COMPOSITIO MATHEMATICA

$\mathbb{A}^1$-homotopy invariants of topological Fukaya categories of surfaces

Tobias Dyckerhoff

Compositio Math. **153** (2017), 1673–1705.

doi:10.1112/S0010437X17007205

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 153 (2017) 1673–1705
doi:10.1112/S0010437X17007205

$\mathbb{A}^1$-homotopy invariants of topological Fukaya categories of surfaces

Tobias Dyckerhoff

Abstract

We provide an explicit formula for localizing $\mathbb{A}^1$-homotopy invariants of topological Fukaya categories of marked surfaces. Following a proposal of Kontsevich, this differential $\mathbb{Z}$-graded category is defined as global sections of a constructible cosheaf of dg categories on any spine of the surface. Our theorem utilizes this sheaf-theoretic description to reduce the calculation of invariants to the local case when the surface is a boundary-marked disk. At the heart of the proof lies a theory of localization for topological Fukaya categories which is a combinatorial analog of Thomason–Trobaugh's theory of localization in the context of algebraic K-theory for schemes.

Contents

	Introduction	**1673**
1	**Cyclic and paracyclic 2-Segal objects**	**1675**
2	**Differential graded categories**	**1685**
3	**Topological Fukaya categories**	**1687**
4	**Localization and Mayer–Vietoris**	**1692**
5	**Delooping of paracyclic 1-Segal objects**	**1700**
6	**Main result and examples**	**1703**
	Acknowledgements	**1703**
	References	**1703**

Introduction

According to Kontsevich's proposal [Kon09], Fukaya categories of Stein manifolds can be described as global sections of a constructible cosheaf of dg categories on a possibly singular Lagrangian spine onto which the manifold retracts. Various approaches to this proposal have been developed, see for example [Nad14, Boc11, STZ14, HKK14, Nad15, DK13, Nad13].

The specific case we focus on in this work is the following. Let S be a compact connected Riemann surface, possibly with boundary, and let $M \subset S$ be a finite non-empty subset of marked points, so that the complement $S \backslash M$ is a Stein manifold. Any spanning graph Γ in $S \backslash M$ provides a Lagrangian spine. In [DK13], the language of *cyclic 2-Segal spaces* was used to realize Kontsevich's proposal in this situation. For every commutative ring k, this theory produces a constructible cosheaf of k-linear differential $\mathbb{Z}/2\mathbb{Z}$-graded categories on any spanning graph Γ, shows that the dg category $F(S, M; k)$ of global sections is independent of the chosen

Received 25 April 2016, accepted in final form 9 March 2017, published online 9 June 2017.
2010 Mathematics Subject Classification 18G55 (primary).
Keywords: topological Fukaya categories, ∞-categories, Segal spaces, K-theory.

graph, and implies a coherent action of the mapping class group of the surface on $F(S, M; k)$. It is expected that the resulting dg category is Morita equivalent to a variant of the wrapped Fukaya category of the surface. We refer to $F(S, M; k)$ as the k-linear *topological Fukaya category of the surface* (S, M). If the surface $S \backslash M$ is equipped with a framing then a *paracyclic* version of the above construction can be used to define a differential $\mathbb{Z}$-graded lift of $F(S, M; k)$ (cf. [Lur14, DK15]). The first part of this paper implements these constructions in the framework of ∞-categories, which provides the flexibility necessary for our purposes.

A functor H defined on the category of small k-linear dg categories with values in a stable ∞-category $\mathcal{C}$ is called:

(1) *localizing* if H inverts Morita equivalences and sends exact sequences of dg categories to exact sequences in $\mathcal{C}$;
(2) $\mathbb{A}^1$-*homotopy invariant* if, for every dg category A, the functor H maps $A \to A[t]$ to an equivalence in $\mathcal{C}$.

Examples of localizing $\mathbb{A}^1$-homotopy invariants are provided by periodic cyclic homology over a field of characteristic 0, homotopy K-theory, K-theory with finite coefficients, and topological K-theory over $\mathbb{C}$ (cf. [Kel98, TVdB15, Bla16]). The main result of this work is the following theorem.

THEOREM 0.1. *Let H be a localizing $\mathbb{A}^1$-homotopy invariant with values in a stable ∞-category $\mathcal{C}$, and let (S, M) be a stable marked surface where $S \backslash M$ is equipped with a framing. Define $E = H(k)$ to be the object of $\mathcal{C}$ obtained by applying H to the dg category with one object and endomorphism ring k. Then there is an equivalence*

$$H(F(S, M; k)) \simeq E(S, M)[-1],$$

where the right-hand side denotes the relative homology of the pair (S, M) with coefficients in $E[-1]$.

As a concrete example, we have the following corollary.

COROLLARY 0.2. *Let k be a field of characteristic 0. We have the formulas*

$$\mathrm{HP}_0(F(S, M; k)) \cong H_1(S, M; k),$$
$$\mathrm{HP}_1(F(S, M; k)) \cong H_2(S, M; k)$$

for periodic cyclic homology over k.

Our proof strategy is as follows: we prove a Mayer–Vietoris type statement for localizing $\mathbb{A}^1$-homotopy invariants of topological Fukaya categories. Roughly speaking, using the local nature of the topological Fukaya category, the result allows us to reduce the calculation of such an invariant to the case when the surface is a boundary-marked disk. The arguments we use are inspired by methods in Thomason–Trobaugh's algebraic K-theory of schemes and may be of independent interest: we establish localization sequences for topological Fukaya categories which are analogous to the ones for derived categories of schemes appearing in [TT90]. Similar localization techniques for Fukaya categories play a role in [HKK14], where the Grothendieck group is computed, and the forthcoming work [PS16]. The Mayer–Vietoris theorem expresses $H(F(S, M))$ as a state sum of a coparacyclic 1-Segal object in a stable ∞-category. We conclude by proving a delooping statement for such objects, which allows us to compute the state sum explicitly. Finally, I would like to point out that Jacob Lurie has communicated to me a version of Theorem 0.1 that does not assume $\mathbb{A}^1$-homotopy invariance but assumes that the surface has no internal marked points.

We will use the language of ∞-categories and refer to [Lur09] as a general reference.

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

1. Cyclic and paracyclic 2-Segal objects

We begin with a translation of some constructions in [DK13] into the context of ∞-categories.

1.1 The Segal conditions

We start by formulating the Segal conditions (cf. [Seg74, Rez01, DK12]).

Remark 1.1. Let $\mathbf{\Delta}$ denote the category of finite non-empty linearly ordered sets. This category contains the simplex category Δ as the full subcategory spanned by the collection of standard ordinals $\{[n], n \geqslant 0\}$. For every object of $\mathbf{\Delta}$, there exists a *unique* isomorphism with an object of Δ so that we can identify any diagram in $\mathbf{\Delta}$ with a unique diagram in Δ. With this in mind, we may use arbitrary finite non-empty linearly ordered sets to describe diagrams in Δ without ambiguity.

Definition 1.2. Let $\mathcal{C}$ be an ∞-category, and let $X^\bullet : \mathrm{N}(\Delta) \to \mathcal{C}$ be a cosimplicial object in $\mathcal{C}$.

(1) The cosimplicial object $X^\bullet$ is called *1-Segal* if, for every $0 < k < n$, the resulting diagram

$$\begin{tikzcd} X^{\{k\}} \arrow[r] \arrow[d] & X^{\{k,k+1,\dots,n\}} \arrow[d] \\ X^{\{0,1,\dots,k\}} \arrow[r] & X^{\{0,1,\dots,n\}} \end{tikzcd}$$

in $\mathcal{C}$ is a pushout diagram.

(2) Let P be a planar convex polygon with vertices labelled cyclically by the set $\{0,1,\dots,n\}$, $n \geqslant 3$. Consider a diagonal of P with vertices labelled by $i < j$ so that we obtain a subdivision of P into two subpolygons with vertex sets $\{0,1,\dots,i,j,\dots,n\}$ and $\{i,i+1,\dots,j\}$, respectively. The resulting triple of numbers $0 \leqslant i < j \leqslant n$ is called a *polygonal subdivision.*

(i) The cosimplicial object $X^\bullet$ is called *2-Segal* if, for every polygonal subdivision $0 \leqslant i < j \leqslant n$, the resulting diagram

$$\begin{tikzcd} X^{\{i,j\}} \arrow[r] \arrow[d] & X^{\{i,i+1,\dots,j\}} \arrow[d] \\ X^{\{0,1,\dots,i,j,\dots,n\}} \arrow[r] & X^{\{0,1,\dots,n\}} \end{tikzcd} \tag{1.3}$$

in $\mathcal{C}$ is a pushout diagram.

(ii) We say $X^\bullet$ is a *unital* 2-Segal object if, in addition to (i), for every $0 \leqslant k < n$, the diagram

$$\begin{tikzcd} X^{\{k,k+1\}} \arrow[r] \arrow[d] & X^{\{0,1,\dots,n\}} \arrow[d] \\ X^{\{k\}} \arrow[r] & X^{\{0,1,\dots,k,k+2,\dots,n\}} \end{tikzcd} \tag{1.4}$$

in $\mathcal{C}$ is a pushout diagram.

Proposition 1.5. *Every* 1*-Segal cosimplicial object* $X^\bullet : \mathrm{N}(\Delta) \to \mathcal{C}$ *is unital* 2*-Segal.*

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Proof. Given a polygonal subdivision $0 \leqslant i \leqslant j \leqslant n$, we augment the corresponding square (1.3) to the following diagram.

$$\begin{tikzcd}
X^{\{i\}} \amalg X^{\{j\}} \arrow[r] \arrow[d] & X^{\{i,j\}} \arrow[r] \arrow[d] & X^{\{i,i+1,\dots,j\}} \arrow[d] \\
X^{\{0,1,\dots,i\}} \amalg X^{\{j,j+1,\dots,n\}} \arrow[r] & X^{\{0,1,\dots,i,j,\dots,n\}} \arrow[r] & X^{\{0,1,\dots,n\}}
\end{tikzcd}$$

The 1-Segal condition on $X^\bullet$ implies that the left-hand square and outer square are pushouts. Hence, by [Lur09, 4.4.2.1], the right-hand square is a pushout. Similarly, to obtain unitality, we augment the square (1.4) to the following diagram.

$$\begin{tikzcd}
X^{\{k\}} \amalg X^{\{k+1\}} \arrow[r] \arrow[d] & X^{\{0,1,\dots,k\}} \amalg X^{\{k+1,k+2,\dots,n\}} \arrow[d] \\
X^{\{k,k+1\}} \arrow[r] \arrow[d] & X^{\{0,1,\dots,n\}} \arrow[d] \\
X^{\{k\}} \arrow[r] & X^{\{0,1,\dots,k,k+2,\dots,n\}}
\end{tikzcd}$$

The 1-Segal condition on $X^\bullet$ implies that the top and outer squares are pushouts so that, again by [Lur09, 4.4.2.1], the bottom square is a pushout. □

1.2 Cyclically ordered sets and ribbon graphs

Let S be a compact oriented surface, possibly with boundary ∂S, together with a chosen finite subset $M \subset S$ of marked points. We call (S, M) stable if:

(1) every connected component of S has at least one marked point;
(2) every connected component of ∂S has at least one marked point;
(3) every connected component of S that is homeomorphic to the 2-sphere has at least two marked points.

