ON RIGHT DUO P.P. RINGS

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Throughout the paper, rings are associative rings with identity. A ring is called *right duo* if every right ideal is two-sided, and it is called *right p.p.* if every principal right ideal is projective. A left duo (p.p.) ring is defined similarly, and a duo (p.p.) ring will mean a ring which is both right and left duo (p.p.). There is a right p.p. ring that is not left p.p. (see Chase [**2**]). Small [**9**] proved that right p.p. implies left p.p. if there are no infinite sets of orthogonal idempotents, and Endo [**5**, Proposition 2] has shown the same implication in the case where each idempotent in the ring is central. Since Courter [**3**, Theorem 1.3] noted that every idempotent in a right duo ring is central, we can simply speak of right duo p.p. rings. A typical example of a right duo ring which is not left duo is the following. Let F be a field and $F(x)$ the field of rational functions over F. Let $R = F(x) \times F(x)$ as an additive group and define the multiplication as follows:

$$(f_1(x), g_1(x))(f_2(x), g_2(x)) = (f_1(x)f_2(x), f_1(x^2)g_2(x) + g_1(x)f_2(x)).$$

Then R is a local artinian ring with $c(R_R) = 2$ and $c({}_RR) = 3$. Thus R is right duo but not left duo.

Vasconcelos [**13**, Theorem 4.2] proved that a commutative ring R is semihereditary if and only if R is p.p. and the weak dimension $\mathrm{wD}(R)$ of R is at most one. Recently Tuganbaev [**12**, Proposition 3] generalized Camillo's result [**1**] by showing that a duo ring is both right and left semihereditary if each two-generated ideal is right projective. It should be noted that Camillo's theorem [**1**] was a generalization of a much older result. Jensen [**6**, Lemma 3] claims that Dedekind [**4**] essentially proved a commutative integral domain is a Prüfer ring provided that every two-generated ideal is projective.

In this paper, we shall establish the following results.

THEOREM 1. *Let R be a right duo ring. The following statements are equivalent:*

(1) *R is right semihereditary;*

(2) *every two-generated ideal is right projective;*

(3) *R is p.p. and* $\mathrm{wD}(R) \le 1$.

THEOREM 2. *Let R be a duo p.p. ring. If I is a finitely generated right projective ideal then I is left projective and a direct summand of an invertible ideal.*

A ring with no non-zero nilpotent elements is called *reduced*. Our results are based on the following key lemma. We note that the right-duo assumption of R in the lemma is essential: let R be the ring of 2 by 2 upper triangular matrices over a field; then R is artinian hereditary indecomposable but not semiprime, and its quotient ring is R itself which is not von Neumann regular.

LEMMA 3. *Let R be a right duo p.p. ring. Then*

(1) *R is reduced and has a right classical quotient ring Q that is von Neumann regular;*

(2) *if I is a right (projective) ideal that is n-generated i.e. $I = \sum_{i=1}^{n} x_iR$, I is a direct summand of a n-generated essential right (projective) ideal.*

Proof. (1) Let K be a nilpotent ideal. The right annihilator $r(K)$ of K is essential as a left ideal, for if L is a non-zero left ideal then there is an integer i such that $K^iL \neq 0$ and $K^{i+1}L = 0$. Hence $r(K)$ is essential as a right ideal since R is right duo. But $K \cdot r(K) = 0$. Therefore $K = 0$ because R is right non-singular. Thus R is semiprime. Suppose that $a^2 = 0$. Then $(aR)^2 \subseteq a^2R = 0$. Hence $a = 0$. Thus R is reduced.

Let $a \in R$. We have $r(a) = eR$ for some idempotent e that is central by [**3**]. Since $aR \cap eR = 0$, we have $aR + eR = (a+e)R$ and $a + e$ is regular (i.e. not a zero divisor). Hence $aR + eR$ is essential. The elements of the form $a + e$ as above are regular, and every regular element c is of this form with $c = a + e$, where $a = c$ and $e = 0$. Because $cR \supseteq Rc$ for every regular element c, we know that R satisfies the right Ore condition with respect to its regular elements. Thus R has a right classical quotient ring Q. In the above notation we have $aQ + eQ = Q$. Let x be an element of Q. Then $x = ac^{-1}$ for some a, c in R with c regular. With e as above we have $xQ = aQ$ and $aQ + eQ = Q$. It follows that Q is von Neumann regular.

