$, $\mathcal{W}(A)$ is not convex.

**2. Generalized numerical range.**

LEMMA 1. *Let* $Q = \begin{pmatrix} U & a \\ b^* & \mu \end{pmatrix}$ *be an* $(n+1) \times (n+1)$ *unitary matrix, where* $U$ *is an* $n \times n$ *matrix. Then there exists a real number* $\gamma$ *such that* $U + \gamma ab^*$ *is unitary.*

*Proof.* Using the property that $Ub + \bar{\mu}a = 0$ and $b^*b + \mu\bar{\mu} = 1$, we can show that $(U+\gamma ab^*)(U+\gamma ab^*)^* = UU^* + [\gamma^2(1-\mu\bar{\mu}) - \gamma(\mu+\bar{\mu})]aa^*$. Since $1 - \mu\bar{\mu} \geqq 0$, it is possible to find $\gamma$ so that $[\gamma^2(1-\mu\bar{\mu}) - \gamma(\mu+\bar{\mu})]aa^* = aa^*$ (when $\mu\bar{\mu} = 1$, $a = 0$). Since $UU^* + aa^* = I_n$, where $I_n$ is the $n \times n$ identity matrix, the lemma is proved.

PROPOSITION 2.

$$\textit{Let } B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n & \\ & & & \lambda_{n+1} \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

*If* $\mathcal{W}(B)$ *is convex and* $\lambda_{n+1}$ *is a vertex of the convex hull of the points* $\lambda_1, \ldots, \lambda_{n+1}$, *then* $\mathcal{W}(A)$ *is convex.*

*Proof.* For any $n \times n$ unitary matrices $U_1$ and $U_2$ and $0 \leqq \alpha \leqq 1$ we can find unitary matrix $Q = \begin{pmatrix} U & a \\ b^* & \mu \end{pmatrix}$ such that

$$\operatorname{diag}\left\{\alpha\begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \lambda_{n+1} \end{pmatrix}\begin{pmatrix} U_1^* & 0 \\ 0 & 1 \end{pmatrix} + (1-\alpha)\begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \lambda_{n+1} \end{pmatrix}\begin{pmatrix} U_2^* & 0 \\ 0 & 1 \end{pmatrix}\right\} = \operatorname{diag} QBQ^*.$$

Hence we have

$$b^*Ab + \lambda_{n+1}\mu\bar{\mu} = \lambda_{n+1}, \tag{1}$$

$$\operatorname{diag}(\alpha U_1AU_1^* + (1-\alpha)U_2AU_2^*) = \operatorname{diag}(UAU^* + \lambda_{n+1}aa^*). \tag{2}$$

From (1), since $\lambda_{n+1}$ is a vertex and $b^*b + \mu\bar{\mu} = 1$, it follows that if $b^* = (b_1, \ldots, b_n)$, we have

$b_i = 0$ for all $i$ such that $\lambda_i \neq \lambda_{n+1}$. This implies that $b^*A = \lambda_{n+1}b^*$ and $Ab = \lambda_{n+1}b$. We can then obtain

$$\begin{aligned}&(U+\gamma ab^*)A(U+\gamma ab^*)^*\\&= UAU^* + \lambda_{n+1}[\gamma^2(1-\mu\bar{\mu}) - \gamma(\mu+\bar{\mu})]aa^*\\&= UAU^* + \lambda_{n+1}aa^*,\end{aligned}$$

where $\gamma$ is chosen as in Lemma 1. From (2) we see that $\mathscr{W}(A)$ is convex.

THEOREM 3.

*Let* $A = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$, *where* $n \geqq 3$.

*Then $\mathscr{W}(A)$ is convex if and only if $\lambda_1, \ldots, \lambda_n$ are collinear.*

*Proof.* First we prove sufficiency. When the eigenvalues of $A$ are collinear, $\alpha A + \beta I$ is Hermitian for some complex numbers $\alpha\,(\neq 0)$ and $\beta$. Since $\mathscr{W}(\alpha A + \beta I)$ is convex, it follows that $\mathscr{W}(A)$ is also convex. To prove necessity we use induction on $n$. The case $n = 3$ is proved in [**1**]. Suppose $\mathscr{W}(A)$ is convex and the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ where $n > 3$. If the eigenvalues are not collinear, we take a vertex, say $\lambda_1$. By Proposition 2 and the induction assumption, $\lambda_2, \lambda_3, \ldots, \lambda_n$ are collinear. Consider a vertex on this line segment, say $\lambda_2$. By the same argument, $\lambda_1, \lambda_3, \ldots, \lambda_n$ are collinear. We must have then $\lambda_3 = \lambda_4 = \ldots = \lambda_n$. (For if $\lambda_i \neq \lambda_j$ for some $i, j \geqq 3$, then $\lambda_1, \lambda_i, \lambda_j$ are not collinear.) This implies $\lambda_3$ is a vertex and hence $\lambda_1, \lambda_2, \lambda_4$ are collinear, which gives a contradiction. Therefore the eigenvalues must be collinear.

From Theorem 3 we have immediately the following result.

COROLLARY 4. *If* $A = \begin{pmatrix}A_1 & \\ & A_2\end{pmatrix}$ *is a normal matrix and $\mathscr{W}(A)$ is convex, then $\mathscr{W}(A_1)$ and $\mathscr{W}(A_2)$ are convex.*

## REFERENCES

**1.** P. A. Fillmore and J. P. Williams, Some convexity theorems for matrices, *Glasgow Math. J.* **12** (1971), 110–116.

**2.** A Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, *Amer. J. Math.* **76** (1954), 620–630.

**3.** L. E. Lerer, On the diagonal elements of normal matrices (Russian), *Mat. Issled.* **2** (1967), 156–163.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HONG KONG

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
PRINTERS TO THE UNIVERSITY OF GLASGOW

https://doi.org/10.1017/S0017089500003244 Published online by Cambridge University Press