In this section, we develop a categorified state sum formalism based on a version of the well-known combinatorial description of stable oriented marked surfaces (S, M) in terms of ribbon graphs.

1.2.1 *Cyclically ordered sets.* In § 1.1, we enlarged the simplex category Δ to the equivalent category $\boldsymbol{\Delta}$ of finite non-empty linearly ordered sets. In analogy, we introduce the category of non-empty finite cyclically ordered sets $\boldsymbol{\Lambda}$ following [DK15], which plays the same role for the cyclic category Λ.

Let J be a finite non-empty set. We define a *cyclic order* on J to be a transitive action of the group $\mathbb{Z}$. Note that any such action induces a simply transitive action of the group $\mathbb{Z}/N\mathbb{Z}$ on J where N denotes the cardinality of the set J.

Example 1.6. Let I be a finite non-empty linearly ordered set. We obtain a cyclic order on I as follows: let $i_0 < i_1 < \cdots < i_n$ denote the elements of I. We set, for $0 \leqslant k < n$, $i_k + 1 = i_{k+1}$, and $i_n + 1 = i_0$. We call the resulting cyclic order on I the *cyclic closure* of the given linear order. We denote the cyclic closure of the standard ordinal $[n]$ by $\langle n \rangle$.

Example 1.7. More generally, let $f : J \to J'$ be a map of finite non-empty sets. Assume that J' carries a cyclic order and that every fiber of f is equipped with a linear order. We define the *lexicographic cyclic order* on J as follows: let $j \in J$. If j is not maximal in its fiber then we define

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

$j+1$ to be the successor to j in its fiber. If j is maximal in its fiber, then we define $j+1$ to be the minimal element of the successor fiber. (The cyclic order on J' induces a cyclic order on the fibers of f where we simply skip empty fibers.)

A *morphism* $J \to J'$ of cyclically ordered sets consists of

(1) a map $f : J \to J'$ of underlying sets,
(2) the choice of a linear order on every fiber of J'

such that the cyclic order on J is the lexicographic order from Example 1.7. We denote the resulting category of cyclically ordered sets by $\mathbf{\Lambda}$. Given a cyclically ordered set J, we define the *set of interstices*

$$J^\vee = \mathrm{Hom}_{\mathbf{\Lambda}}(J, \langle 0 \rangle),$$

which, by definition, is the set of linear orders on J whose cyclic closure agrees with the given cyclic order on J. If the set J has cardinality $n+1$, then we may identify $J^\vee$ with the set of isomorphisms $J \to \langle n \rangle$ in $\mathbf{\Lambda}$. The $\mathbb{Z}$-action on $\langle n \rangle$ induces a $\mathbb{Z}$-action on $J^\vee$ that defines a cyclic order.

PROPOSITION 1.8. *The association $J \mapsto J^\vee$ extends to an equivalence of categories $\mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$.*

Proof. Given a morphism $f : J \to J'$ in $\mathbf{\Lambda}$, we have to define a dual $f^\vee : (J')^\vee \to J^\vee$. The datum of f includes a choice of linear order on each fiber of f. Given a linear order on J', we can form the lexicographic linear order on J by a linear analog of the construction in Example 1.7. This defines the map $f^\vee$ on underlying sets. We further have to define a linear order on the fibers of $f^\vee$. Given linear orders $h : J' \cong [n]$ and $h' : J' \cong [n]$ such that $f^\vee(h) = f^\vee(h')$, we fix any $j \in J$ and declare $h \leqslant h'$ if $h(j) \leqslant h'(j)$. This defines a linear order on each fiber of $f^\vee$ which does in fact not depend on the chosen element $j \in J$. To verify that $J \mapsto J^\vee$ is an equivalence, we observe that the double dual is naturally equivalent to the identity functor: an element $j \in J$ determines a linear order on $J^\vee$ by declaring, for $h : J \cong [n]$ and $h' : J \cong [n]$, $h \leqslant h'$ if $h(j) \leqslant h'(j)$. We leave to the reader the verification that this association defines an isomorphism

$$J \to (J^\vee)^\vee$$

in $\mathbf{\Lambda}$ that extends to a natural isomorphism between the identity functor and the double dual. $\square$

We refer to the equivalence $\mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$ as *interstice duality*. The following lemma will be important for the interplay between cyclic 2-Segal objects and ribbon graphs.

LEMMA 1.9. *Let I, J be finite sets with elements $i \in I$, $j \in J$, and consider the pullback square*

$$\begin{tikzcd} K \arrow[r] \arrow[d] & J \arrow[d, "q"] \\ I \arrow[r, "p"] & \{i, j\} \end{tikzcd} \tag{1.10}$$

where the maps p and q are determined by $p^{-1}(i) = \{i\}$ and $q^{-1}(j) = \{j\}$. Assume that K is non-empty and that the sets I and J are equipped with cyclic orders. Then the following hold.

(1) *The above diagram lifts uniquely to a diagram in $\mathbf{\Lambda}$ such that the induced cyclic orders in I and J are the given ones.*
(2) *The resulting square in $\mathbf{\Lambda}$ is a pullback square.*

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Proof. Follows by direct inspection. □

Example 1.11. Consider the diagram of linearly ordered sets

$$\begin{tikzcd} \{i,j\} \arrow[r] \arrow[d] & \{i,i+1,\dots,j\} \arrow[d] \\ \{0,1,\dots,i,j,\dots,n\} \arrow[r] & \{0,1,\dots,n\} \end{tikzcd} \tag{1.12}$$

corresponding to a polygonal subdivision $0 \leqslant i < j \leqslant n$ of a planar convex polygon as in § 1.1. Passing to cyclic closures we obtain a diagram in $\mathbf{\Lambda}$. By applying interstice duality we obtain a diagram in $\mathbf{\Lambda}$ of the form (1.10) which is hence a pullback diagram. We deduce that the original diagram (1.12) is a pushout diagram in $\mathbf{\Lambda}$. Further, the same argument implies that, for every $0 \leqslant k < n$, the diagram

$$\begin{tikzcd} \{k,k+1\} \arrow[r] \arrow[d] & \{0,1,\dots,n\} \arrow[d] \\ \{k\} \arrow[r] & \{0,1,\dots,k,k+2,\dots,n\} \end{tikzcd} \tag{1.13}$$

is a pushout diagram in $\mathbf{\Lambda}$.

Definition 1.14. Let $\mathcal{C}$ be an ∞-category. A cocyclic object $X : \mathrm{N}(\mathbf{\Lambda}) \to \mathcal{C}$ is called (unital) 2-Segal (respectively 1-Segal) if the underlying cosimplicial object is (unital) 2-Segal (respectively 1-Segal).

Remark 1.15. Note, that the diagram

$$\begin{tikzcd} \{k\} \arrow[r] \arrow[d] & \{k,k+1,\dots,n\} \arrow[d] \\ \{0,1,\dots,k\} \arrow[r] & \{0,1,\dots,n\} \end{tikzcd} \tag{1.16}$$

is a pushout diagram in $\mathbf{\Delta}$ and the 1-Segal condition requires a cosimplicial object $X : \Delta \to \mathcal{C}$ to preserve this pushout. In light of this observation, the 2-Segal condition becomes very natural for cocyclic objects: while the cyclic closure of (1.16) is *not* a pushout square in $\mathbf{\Lambda}$, the squares (1.12) and (1.13) are pushout squares. The 2-Segal condition requires that these pushouts are preserved.

1.2.2 *Ribbon graphs.* A *graph* Γ is a pair of finite sets (H,V) equipped with an involution $\tau : H \to H$ and a map $s : H \to V$. The elements of the set H are called *half-edges*. We call a half-edge *external* if it is fixed by τ, and *internal* otherwise. A pair of τ-conjugate internal half-edges is called an *edge* and we denote the set of edges by E. The elements of V are called *vertices*. Given a vertex v, the half-edges in the set $H(v) = s^{-1}(v)$ are said to be *incident to* v. A graph with one vertex and n half-edges equipped with the trivial involution is called an *n-corolla*.

A *ribbon graph* is a graph Γ where, for every vertex v of Γ, the set $H(v)$ of half-edges incident to v is equipped with a cyclic order. We give an interpretation of this datum in terms of the category of cyclically ordered sets: let Γ be a graph. We define the incidence category $I(\Gamma)$ to

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

have set of objects given by $V \cup E$ and, for every internal half-edge h, a unique morphism from the vertex $s(h)$ to the edge $\{h, \tau(h)\}$. We define a functor

$$\gamma : I(\Gamma) \longrightarrow \mathcal{S}et$$

that, on objects, associates to a vertex v the set $H(v)$ and to an edge e the set of half-edges underlying e. To a morphism $v \to e$, given by a half-edge $h \in H(v)$ with $h \in e$, we associate the map $\pi : H(v) \to e$ which is determined by $\pi^{-1}(h) = \{h\}$ so that π maps $H(v)\backslash h$ to $\tau(h)$. We call the functor γ the *incidence diagram* of the graph Γ.

PROPOSITION 1.17. *Let Γ be a graph. A ribbon structure on Γ is equivalent to a lift*

$$\begin{tikzcd} & \mathbf{\Lambda} \arrow[d] \\ I(\Gamma) \arrow[r] \arrow[ur, dashed] & \mathcal{S}et \end{tikzcd}$$

of the incidence diagram of Γ where $\mathbf{\Lambda}$ denotes the category of cyclically ordered sets.