(2) Suppose $I = \sum_{i=1}^{n} x_iR$. Let $r(x_i) = e_iR$ with e_i central idempotents. Then $e = e_1 \cdots e_n$ is a central idempotent and $Ie = 0$. It follows that the sum $I + eR$ is direct. In fact $eR = r(I)$ and $I + eR$ is essential. Also $I + eR$ is generated by the n elements $x_i + e$, for if an ideal contains $x_i + e$ it also contains $(x_i + e)e = e$ and hence also x_i.

Proof of Theorem 1. Clearly we have (1)$\Rightarrow$(2), (3). The implication (2)$\Rightarrow$(1) follows from Lemma 3(1), [**12**, Corollary 2] and [**10**, Corollary 2], and (3)$\Rightarrow$(1) follows from Lemma 3(1), [**11**, Lemma 12(b)] and [**10**, Corollary 2].

There exists a ring R such that all 2-generated right ideals are projective but R has a nonflat 3-generated right ideal. (See Jøndrup [**7**, p. 434, Example]. This example was found jointly with P. M. Cohn, as mentioned in [**7**].) Hence the implications (2)$\Rightarrow$(1) and (2)$\Rightarrow$(3) in Theorem 1 are false if one drops the assumption that R is right duo.

We need the following proposition to prove Theorem 2. Again, we can not remove the right duo hypothesis of R. For example, take R to be any simple noetherian non-artinian domain such as the first Weyl algebra, I any nonzero proper right ideal; then $I^2 = I$ but I is not generated by an idempotent. The question is more interesting perhaps and more relevant for two-sided ideals, and an easy example is to take $R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$; then R is prime noetherian hereditary; set $I = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$; then I is an idempotent two-sided ideal with zero annihilator so not generated by an idempotent.

PROPOSITION 4. *Let R be a right duo ring with a finitely generated idempotent ideal A. Then $A = eR$ for some idempotent e.*

Proof. Let $A = \sum_{i=1}^{n} x_iR$. Because $A^2 = A$, we know that each $x_i \in \sum_{i=1}^{n} x_iA$. So we get equations of the form

$$x_1(1 - a_{11}) + x_2a_{12} + \ldots + x_na_{1n} = 0, \tag{1}$$

$$x_1a_{21} + x_2(1 - a_{22}) + \ldots + x_na_{2n} = 0, \tag{2}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_1a_{n1} + x_2a_{n2} + \ldots + x_n(1 - a_{nn}) = 0. \tag{n}$$

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Since R is right duo, we have $a_{1n}(1-a_{nn})=(1-a_{nn})b_{1n}$ for some $b_{1n}\in R$, in fact $b_{1n}\in A$. Now (1) multiplied by $(1-a_{nn})$ minus (n) multiplied by b_{1n} gives

$$x_1(1-b_{11})+x_2b_{12}+\ldots+x_{n-1}b_{1,n-1}=0, \tag{1)'}$$

with $b_{1j}\in A$. Similarly, we get

$$x_1b_{21}+x_2(1-b_{22})+\ldots+x_{n-1}b_{2,n-1}=0, \qquad (2)'$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$x_1b_{n-1,1}+x_2b_{n-1,2}+\ldots+x_{n-1}(1-b_{n-1,n-1})=0, \qquad (n-1)'$$

with $b_{ij}\in A$. Continue this until we get $x_1(1-u_1)=0$ with $u_1\in A$. Using the same method, we have

$$x_i(1-u_i)=0 \qquad \text{for all } i \text{ with } u_i\in A.$$

Since R is right duo and $A=\sum_{i=1}^{n}x_iR$, we have $A(1-u_1)\ldots(1-u_n)=0$. Let $(1-u_1)\ldots(1-u_n)=1-e$ with $e\in A$. In particular $e(1-e)=0$ so that $e=e^2$. Also $A(1-e)=0$; so that $A=Re=eR$, since e is central by [**3**].

Proof of Theorem 2. If I is, in addition, essential, we shall show that I is invertible.