The advantage of the interpretation of a ribbon structure given in Proposition 1.17 is that it facilitates the passage to interstices: given a ribbon graph Γ with corresponding incidence diagram

$$\gamma : I(\Gamma) \to \mathbf{\Lambda},$$

we introduce the *coincidence diagram*

$$\delta : I(\Gamma)^{\mathrm{op}} \longrightarrow \mathbf{\Lambda}$$

obtained by postcomposing γ^{op} with the interstice duality functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathbf{\Lambda}$.

A morphism $(f, \eta) : \Gamma \to \Gamma'$ of ribbon graphs consists of:

(1) a functor $f : I(\Gamma) \to I(\Gamma')$ of incidence categories;

(2) a natural transformation $\eta : f^*\gamma' \to \gamma$ of incidence diagrams.

We denote the resulting category of ribbon graphs by $\mathcal{R}ib$.

Example 1.18. Let Γ be a graph and let e be an edge incident to two distinct vertices v and w. We define a new graph Γ' obtained from Γ by contracting e as follows: the set of half-edges H' is given by $H\backslash e$ and the set of vertices V' is obtained from V by identifying v and w. The involution τ on H restricts to an involution τ' on H'. We define $s : H' \to V'$ as the composite of the restriction of $s : H \to V$ to H' and the quotient map $V \to V'$.

We obtain a natural functor $f : I(\Gamma) \to I(\Gamma')$ of incidence categories which collapses the objects v, w, and e to $\overline{v}$. Denoting by γ and γ' the set-valued incidence diagrams, we construct a natural transformation $\eta : f^*\gamma' \to \gamma$ as follows. On objects of $I(\Gamma)$ different from v, w, and e, we define η to be the identity map. To obtain the values of η at v, w, and e, note that we have a natural commutative diagram

$$\begin{tikzcd} H(v) \arrow[d] & \\ e & H(\overline{v}) \arrow[l, "\eta_e"'] \arrow[ul, "\eta_v"'] \arrow[dl, "\eta_w"] \\ H(w) \arrow[u] & \end{tikzcd}$$

as in Lemma 1.9. The maps in the diagram determine the values of η as indicated.

Finally, assume that Γ carries a ribbon structure. Then Lemma 1.9(1) implies that Γ' carries a unique ribbon structure such that the natural transformation η lifts to $\mathbf{\Lambda}$-valued incidence diagrams.

A morphism of ribbon graphs $\Gamma \to \Gamma'$ as constructed in Example 1.18 is called an *edge contraction*. Note that the category $\mathcal{R}ib$ contains morphisms that cannot be obtained as compositions of edge contractions: for example, it contains a copy of the cyclic category Λ, given by the full subcategory spanned by the corollas. The following proposition indicates the relevance of Lemma 1.9(2).

Proposition 1.19. *Let Γ be a ribbon graph and let $(f,\eta) : \Gamma \to \Gamma'$ be an edge contraction. Then the natural transformation*

$$\eta : f^*\gamma' \longrightarrow \gamma$$

exhibits γ' as a right Kan extension of γ along f.

Proof. By the pointwise formula for right Kan extensions, it suffices to verify that, for every object $y \in I(\Gamma')$, the natural transformation η exhibits $\gamma'(y)$ as the limit of the diagram

$$y/f \longrightarrow \Lambda, \quad (x, y \to f(x)) \mapsto \gamma(x).$$

Unravelling the definitions, this is a trivial condition unless y is the object corresponding to the vertex $\overline{v}$ under the contracted edge. For $y = \overline{v}$ the condition reduces to Lemma 1.9(2). □

1.2.3 *State sums on ribbon graphs.* We introduce a category $\mathcal{R}ib^*$ with objects given by pairs (Γ, x) where Γ is a ribbon graph and x is an object of the incidence category $I(\Gamma)$. A morphism $(\Gamma, x) \to (\Gamma', y)$ consists of a morphism $(f, \eta) : \Gamma \to \Gamma'$ of ribbon graphs together with a morphism $y \to f(x)$ in $I(\Gamma')$. The category $\mathcal{R}ib^*$ comes equipped with a forgetful functor $\pi : \mathcal{R}ib^* \to \mathcal{R}ib$ and an evaluation functor

$$\mathrm{ev} : \mathcal{R}ib^* \longrightarrow \mathbf{\Lambda}, \quad (\Gamma, x) \mapsto \delta(x),$$

where δ denotes the coincidence diagram of Γ. Let $\mathcal{C}$ be an ∞-category with colimits, and let $X : \mathrm{N}(\mathbf{\Lambda}) \to \mathcal{C}$ be a cocyclic object in $\mathcal{C}$. The functor

$$\rho_X = \mathrm{N}(\pi)_!(X \circ \mathrm{N}(\mathrm{ev})) : \mathrm{N}(\mathcal{R}ib) \longrightarrow \mathcal{C}$$

is called the *state sum functor of X*. Here, $\mathrm{N}(\pi)_!$ denotes the ∞-categorical left Kan extension defined in [Lur09, 4.3.3.2]. For a ribbon graph Γ, the object $X(\Gamma) := \rho_X(\Gamma)$ is called the *state sum of X on Γ*.

Proposition 1.20. *The state sum of X on Γ admits the formula*

$$X(\Gamma) \simeq \operatorname{colim} X \circ \mathrm{N}(\delta),$$

where $\delta : I(\Gamma)^{\mathrm{op}} \to \mathbf{\Lambda}$ denotes the coincidence diagram of Γ.

Proof. By the pointwise formula for left Kan extensions, we have

$$X(\Gamma) = \operatorname{colim} X \circ \mathrm{N}(\mathrm{ev})_{|\mathrm{N}(\pi/\Gamma)}.$$

The nerve of the functor

$$I(\Gamma)^{\mathrm{op}} \longrightarrow \pi/\Gamma, \quad x \mapsto ((\Gamma, x), \Gamma \xrightarrow{\mathrm{id}} \Gamma)$$

is cofinal, which implies the claim. □

Example 1.21. The universal example of a cocyclic object with values in an ∞-category with colimits is the Yoneda embedding $j : \mathrm{N}(\mathbf{\Lambda}) \to \mathcal{P}(\mathbf{\Lambda})$ where $\mathcal{P}(\mathbf{\Lambda})$ denotes the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Lambda})^{\mathrm{op}}, \mathcal{S})$ of cyclic spaces. We obtain a functor

$$\rho_j : \mathrm{N}(\mathcal{R}ib) \longrightarrow \mathcal{P}(\mathbf{\Lambda})$$

that realizes a ribbon graph as a cyclic space. This functor is the universal state sum: given any cocyclic object $X : \mathrm{N}(\mathbf{\Lambda}) \to \mathcal{C}$ where $\mathcal{C}$ has colimits, we have

$$\rho_X \simeq j_! X \circ \rho_j, \tag{1.22}$$

where we use Proposition 1.20 and the fact [Lur09, 5.1.5.5] that $j_! X$ commutes with colimits. We will use the notation

$$\Lambda(\Gamma) := \rho_j(\Gamma)$$

for the state sum of j on Γ.

The following proposition explains the relevance of the 2-Segal condition for state sums.

PROPOSITION 1.23. *Let $\mathcal{C}$ be an ∞-category with colimits and let $X : \mathrm{N}(\mathbf{\Lambda}) \to \mathcal{C}$ be a cocyclic object. Then X is unital 2-Segal if and only if the state sum functor*

$$\rho_X : \mathrm{N}(\mathcal{R}ib) \longrightarrow \mathcal{C}, \quad \Gamma \mapsto X(\Gamma)$$

maps edge contractions in $\mathcal{R}ib$ to equivalences in $\mathcal{C}$.

Proof. Let $(f, \eta) : \Gamma \to \Gamma'$ be a morphism of ribbon graphs. The associated morphism $\rho_X(f, \eta) : X(\Gamma) \to X(\Gamma')$ is given by the composite

$$\mathrm{colim}(X \circ \delta) \xrightarrow{X \circ \eta^\vee} \mathrm{colim}(X \circ \delta' \circ f^{\mathrm{op}}) \longrightarrow \mathrm{colim}(X \circ \delta').$$

We claim that, if (f, η) is an edge contraction, then $X \circ \eta^\vee$ exhibits $X \circ \delta'$ as a left Kan extension of $X \circ \delta$. This implies the result since a colimit is given by the left Kan extension to the final category and left Kan extension functors are functorial in the sense $f_! \circ g_! \simeq (f \circ g)_!$. The claim follows immediately from the argument of Proposition 1.19, Lemma 1.9, and Remark 1.15. $\square$

Remark 1.24. In the situation of Proposition 1.23, we may restrict ourselves to the subcategory $\mathcal{R}ib' \subset \mathcal{R}ib$ generated by edge contractions and isomorphisms. Then by the statement of the theorem, we obtain a functor

$$\mathrm{N}(\mathcal{R}ib')_{\simeq} \to \mathcal{C}, \quad \Gamma \mapsto X(\Gamma),$$

where $\mathrm{N}(\mathcal{R}ib')_{\simeq} = \mathrm{Sing}\,|\,\mathrm{N}(\mathcal{R}ib')|$ denotes the ∞-groupoid completion of the ∞-category $\mathrm{N}(\mathcal{R}ib')$. The automorphism group in $\mathrm{N}(\mathcal{R}ib')_{\simeq}$ of a ribbon graph Γ that represents a stable oriented marked surface (S, M) can be identified with the mapping class group $\mathrm{Mod}(S, M)$. The above functor implies the existence of an ∞-categorical action of $\mathrm{Mod}(S, M)$ on $X(\Gamma)$, which is a main result of [DK13].

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

1.3 Paracyclically ordered sets and framed graphs

Let (S, M) be a stable oriented marked surface. We may interpret the orientation as a reduction of the structure group of the tangent bundle of $S\backslash M$ along $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$. We define a *framing* of (S, M) as a further lift of the structure group along the universal cover

$$\widetilde{\mathrm{GL}_2^+(\mathbb{R})} \longrightarrow \mathrm{GL}_2^+(\mathbb{R}).$$

Up to contractible choice, this datum is equivalent to a trivialization of the tangent bundle of $S\backslash M$. In this section, we describe a state sum formalism based on a combinatorial model for stable framed marked surfaces as developed in [DK15]. This amounts to a variation of the constructions in the previous section, obtained by replacing the cyclic category by the *paracyclic category*.

1.3.1 *Paracyclically ordered sets.* Let J be a finite non-empty set. We define a *paracyclic order* on J to be a cyclic order on J together with the choice of a $\mathbb{Z}$-torsor $\widetilde{J}$ and a $\mathbb{Z}$-equivariant map $\widetilde{J} \to J$. A morphism of paracyclically ordered sets $(J, \widetilde{J}) \to (J', \widetilde{J}')$ consists of a commutative diagram of sets

$$\begin{tikzcd} \widetilde{J} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{J}' \arrow[d] \\ J \arrow[r, "f"] & J' \end{tikzcd}$$

such that $\widetilde{f}$ is monotone with respect to the $\mathbb{Z}$-torsor linear orders. The lift $\widetilde{f}$ equips f naturally with the structure of a morphism of cyclically ordered sets so that we obtain a forgetful functor

$$\mathbf{\Lambda}_\infty \longrightarrow \mathbf{\Lambda},$$

where $\mathbf{\Lambda}_\infty$ denotes the category of paracyclically ordered sets. As for cyclically ordered sets, there is a skeleton $\Lambda_\infty \subset \mathbf{\Lambda}_\infty$ consisting of standard paracyclically ordered sets $(\langle n\rangle, \widetilde{\langle n\rangle})$ where we define $\widetilde{\langle n\rangle} = \mathbb{Z}$ and $\widetilde{\langle n\rangle} \to \langle n\rangle$ is given by the natural quotient map.

Given a paracyclically ordered set $(J, \widetilde{J})$, then the cyclic order on the interstice dual $J^\vee$ lifts to a natural paracyclic order given by

$$\widetilde{J^\vee} = \mathrm{Hom}_{\mathbf{\Lambda}_\infty}((J, \widetilde{J}), (\langle 0\rangle, \widetilde{\langle 0\rangle})).$$

This construction extends the self duality of $\mathbf{\Lambda}$ to one for $\mathbf{\Lambda}_\infty$. All statements of § 1.2.1 hold *mutatis mutandis* for $\mathbf{\Lambda}_\infty$.

1.3.2 *Framed graphs and state sums.* We define a *framed graph* Γ to be a graph Γ equipped with a lift

$$\begin{tikzcd} & \mathbf{\Lambda}_\infty \arrow[d] \\ I(\Gamma) \arrow[r] \arrow[ur, dashed] & \mathcal{S}et \end{tikzcd}$$

of the incidence diagram of Γ where $\mathbf{\Lambda}_\infty$ denotes the category of paracyclically ordered sets. Framed graphs form a category $\mathcal{R}ib_\infty$ which is defined in complete analogy with $\mathcal{R}ib$.

Let Γ be a framed graph with incidence diagram

$$\gamma : I(\Gamma) \to \mathbf{\Lambda}_\infty.$$

Let $e = \{h, \tau(h)\}$ be an edge in Γ incident to the vertices $v = s(h)$ and $w = s(\widetilde{\tau(h)})$, and let h' be a half-edge incident to w. Let $\widetilde{h}$ be a lift of h to an element of the $\mathbb{Z}$-torsor $\widetilde{H(v)}$ that is part of the paracyclic structure on $H(v)$. Then we may transport this lift along the edge e to obtain a lift of h' to an element $\widetilde{h}'$ of $\widetilde{H(w')}$ as follows: there is a unique lift of $\widetilde{\tau(h)} \in \widetilde{H(w')}$ of $\tau(h)$ which maps to $\widetilde{h} - 1$ under $\gamma(\tau(h))$. We then set $\widetilde{h}' = \widetilde{\tau(h)} + i$ where $i \geqslant 0$ is minimal such that $\widetilde{h}'$ lifts h'. Iterating this transport along a loop l returning to the half-edge h, we obtain another lift $\overline{h}$ of h. The integer $\overline{h} - \widetilde{h}$ only depends on l and we refer to it as the *winding number of l*.

Remark 1.25. As explained in [DK15], framed graphs provide a combinatorial model for stable marked surfaces (S, M) equipped with a trivialization of the tangent bundle of $S\backslash M$. The above combinatorial construction then coincides with the geometric concept of winding number computed with respect to the framing.

Let $\mathcal{C}$ be an ∞-category with colimits, and let $X^\bullet : \mathrm{N}(\mathbf{\Lambda}_\infty) \to \mathcal{C}$ be a coparacyclic object in $\mathcal{C}$. Then, given a framed graph Γ, we have a state sum

$$X(\Gamma) = \operatorname{colim} X \circ \delta,$$

where $\delta : I(\Gamma)^{\mathrm{op}} \to \mathbf{\Lambda}_\infty$ denotes the coincidence diagram of Γ. The various state sums naturally organize into a functor

$$\mathrm{N}(\mathcal{R}ib_\infty) \longrightarrow \mathcal{C}, \quad \Gamma \mapsto X(\Gamma).$$

Remark 1.26. As shown in [DK15], the state sum $X(\Gamma)$ of a framed graph with values in a coparacyclic unital 2-Segal object X admits an action of the *framed* mapping class group of the surface.

1.4 The universal loop space

We give a first example of a state sum that will play an important role later on. Consider the functor

$$\Lambda \longrightarrow \mathrm{Grp}, \quad \langle n \rangle \mapsto \pi_1(D/\{0, 1, \ldots, n\}),$$

where $D/\{0, 1, \ldots, n\}$ denotes the quotient of the unit disk by $n + 1$ marked points on the boundary. Replacing the fundamental groups by their nerves, we obtain a functor

$$L^\bullet : \mathrm{N}(\Lambda) \longrightarrow \mathcal{S}_*, \tag{1.27}$$

where $\mathcal{S}_*$ denotes the ∞-category of pointed spaces. The cosimplicial pointed space underlying $L^\bullet$ is 1-Segal and hence, by Proposition 1.5, unital 2-Segal.

Remark 1.28. Note that the group $\pi_1(D/\{0, 1, \ldots, n\})$ is a free group on n generators so that L^n is equivalent to a bouquet of n one-dimensional spheres. Given any pointed space X, the cyclic pointed 1-Segal space $\mathrm{Map}(L^\bullet, X)$ describes the loop space ΩX together with its natural group structure. Similarly, the cocyclic pointed 1-Segal space $L^\bullet \otimes X$ describes the suspension of X together with its natural cogroup structure.

PROPOSITION 1.29. *Let (S, M) be a stable marked oriented surface represented by a ribbon graph Γ. Then we have*

$$L(\Gamma) \simeq S/M.$$

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Proof. We can compute the state sum defining $L(\Gamma)$ explicitly as a homotopy colimit in the category of pointed spaces. We may replace the diagram by a homotopy equivalent one which assigns to each $(n+1)$-corolla of the graph Γ the space $D/\{0,1,\dots,n\}$. The homotopy colimit is then obtained by identifying the boundary cycles of the various spaces $D/\{0,1,\dots,n\}$ according to the incidence relations given by the edges of the ribbon graphs. It is apparent that the resulting space is equivalent to the quotient space S/M. $\square$

Example 1.30. As an illustration of the gluing procedure described in the proof of Proposition 1.29, consider the ribbon graph with one vertex and two half-edges forming a loop at the vertex. This graph models a sphere S^2 with two marked points M. The colimit in the proof describes the space S^2/M as obtained from $D/\{0,1\}$ by identifying the two boundary circles given by the images of the two boundary arcs connecting the points 0 and 1 on ∂D.

For a slight elaboration on Proposition 1.29, let $\mathcal{C}$ be a stable ∞-category with colimits and let E be an object of $\mathcal{C}$. By [Lur11, 1.4.2.21], the functor

$$\mathrm{Sp}(\mathcal{C}) = \mathrm{Fun}^c(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \to \mathcal{C}, \quad f \mapsto f(S^0)$$

from the ∞-category of spectrum objects in $\mathcal{C}$ to $\mathcal{C}$ is an equivalence. Therefore, the object E defines an essentially unique functor

$$E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}, \tag{1.31}$$

which we still denote by E.

Definition 1.32. Let (X,Y) be a pair of finite spaces. We introduce the pointed quotient space X/Y as the pushout

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/Y \end{tikzcd}$$

in $\mathcal{S}$. The object $E(X/Y)$ of $\mathcal{C}$ is called the *relative homology of the pair* (X,Y) *with coefficients in* E. In the case when $\mathcal{C}$ is the category of spectra, this terminology agrees with the customary one.

We introduce the cocyclic object

$$L_E = E(L^\bullet) : \mathrm{N}(\Lambda) \longrightarrow \mathcal{C} \tag{1.33}$$

obtained from (1.27) by postcomposing with $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$.

Proposition 1.34. *Let* (S,M) *be a stable marked oriented surface represented by a ribbon graph* Γ. *Then we have an equivalence*

$$L_E(\Gamma) \simeq E(S/M)$$

in $\mathcal{C}$.

Proof. The functor $E(-)$ commutes with finite colimits so that the statement follows immediately from Proposition 1.29. $\square$

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Remark 1.35. We may pullback a cocyclic 2-Segal object $X^\bullet$ along the functor $\Lambda_\infty \to \Lambda$ to obtain a coparacyclic 2-Segal object $\widetilde{X}^\bullet$. Given a framed graph Γ, we have

$$\widetilde{X}(\Gamma) \simeq X(\overline{\Gamma}),$$

where $\overline{\Gamma}$ is the ribbon graph underlying Γ. In particular, in the context of Proposition 1.34, we obtain

$$\widetilde{L_E}(\Gamma) \simeq E(S/M).$$

2. Differential graded categories

2.1 Morita equivalences

We introduce some terminology for the derived Morita theory of differential graded categories and refer the reader to [Tab05, Toë07] for more detailed treatments. Let k be a commutative ring, and let $\mathcal{C}at_{\mathrm{dg}}$ be the category of small k-linear differential $\mathbb{Z}$-graded categories. Recall that a functor $f : A \to B$ is called a *quasi-equivalence* if:

(1) the functor $H^0(f) : H^0(A) \to H^0(B)$ of homotopy categories is an equivalence of categories;

(2) for every pair of objects (x, y) in A, the morphism $f : \mathrm{Hom}_A(x, y) \to \mathrm{Hom}_B(f(x), f(y))$ of complexes is a quasi-isomorphism.

We denote by $\mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}})$ the ∞-category obtained by localizing $\mathcal{C}at_{\mathrm{dg}}$ along quasi-equivalences [Lur11, 1.3.4.1]. The collection of quasi-equivalences can be supplemented to a combinatorial model structure on $\mathcal{C}at_{\mathrm{dg}}$ which facilitates calculations in $\mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}})$.