By Lemma 3(1), R has a right classical quotient ring Q that is von Neumann regular, and then IQ is a finitely generated essential right ideal of the regular ring Q. Hence $IQ=fQ$ for some idempotent $f\in Q$. Because IQ is essential, it follows that $IQ=Q$. Hence I contains a regular element of R. Also any right R-module homomorphism from I to R can be extended to a right Q-homomorphism from IQ to Q, i.e. from Q to Q. Thus we can identify $\mathrm{Hom}_R(I,R)$ with the set $I^*=\{q\in Q \mid qI\subseteq R\}$. Because I_R is finitely generated projective, we have $1\in II^*$. Let d be a regular element of R. Since R is a duo ring, we have $dR=Rd$; so $Rd^{-1}=d^{-1}R$. Hence $dI^*\subseteq R$ if and only if $I^*d\subseteq R$. We know that I contains a regular element c. Let $I=x_1R+\ldots+x_nR$. We do not know that the x_i are regular. With $r(x_i)=e_iR$ as usual, it is easy to show that each ce_i+x_i is regular and that I is generated by c and the ce_i+x_i. Thus I is generated by regular elements. We have $I^*I\subseteq R$, and then $II^*\subseteq R$. Therefore $II^*=R$, and then I^*I is an idempotent ideal of R. With $I=x_1R+\ldots+x_nR$, we get $I^*=Ry_1+\ldots+Ry_n$ for some y_i. Each $y_ix_j\in R$; so $Ry_ix_jR=y_ix_jR$. So I^*I is finitely generated by y_ix_j. By Proposition 4, $I^*I=eR$ for some idempotent e. But I^*I contains I and so is essential. Therefore $I^*I=R$. Hence I is invertible and in particular I is left projective.

Now let I be a finitely generated right projective ideal. By Lemma 3(2), I is a direct summand of a finitely generated essential right projective ideal K, say $I+J=K$ with $I+J$ direct. From above K is invertible and left projective. Hence I is left projective.

The next lemma is known, but it is included here since it is unavailable in the literature.

LEMMA 5. *Let M be a finitely generated module. If there is an exact sequence $0\rightarrow N\rightarrow P\rightarrow M\rightarrow 0$, where N is finitely generated and P is finitely related, then M is finitely related.*

Proof. Let $f:F\rightarrow P$ be an onto homomorphism, where F is a finitely generated free module with $\mathrm{Ker}(f)$ finitely generated. Assume $N\leq P$ and let $g=f|_{f^{-1}(N)}$. We have a

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commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}(N) & \hookrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \| & & \\ 0 & \longrightarrow & N & \hookrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$
with exact rows, where g is onto, and both N and $\mathrm{Ker}(g)=\mathrm{Ker}(f)$ are finitely generated. So $f^{-1}(N)$ is finitely generated, and hence M is finitely related.

Let I and J be two ideals in a ring. We always have an exact sequence of bimodules
$$0 \to I \cap J \xrightarrow{\alpha} I \oplus J \xrightarrow{\beta} I + J \to 0,$$
where $\alpha(x)=(x, -x)$ and $\beta((x, y))=x+y$. We shall use this fact without reference.

A ring R is called *z.c.* (zero commutative) if $l(a)=r(a)$ for all $a \in R$. A reduced ring is z.c..

THEOREM 6. *Let R be a z.c. duo ring. If every n-generated ideal is finitely related as a right module then every n-generated ideal is finitely related as a left module.*

Proof. The ideal generated by a subset $X \subseteq R$ is denoted by (X). Let $a \in R$. Since $l(a)=r(a)$ and R is duo, ${}_R l(a)$ is finitely generated if and only if $r(a)_R$ is finitely generated. It follows that ${}_R(a)$ is finitely related if and only if $(a)_R$ is finitely related. This proves the case when $n=1$.

Suppose $n>1$. Now let $I=(a_1, \ldots, a_n)$, and assume that $I'=(a_1, \ldots, a_{n-1})$ is finitely related as a left module. We shall show that ${}_R I$ is finitely related.

Since I_R is finitely related, the exact sequence of bimodules
$$0 \to I' \cap (a_n) \to I' \oplus (a_n) \to I \to 0$$
implies that $I' \cap (a_n)$ is finitely generated as a right module by Rotman [**8**, Corollary 3.63]. Therefore $I' \cap (a_n)$ is finitely generated as a left module, since R is duo. Now the result follows from Lemma 5.

COROLLARY 7. *A z.c. duo ring is right coherent if and only if it is left coherent.*

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