Given dg categories A, B, we denote the dg category of enriched functors from A to B by $\underline{\mathrm{Hom}}(A, B)$. We denote by Mod_k the dg category of unbounded complexes of k-modules, and further, by Mod_A the dg category $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathrm{Mod}_k)$. We equip Mod_A with the projective model structure and denote by $\mathrm{Perf}_A \subset \mathrm{Mod}_A$ the full dg category spanned by those objects x such that:

(1) x is cofibrant;

(2) the image of x in $H^0(\mathrm{Mod}_A)$ is compact, i.e. $\mathrm{Hom}(x, -)$ commutes with coproducts.

Given a dg functor $f : A \to B$, we have a Quillen adjunction

$$f_! : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B : f^*$$

and obtain an induced functor

$$f_! : \mathrm{Perf}_A \longrightarrow \mathrm{Perf}_B. \tag{2.1}$$

The functor $f : A \to B$ is called a *Morita equivalence* if the induced functor (2.1) is a quasi-equivalence. We denote by $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ the ∞-category obtained by localizing $\mathcal{C}at_{\mathrm{dg}}$ along Morita equivalences. We have an adjunction

$$l : \mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}}) \longleftrightarrow \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) : i, \tag{2.2}$$

where i is fully faithful so that l is a localization functor.

Let $\mathcal{C}at_{\mathrm{dg}}^{(2)}$ denote the category of small $\mathbf{k}$-linear differential $\mathbb{Z}/2\mathbb{Z}$-graded categories. All of the above theory can be translated *mutatis mutandis* via the adjunction

$$P : \mathcal{C}at_{\mathrm{dg}} \longleftrightarrow \mathcal{C}at_{\mathrm{dg}}^{(2)} : Q,$$

which is a Quillen adjunction with respect to an adaptation of the quasi-equivalence model structure on $\mathcal{C}at_{\mathrm{dg}}^{(2)}$. The periodization functor P associates to a differential $\mathbb{Z}$-graded category the differential $\mathbb{Z}/2\mathbb{Z}$-graded category with the same objects and $\mathbb{Z}/2\mathbb{Z}$-graded mapping complexes obtained by summing over all even (respectively odd) terms of the $\mathbb{Z}$-graded mapping complexes. We will refer to the $\mathbb{Z}/2\mathbb{Z}$-graded analogs of the above constructions via the superscript (2).

Remark 2.3. Note that, due to the adjunction (2.2), the functor l commutes with colimits so that we may compute colimits in the category $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ as colimits in $\mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}})$. The latter category is equipped with the quasi-equivalence model structure so that, by [Lur09], we can compute colimits as homotopy colimits with respect to this model structure. The analogous statement holds for the $\mathbb{Z}/2\mathbb{Z}$-graded variants.

2.2 Exact sequences of dg categories

A morphism in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ is called quasi-fully faithful if it is equivalent to the image of a quasi-fully faithful morphism under the localization functor $\mathrm{N}(\mathcal{C}at_{\mathrm{dg}}) \to \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$. A pushout square

$$\begin{tikzcd} S \arrow[r, hook, "g"] \arrow[d] & T \arrow[d, two heads, "f"] \\ 0 \arrow[r] & U \end{tikzcd}$$

in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ with g quasi-fully faithful is called an *exact sequence*. The following technical statement will be used below.

Lemma 2.4. *Quasi-fully faithful morphisms are stable under pushouts in* $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$.

Proof. The adjunction (2.2) implies that the left adjoint $l : \mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ preserves colimits so that it suffices to prove the corresponding statement for $\mathrm{L}_{\mathrm{qe}}(\mathcal{C}at_{\mathrm{dg}})$. To this end it suffices to show that quasi-fully faithful functors in the category $\mathcal{C}at_{\mathrm{dg}}$ are stable under homotopy pushouts with respect to the quasi-equivalence model structure defined in [Tab05]. Given a diagram

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"] & T \\ S' & \end{tikzcd} \tag{2.5}$$

with g quasi-fully faithful, we may assume that all objects are cofibrant, and f, g are cofibrations so that the homotopy pushout is given by an ordinary pushout. Denoting by I the set of generating cofibrations of $\mathcal{C}at_{\mathrm{dg}}$, we may, by Quillen's small object argument, factor the morphism f as

$$S \xrightarrow{f_1} \widetilde{S}' \xrightarrow{f_2} S',$$

where f_1 is a relative I-cell complex and f_2 is a trivial fibration. Forming pushouts we obtain a diagram

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f_1"] & T \arrow[d] \\ \widetilde{S}' \arrow[r, "\widetilde{g}"] \arrow[d, "f_2"] & T' \arrow[d, "r"] \\ S' \arrow[r, "g'"] & T'' \end{tikzcd} \tag{2.6}$$

Since $\widetilde{g}$ is a cofibration, the bottom square is a homotopy pushout square and thus r is a quasi-equivalence. It hence suffices to show that $\widetilde{g}$ is quasi-fully faithful so that we may assume that f is a relative I-cell complex. Using that filtered colimits of complexes are homotopy colimits [TV07], and hence preserve quasi-isomorphisms, we reduce to the case that f is the pushout of a single generating cofibration from I. This leaves us with the following two cases.

(1) S' is obtained from S by adjoining one object, and the pushout T' of (2.5) is obtained from T by adjoining one object. Clearly, the functor $S' \to T'$ is quasi-fully faithful.

(2) S' is obtained from S by freely adjoining a morphism $p : a \to b$ of some degree n between objects a, b of S where $d(p)$ is a prescribed morphism q of S. The morphism complex between objects x, y in S' can be described explicitly as

$$S'(x,y) = \bigoplus_{n\geqslant 0} S(x,a) \otimes kp \otimes S(b,a) \otimes kp \otimes \cdots \otimes S(b,y),$$

where n copies of kp appear in the nth summand. The differential is given by the Leibniz rule where, upon replacing p by $d(p) = q$, we also compose with the neighboring morphisms so that the level is decreased from n to $n-1$. The morphism complexes of the pushout T' admit an analogous expression with p replaced by $g(p)$. We have to show that, for every pair of objects, the morphism of complexes

$$S'(x,y) \to T'(g(x), g(y))$$

is a quasi-isomorphism. To this end, we filter both complexes by the level n. On the associated graded complexes we have a quasi-isomorphism, since g is quasi-fully faithful. The corresponding spectral sequence converges, which yields the desired quasi-isomorphism.

□

Remark 2.7. The proof of the lemma works verbatim for $\mathcal{C}at_{\mathrm{dg}}^{(2)}$ instead of $\mathcal{C}at_{\mathrm{dg}}$.

3. Topological Fukaya categories

3.1 $\mathbb{Z}/2\mathbb{Z}$-graded

Let k be a commutative ring, and let $R = k[z]$ denote the polynomial ring with coefficients in k, considered as a $\mathbb{Z}/(n+1)$-graded k-algebra with $|z| = 1$. A matrix factorization (X, d_X) of $w = z^{n+1}$ consists of:

– a pair X^0, X^1 of $\mathbb{Z}/(n+1)$-graded R-modules;

– a pair of homogeneous R-linear homomorphisms

$$\begin{tikzcd} X^0 \arrow[r, bend right, "d^0"'] & X^1 \arrow[l, bend right, "d^1"'] \end{tikzcd}$$

of degree 0;

such that

– $d^1 \circ d^0 = w\,\mathrm{id}_{X^0}$ and $d^0 \circ d^1 = w\,\mathrm{id}_{X^1}$.

Example 3.1. For $i, j \in \mathbb{Z}/(n+1)$, $i \neq j$, we have a corresponding *scalar* matrix factorization $[i, j]$ defined as

$$k[z](i) \underset{z^{j-i}}{\overset{z^{i-j}}{\leftrightarrows}} k[z](j),$$

where the exponents of z are to be interpreted via their representatives in $\{1, 2, \ldots, n\}$. For $i = j$, we have two scalar matrix factorizations

$$k[z](i) \underset{1}{\overset{z^{n+1}}{\leftrightarrows}} k[z](i)$$

and

$$k[z](i) \underset{z^{n+1}}{\overset{1}{\leftrightarrows}} k[z](i),$$

which we denote by $[i, i]_r$ and $[i, i]_l$, respectively.

Given matrix factorizations X, Y of w, we form the $\mathbb{Z}/2\mathbb{Z}$-graded k-module $\mathrm{Hom}^\bullet(X, Y)$ with

$$\begin{aligned}\mathrm{Hom}^0(X, Y) &= \mathrm{Hom}_R(X^0, Y^0) \oplus \mathrm{Hom}_R(X^1, Y^1),\\ \mathrm{Hom}^1(X, Y) &= \mathrm{Hom}_R(X^0, Y^1) \oplus \mathrm{Hom}_R(X^1, Y^0),\end{aligned}$$

where Hom_R denotes homogeneous R-linear homomorphisms of degree 0. It is readily verified that the formula

$$d(f) = d_Y \circ f - (-1)^{|f|} f \circ d_X$$

defines a differential on $\mathrm{Hom}^\bullet(X, Y)$, i.e. $d^2 = 0$. Therefore, the collection of all matrix factorizations of w organizes into a differential $\mathbb{Z}/2\mathbb{Z}$-graded k-linear category which we denote by $\mathrm{MF}^{\mathbb{Z}/(n+1)}(k[z], z^{n+1})$. We further define

$$\bar{\mathrm{F}}^n \subset \mathrm{MF}^{\mathbb{Z}/(n+1)}(k[z], z^{n+1})$$

to be the full dg subcategory spanned by the scalar matrix factorizations from Example 3.1.

Theorem 3.2. *The association $n \mapsto \bar{\mathrm{F}}^n$ extends to a cocyclic object*

$$\bar{\mathrm{F}} : \mathrm{N}(\Lambda) \to \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}^{(2)})$$

which is unital 2-Segal.

Proof. This is the content of [DK13, 2.4.1] where the dg category $\bar{\mathrm{F}}^n$ is denoted by $\mathcal{E}^n$. □

Remark 3.3. Let A^n denote the k-linear envelope of the category associated to the linearly ordered set $\{1, 2, \ldots, n\}$, considered as a differential $\mathbb{Z}/2\mathbb{Z}$-graded category concentrated in degree 0. There is a dg functor

$$g : A^n \to \bar{\mathrm{F}}^n, \quad i \mapsto [0, i]$$

that maps the generating morphism $i \to j$ to the closed morphism of matrix factorizations

$$\begin{tikzcd} k[z] \arrow[r, shift right, "z^i"'] \arrow[d, "1"'] & k[z](i) \arrow[l, shift right, "z^{n+1-i}"'] \arrow[d, "z^{j-i}"] \\ k[z] \arrow[r, shift right, "z^j"'] & k[z](j) \arrow[l, shift right, "z^{n+1-j}"'] \end{tikzcd}$$

An explicit calculation enables the following observations to be made.

- The functor g is quasi-fully faithful.
- The object $[i, j]$ is a cone over the above morphism $[0, i] \to [0, j]$.
- The objects $[i, i]_l$ and $[i, i]_r$ are zero objects.

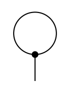
These observations imply that the functor g is a Morita equivalence (cf. [DK13, 2.3.6]). The reason for using $\bar{\mathrm{F}}^n$ instead of the much simpler dg category A^n is the following: the cocyclic object in Theorem 3.2 is difficult to describe in terms of A^n while the association $n \mapsto \bar{\mathrm{F}}^n$ defines a strict functor $\Lambda \to \mathcal{C}at_{\mathrm{dg}}^{(2)}$ which induces $\bar{\mathrm{F}}^{\bullet}$ by passing to the Morita localization.

The state sum formalism of § 1.2.3 yields a functor

$$\rho_{\bar{\mathrm{F}}} : \mathrm{N}(\mathcal{R}ib) \longrightarrow \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}^{(2)}), \quad \Gamma \mapsto \bar{\mathrm{F}}(\Gamma).$$

The state sum $\bar{\mathrm{F}}(\Gamma)$ of $\bar{\mathrm{F}}$ on a ribbon graph Γ is called the ($\mathbb{Z}/2\mathbb{Z}$-graded) *topological Fukaya category* of Γ.

Example 3.4. Consider the ribbon graph Γ given by



The corresponding topological Fukaya category $\bar{\mathrm{F}}(\Gamma)$ can, by Proposition 4.4(i), Remarks 2.3, and 3.3, be computed as the homotopy pushout of the diagram

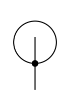
$$\begin{tikzcd} A^0 \amalg A^0 \arrow[r] \arrow[d] & A^0 \\ A^1 & \end{tikzcd}$$

with respect to the quasi-equivalence model structure on $\mathcal{C}at_{\mathrm{dg}}^{(2)}$. All objects are cofibrant and the vertical functor is a cofibration so that the homotopy pushout can be computed as an ordinary pushout. Therefore, we obtain

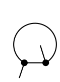
$$\bar{\mathrm{F}}(\Gamma) \simeq k[t]$$

the k-linear category with one object and endomorphism ring $k[t]$, considered as a differential $\mathbb{Z}/2\mathbb{Z}$-graded category with zero differential. We can therefore interpret $\bar{\mathrm{F}}(\Gamma)$ as the $\mathbb{Z}/2\mathbb{Z}$-folding of the bounded derived dg category of coherent sheaves on the affine line $\mathbb{A}_k^1$ over k.

Example 3.5. Consider the ribbon graph Γ given by



We replace Γ by the ribbon graph Γ'



https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

which, by Proposition 1.23, has an equivalent topological Fukaya category. Using [Lur09, Proposition 4.2.3.8], Remarks 2.3, and 3.3, we can obtain $\bar{\mathrm{F}}(\Gamma')$ as the homotopy pushout of

$$\begin{tikzcd} A^0 \amalg A^0 \arrow[r] \arrow[d] & A^1 \\ A^1 & \end{tikzcd}$$

with respect to the quasi-equivalence model structure on $\mathcal{C}at_{\mathrm{dg}}^{(2)}$. Since all objects in this diagram are cofibrant and all functors cofibrations, we can form the ordinary pushout to obtain a description of $\bar{\mathrm{F}}(\Gamma)$ as the k-linear category generated by the Kronecker quiver with two vertices 0 and 1 and two edges from 0 to 1. In virtue of Beilinson's famous result [Bei78], we can therefore interpret $\bar{\mathrm{F}}(\Gamma)$ as the $\mathbb{Z}/2\mathbb{Z}$-folding of the bounded derived dg category of coherent sheaves on the projective line $\mathbb{P}^1_k$ over k.

3.2 $\mathbb{Z}$-graded

We discuss a $\mathbb{Z}$-graded variant of the topological Fukaya category which can be associated to any framed stable marked surface and provides a lift of the $\mathbb{Z}/2\mathbb{Z}$-graded category associated to the underlying oriented surface. It can be obtained via a minor modification of the constructions in § 3.1: we introduce a differential $\mathbb{Z}$-graded category $\mathrm{MF}^{\mathbb{Z}}(k[z], z^{n+1})$ of $\mathbb{Z}$-graded matrix factorizations.

Let k be a commutative ring, and let $R = k[z]$ denote the polynomial ring, considered as a $\mathbb{Z}$-graded ring with $|z| = 1$. A $\mathbb{Z}$-graded matrix factorization (X, d) of $w = z^{n+1}$ consists of:

- a pair X^0, X^1 of $\mathbb{Z}$-graded R-modules;
- a pair of homogeneous R-linear homomorphisms

  $$\begin{tikzcd} X^0 \arrow[r, shift right, "d^0"'] & X^1, \arrow[l, shift right, "d^1"'] \end{tikzcd}$$

  where $|d^0| = 0$ and $|d^1| = n + 1$;

  such that
- $d^1 \circ d^0 = w\,\mathrm{id}_{X^0}$ and $d^0 \circ d^1 = w\,\mathrm{id}_{X^1}$.

Example 3.6. For $i, j \in \mathbb{Z}$, $0 \leqslant j - i \leqslant n + 1$, we have a corresponding *scalar* matrix factorization $[i, j]$ defined as

$$\begin{tikzcd} k[z](i) \arrow[r, shift right, "z^{j-i}"'] & k[z](j). \arrow[l, shift right, "z^{i-j+n+1}"'] \end{tikzcd}$$

Given matrix factorizations X, Y, we define a $\mathbb{Z}$-graded mapping complex $\mathrm{Hom}^\bullet(X, Y)$ as follows. We extend X to a $\mathbb{Z}$-sequence

$$\cdots \xrightarrow{d} \widetilde{X}^{i-1} \xrightarrow{d} \widetilde{X}^{i} \xrightarrow{d} \widetilde{X}^{i+1} \longrightarrow \cdots$$

setting

$$\widetilde{X}^i := X^{\bar{i}}\left((n+1)\left\lfloor \frac{i}{2} \right\rfloor\right),$$

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

where $\bar{i}$ denotes the residue of i modulo 2. In particular, we have $\widetilde{X}^{i+2} = \widetilde{X}^i(n+1)$. The morphisms d in the sequence $\widetilde{X}$ are homogeneous of degree 0 and satisfy $d^2 = w$. Given matrix factorizations X, Y, we define the $\mathbb{Z}$-graded complex $\mathrm{Hom}^\bullet(X,Y)$ by setting

$$\mathrm{Hom}^j(X,Y) = \{(f_i)_{i\in\mathbb{Z}} \mid f_{i+2} = f_i(n+1)\} \subset \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(\widetilde{X}^i, \widetilde{X}^{i+j})$$

equipped with the differential given by the formula

$$d(f) = d_{\widetilde{Y}} f - (-1)^{|f|} f \circ d_{\widetilde{X}}.$$

Therefore, the collection of all $\mathbb{Z}$-graded matrix factorizations of w organizes into a differential $\mathbb{Z}$-graded k-linear category which we denote by $\mathrm{MF}^{\mathbb{Z}}(k[z], z^{n+1})$. In analogy with the $\mathbb{Z}/2\mathbb{Z}$-graded case, we further define

$$\mathrm{F}^n \subset \mathrm{MF}^{\mathbb{Z}}(k[z], z^{n+1})$$

to be the full dg subcategory spanned by the scalar matrix factorizations from Example 3.6.

THEOREM 3.7. *The association $n \mapsto \mathrm{F}^n$ extends to a coparacyclic object*

$$\mathrm{F} : \mathrm{N}(\Lambda_\infty) \to \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$$

which is unital 2*-Segal.*

Proof. See [Lur14, DK15]. □

Remark 3.8. The statement of Remark 3.3 has a $\mathbb{Z}$-graded analog: for every $n \geqslant 0$, the association $i \mapsto [0, i]$ defines a Morita equivalence of $\mathbb{Z}$-graded categories

$$A^n_{\mathbb{Z}} \longrightarrow \mathrm{F}^n,$$

where $A^n_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-graded variant of A^n.

We obtain a state sum functor

$$\rho_{\mathrm{F}} : \mathrm{N}(\mathcal{R}ib_\infty) \longrightarrow \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}), \quad \Gamma \mapsto \mathrm{F}(\Gamma).$$

The state sum $\mathrm{F}(\Gamma)$ of F on a framed graph Γ is called the ($\mathbb{Z}$-graded) *topological Fukaya category* of Γ.

Remark 3.9. It is immediate from the definitions that we have a commutative square

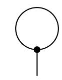
$$\begin{tikzcd} \mathrm{N}(\Lambda_\infty) \arrow[r, "\mathrm{F}"] \arrow[d] & \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \arrow[d, "P"] \\ \mathrm{N}(\Lambda) \arrow[r, "\bar{\mathrm{F}}"] & \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at^{(2)}_{\mathrm{dg}}) \end{tikzcd}$$

in $\mathcal{C}at_\infty$ where the left vertical arrow is the natural forgetful functor. Since the periodization functor P commutes with colimits it follows that, for a framed graph Γ, we have

$$P(\mathrm{F}(\Gamma)) \simeq \bar{\mathrm{F}}(\bar{\Gamma}),$$

where $\bar{\Gamma}$ denotes the ribbon graph underlying Γ. In other words, $\mathrm{F}(\Gamma)$ provides a lift of the differential $\mathbb{Z}/2\mathbb{Z}$-graded category $\bar{\mathrm{F}}(\bar{\Gamma})$ to a differential $\mathbb{Z}$-graded category.

Example 3.10. Consider the ribbon graph Γ given by

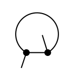


equipped with framing corresponding to the winding number r around the loop. A calculation analogous to that in Example 3.4 yields

$$\mathrm{F}(\Gamma) \simeq k[t],$$

the k-linear category with one object and endomorphism ring $k[t]$, $|t| = 2r$, considered as a differential $\mathbb{Z}$-graded category with zero differential. The endomorphism dga of the $k[t]$-module $k \cong k[t]/(t)$ is given by the graded ring $k[\delta]$, $\delta^2 = 0$, $|\delta| = 1 - 2r$.

Example 3.11. Consider the ribbon graph Γ



with framing corresponding to the winding number r. An analogous calculation to Example 3.5 gives a description of $\mathrm{F}(\Gamma)$ as the k-linear category generated by the *graded* Kronecker quiver with two vertices 0 and 1 and two edges from 0 to 1 where the edges have degree 0 and $2r$, respectively. Therefore, $\mathrm{F}(\Gamma)$ can be interpreted as a twisted version of the derived dg category of $\mathbb{P}^1_k$ (cf. [Sei04] for an appearance of this quiver in another Fukaya-categorical context). The objects of $\mathrm{Perf}_{\mathrm{F}(\Gamma)}$ given by the cones of the two edges of the quiver have endomorphism algebras given by $k[\delta]$, $\delta^2 = 0$, where δ has degree $1 - 2r$ and $1 + 2r$, respectively.

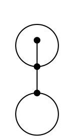
4. Localization and Mayer–Vietoris

4.1 Localization for topological Fukaya categories

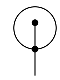
We construct localization sequences for topological Fukaya categories which are analogous to the proto-localization sequences of Thomason–Trobaugh for derived categories of schemes. A refined analysis of certain such sequences will feature in our proof of the Mayer–Vietoris theorem in § 4.2. We focus on the $\mathbb{Z}$-graded case, the $\mathbb{Z}/2\mathbb{Z}$-graded case can be treated *mutatis mutandis*.

Let Γ be a graph. A subgraph $\Gamma' \subset \Gamma$ is called *open* if, for every vertex $v \in \Gamma'$, the graph Γ' contains all half-edges incident to v in Γ. The *complement* $\Gamma'' = \Gamma \backslash \Gamma'$ of an open graph $\Gamma' \subset \Gamma$ is the subgraph of Γ with set of vertices $V \backslash V'$ and set of half-edges $H \backslash H'$. Note that the complement of an open graph is open. Given an open subgraph $\Gamma' \subset \Gamma$, we define the *closure* $\overline{\Gamma'}$ to be the graph which is obtained from Γ' by adding, for every external half-edge h of Γ' which becomes internal in Γ, a new half-edge $\tau(h)$ and vertex v which is declared incident to $\tau(h)$. We further define the *retract* $\underline{\Gamma'}$ of Γ' to be the graph obtained from Γ' by removing all half-edges which become internal in Γ.

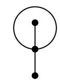
Example 4.1. Consider the graph Γ depicted by



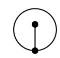
It contains the open subgraph Γ' given by



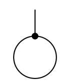
with closure $\overline{\Gamma'}$



retract $\underline{\Gamma'}$



and complement $\Gamma \backslash \Gamma'$



Note that if Γ is a framed (respectively ribbon) graph then any open subgraph inherits a canonical framing (respectively ribbon structure). We will need the following special case of a general descent statement which can be proved with the same technique.

Proposition 4.2. *Let $\mathcal{C}$ be an ∞-category with finite colimits. Let Γ be a graph, and let $\Gamma' \subset \Gamma$ be an open subgraph of Γ with complement Γ''. Assume that Γ carries a ribbon (respectively framed) structure and X is a co(para)cyclic object in $\mathcal{C}$. Then there is a canonical pushout square*

$$\begin{tikzcd} X(\Xi) \arrow[r] \arrow[d] & X(\Gamma'') \arrow[d] \\ X(\Gamma') \arrow[r] & X(\Gamma) \end{tikzcd} \tag{4.3}$$

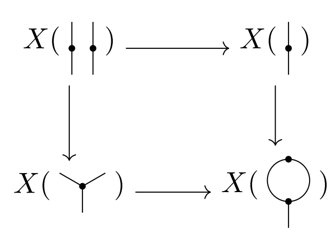
in $\mathcal{C}$, where Ξ denotes the graph given by the disjoint union of copies of the corolla

indexed by those half-edges in Γ' which become internal in Γ, equipped with the (para)cyclic order induced from the corresponding edges of Γ.

Proof. This is an immediate consequence of [Lur09, 4.2.3.8], which allows us to compute the state sum colimit $X(\Gamma)$ by covering the state sum diagram with two subdiagrams whose corresponding colimits yield $X(\Gamma'')$ and $X(\Gamma')$, respectively. □

Example 4.4. The following examples of pushout squares as given by Proposition 4.2 will be used below:

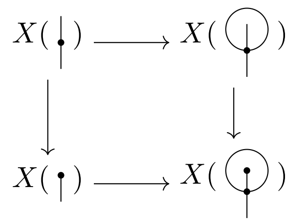
(1)



(4.5)

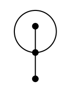
https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

(2)

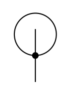


(4.6)

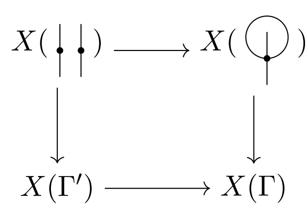
(3) Let Γ be a ribbon (framed) graph which contains the graph



as a subgraph and so that the central vertex has valency 4 in Γ. Let Γ' denote the ribbon (framed) graph obtained by removing the graph

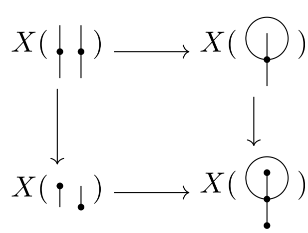


Then we have a pushout square



(4.7)

(4) The pushout square



(4.8)

is a special case of (4.7).

We use Proposition 4.2 to obtain localization sequences for topological Fukaya categories.

Proposition 4.9 (Localization). *Let Γ be a ribbon graph and $\Gamma' \subset \Gamma$ an open subgraph with complement Γ''.*

(1) *There is a canonical pushout square*

$$\begin{tikzcd} \bar{\mathrm{F}}(\Gamma') \arrow[r] \arrow[d] & \bar{\mathrm{F}}(\Gamma) \arrow[d] \\ 0 \arrow[r] & \bar{\mathrm{F}}(\underline{\Gamma''}) \end{tikzcd} \tag{4.10}$$

in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}^{(2)})$.

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

(2) *Assume that* Γ *carries the structure of a framed graph. Then there is a lift of (4.10) to a pushout diagram of* $\mathbb{Z}$*-graded Fukaya categories*

$$\begin{tikzcd} \mathrm{F}(\Gamma') \arrow[r] \arrow[d] & \mathrm{F}(\Gamma) \arrow[d] \\ 0 \arrow[r] & \mathrm{F}(\underline{\Gamma''}) \end{tikzcd}$$

in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$.

Proof. We treat the framed case; the ribbon case is analogous. By Proposition 4.2, we have a pushout square

$$\begin{tikzcd} \mathrm{F}(\Xi) \arrow[r] \arrow[d] & \mathrm{F}(\Gamma'') \arrow[d] \\ \mathrm{F}(\Gamma') \arrow[r] & \mathrm{F}(\Gamma) \end{tikzcd} \tag{4.11}$$

Another application of Proposition 4.2 yields a pushout square

$$\begin{tikzcd} \mathrm{F}(\Xi) \arrow[r] \arrow[d] & \mathrm{F}(\Gamma'') \arrow[d] \\ \mathrm{F}(\Psi) \arrow[r] & \mathrm{F}(\overline{\Gamma''}) \end{tikzcd} \tag{4.12}$$

where Ψ denotes a disjoint union of copies of the corolla

again indexed by those half-edges in Γ' which become internal in Γ. We have $\mathrm{F}(\Psi) \simeq 0$ so that, using the universal property of the pushout (4.11), we may combine (4.11) and (4.12) to obtain a canonical diagram

$$\begin{tikzcd} \mathrm{F}(\Xi) \arrow[r] \arrow[d] & \mathrm{F}(\Gamma'') \arrow[d] \\ \mathrm{F}(\Gamma') \arrow[r] \arrow[d] & \mathrm{F}(\Gamma) \arrow[d] \\ 0 \arrow[r] & \mathrm{F}(\overline{\Gamma''}) \end{tikzcd}$$

The top and exterior square are pushouts so that, by [Lur09, 4.4.2.1], the bottom square is a pushout square as well. To obtain the final statement, we note that, by the 2-Segal property of F, we have an equivalence $\mathrm{F}(\overline{\Gamma''}) \simeq \mathrm{F}(\underline{\Gamma''})$. □

Remark 4.13. Note that the argument of Proposition 4.9 generalizes to any co(para)cyclic 2-Segal object X with values in a *pointed* ∞-category satisfying $X^0 \simeq 0$.

Remark 4.14. Define $\mathrm{F}(\Gamma \text{ on } \Gamma')$ to be the full dg subcategory of $\mathrm{F}(\Gamma)$ spanned by the image of $\mathrm{F}(\Gamma')$. We call $\mathrm{F}(\Gamma \text{ on } \Gamma')$ the *topological Fukaya category of* Γ *with support in* Γ'. Then, in the terminology of § 2.2, we have an exact sequence

$$\begin{tikzcd} \mathrm{F}(\Gamma \text{ on } \Gamma') \arrow[r] \arrow[d] & \mathrm{F}(\Gamma) \arrow[d] \\ 0 \arrow[r] & \mathrm{F}(\underline{\Gamma''}) \end{tikzcd}$$

which should be regarded as an analog of Thomason–Trobaugh's proto-localization sequence for derived categories of perfect complexes in the context of algebraic K-theory of schemes. It is an interesting question whether there is a general dévissage statement saying that $H(\Gamma') \to H(\mathrm{F}(\Gamma \text{ on } \Gamma'))$ is an equivalence for a suitable class of additive invariants H. Lemma 4.21 can be regarded as the simplest instance of such a result.

4.2 A Mayer–Vietoris theorem

In this section, we use localization sequences for topological Fukaya categories to prove a Mayer–Vietoris theorem. The argument is inspired by similar techniques in the context of algebraic K-theory for schemes [TT90]. We focus on the $\mathbb{Z}$-graded version but all results have obvious $\mathbb{Z}/2\mathbb{Z}$-graded analogs that admit similar proofs.

Definition 4.15. Let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a functor with values in a stable ∞-category $\mathcal{C}$.

(1) We say H is *localizing* if it preserves finite sums and exact sequences.
(2) The functor H is called $\mathbb{A}^1$-*homotopy invariant* if, for every dg category A, the morphism

$$A \longrightarrow A[t]$$

is an H-equivalence (i.e. is mapped to an equivalence under H). Here, the morphism $A \to A[t]$ is defined as the tensor product of A with the morphism $k \to k[t]$ of k-algebras, interpreted as a dg functor of dg categories with one object.

Let $\mathcal{C}$ be an ∞-category with finite colimits and let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a functor, we have a canonical morphism

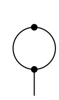
$$(HF)(\Gamma) \longrightarrow H(F(\Gamma)) \tag{4.16}$$

in $\mathcal{C}$ which is obtained by applying H to the colimit cone over the diagram $F \circ \delta$.

Definition 4.17. We say a framed graph Γ *satisfies Mayer–Vietoris with respect to H* if the morphism (4.16) is an equivalence. In other words, a framed graph Γ satisfies Mayer–Vietoris with respect to H, if H commutes with the state sum colimit parametrized by the incidence category of Γ.

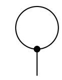
Theorem 4.18 (Mayer–Vietoris). *Let $\mathcal{C}$ be a stable ∞-category and let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a localizing $\mathbb{A}^1$-homotopy invariant functor. Then every framed graph satisfies Mayer–Vietoris with respect to H.*

Proof. The proof will use the results from § 4.3 below. Note that, by definition, every framed corolla satisfies Mayer–Vietoris. By Lemma 4.27, with respect to (4.5), and Lemma 4.25(2), we deduce that the graph

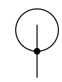


https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

provided with any framing, satisfies Mayer–Vietoris. By Proposition 1.23, we may contract an internal edge to obtain that

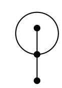


satisfies Mayer–Vietoris. Similarly, by Lemma 4.27, with respect to (4.6), and Lemma 4.25(2), we obtain that the graph



with any framing, satisfies Mayer–Vietoris. Here, we again use Proposition 1.23 to contract the internal edge of the graph in the bottom right corner of (4.6).

Let Γ be any framed graph. Since H commutes with finite sums, we may assume that Γ is connected. By Proposition 1.23, Γ satisfies Mayer–Vietoris if and only if the graph Γ' with one vertex obtained by collapsing a maximal forest in Γ satisfies Mayer–Vietoris. Therefore, we may assume that Γ has one vertex v. We now proceed inductively on the number of loops in Γ. If Γ does not have any loops then Γ is a corolla and satisfies Mayer–Vietoris. Assume Γ has a loop l. We isolate the loop l by blowing up two edges so that we obtain a graph $\widetilde{\Gamma}$ with three vertices that contains the subgraph



with loop l, satisfies the conditions of Example 4.4(3), and Γ is obtained from $\widetilde{\Gamma}$ by contracting the two internal edges. We now apply the induction hypothesis, Lemma 4.27 with respect to (4.7), and Lemma 4.25(1) to deduce that $\widetilde{\Gamma}$ and hence, by Proposition 1.23, Γ satisfy Mayer–Vietoris, concluding the argument. $\square$

Remark 4.19. Let k be a field of characteristic 0. An example of a localizing functor for which Mayer–Vietoris fails (since $\mathbb{A}^1$-homotopy invariance does not hold) is given by Hochschild homology. Applying $HH_* \circ j_! F$ to the square (4.5), we obtain

$$\begin{tikzcd} HH_*(k \amalg k) \arrow[r] \arrow[d, "f"] & HH_*(k) \arrow[d, "f'"] \\ HH_*(A^1) \arrow[r] & HH_*(k[t]) \end{tikzcd}$$

where the left vertical morphism becomes an equivalence by Proposition 4.20. However, the right vertical morphism is *not* an equivalence: by the Hochschild–Kostant–Rosenberg theorem, we have $HH_1(k[t]) \cong k[t]$ while $HH_1(k) \cong 0$. In this situation, replacing Hochschild homology by periodic cyclic homology leads to an $\mathbb{A}^1$-homotopy invariant functor that therefore satisfies Mayer–Vietoris.

4.3 Lemmas

We collect technical results for the proof of the Mayer–Vietoris theorem.

PROPOSITION 4.20. *Let $\mathcal{C}$ be a stable ∞-category and let $H : \mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a localizing functor. Then the coparacyclic object*

$$HF : \mathrm{N}(\Lambda_\infty) \longrightarrow \mathcal{C}$$

is 1-Segal.

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Proof. We have to show that, for every $n \geqslant 1$, the natural map

$$H(\mathrm{F}^{\{0,1\}}) \amalg H(\mathrm{F}^{\{1,2\}}) \amalg \cdots \amalg H(\mathrm{F}^{\{n-1,n\}}) \to H(\mathrm{F}^{\{0,1,\ldots,n\}})$$

is an equivalence in $\mathcal{C}$. We have a diagram in $\mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}})$

$$\begin{tikzcd}
\mathrm{F}^{\{0,1,\ldots,n-1\}} \arrow[r] \arrow[d] & \mathrm{F}^{\{0,1,\ldots,n-1\}} \amalg \mathrm{F}^{\{n-1,n\}} \arrow[r] \arrow[d, "f"] & \mathrm{F}^{\{n-1,n\}} \arrow[d] \\
\mathrm{F}^{\{0,1,\ldots,n-1\}} \arrow[r] & \mathrm{F}^{\{0,1,\ldots,n\}} \arrow[r] & \mathrm{F}^{\{n-1,n\}}
\end{tikzcd}$$

with exact rows. After applying H, the rows stay exact, and $H(f)$ must be an equivalence, since its cofiber is 0. We now proceed by induction on n. $\square$

Lemma 4.21. *Let $k[\delta]$ denote the differential $\mathbb{Z}$-graded k-algebra generated by δ in some degree with relation $\delta^2 = 0$ and zero differential. Let A be a dg category and consider a pushout diagram*

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[\delta] \arrow[d] \\
A \arrow[r, "i"] & A^{\delta}
\end{tikzcd} \tag{4.22}$$

so that A^{δ} is obtained from A by adjoining the endomorphism δ to a fixed object in A. Then, for any localizing $\mathbb{A}^1$-homotopy invariant $H : \mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$, the morphism $H(i)$ is an equivalence.

Proof. We may extend (4.22) to the diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[\delta] \arrow[r, "\delta\mapsto 0"] \arrow[d] & k \arrow[d] \\
A \arrow[r, "i"] & A^{\delta} \arrow[r, "p"] & A
\end{tikzcd} \tag{4.23}$$

in which both squares are pushout squares so that $pi \simeq \mathrm{id}_A$. We will construct a commutative diagram

$$\begin{tikzcd}
 & A^{\delta} \\
A^{\delta} \arrow[ur, "\mathrm{id}"] \arrow[r, "h"] \arrow[dr, "i\circ p"'] & A^{\delta}[t] \arrow[u, "\mathrm{ev}_1"'] \arrow[d, "\mathrm{ev}_0"] \\
 & A^{\delta}
\end{tikzcd} \tag{4.24}$$

in $\mathrm{L_{mo}}(\mathcal{C}at_{\mathrm{dg}})$, where ev_0 and ev_1 are H-equivalences. This will imply that $i \circ p$ is an H-equivalence so that the diagram exhibits $p : A^{\delta} \to A$ as an $\mathbb{A}^1$-homotopy inverse of $i : A \to A^{\delta}$. In particular, we have that i is an H-equivalence.

To construct (4.24), consider the morphism $\lambda : k[\delta] \to k[\delta, t]$ determined by $\lambda(\delta) = \delta t$. Further, let $g : k[\delta, t] \to A^{\delta}[t]$ be the morphism obtained by tensoring $k[\delta] \to A^{\delta}$ with $k[t]$. Further, let $f : A \to A^{\delta}[t]$ be the composite of the bottom horizontal morphisms in the diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[t] \arrow[r] \arrow[d] & k[\delta, t] \arrow[d] \\
A \arrow[r] & A[t] \arrow[r] & A^{\delta}[t]
\end{tikzcd}$$

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

where all morphisms are the apparent ones. By construction, the above morphisms fit into a cone diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k[\delta] \arrow[d] \arrow[ddr, bend left, "g\lambda"] & \\ A \arrow[r, "i"] \arrow[drr, bend right, "f"'] & A^\delta \arrow[dr, dashed, "h"] & \\ & & A^\delta[t] \end{tikzcd}$$

that determines the dashed morphism h (up to contractible choice). It is now immediate to verify that the morphism h fits into a diagram of the form (4.24), where ev_0 and ev_1 are the morphisms obtained by applying $A^\delta \otimes -$ to the morphisms $k[t] \to k, t \mapsto 0$, and $k[t] \to k, t \mapsto 1$, respectively. $\square$

LEMMA 4.25. *Let $\mathcal{C}$ be a stable ∞-category, and let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a localizing functor. Suppose*

$$\begin{tikzcd} \Lambda_\infty(\Gamma_1) \arrow[r] \arrow[d] & \Lambda_\infty(\Gamma_2) \arrow[d] \\ \Lambda_\infty(\Gamma_3) \arrow[r] & \Lambda_\infty(\Gamma_4) \end{tikzcd} \tag{4.26}$$

is a pushout diagram in $\mathcal{P}(\Lambda_\infty)$ that stays a pushout diagram after application of $H \circ j_! \mathrm{F}$.

(1) *Assume that Γ_1, Γ_2, and Γ_3 satisfy Mayer–Vietoris. Then Γ_4 satisfies Mayer–Vietoris.*

(2) *Suppose $F(\Gamma_3) \simeq 0$ and assume that any two among the ribbon graphs Γ_1, Γ_2, Γ_4 satisfy Mayer–Vietoris. Then all ribbon graphs in (4.26) satisfy Mayer–Vietoris.*

Proof. Since $H \circ j_! \mathrm{F}$ is a left extension of $H\,\mathrm{F} : \mathrm{N}(\Lambda_\infty) \to \mathcal{C}$ along $j : \Lambda_\infty \to \mathcal{P}(\Lambda_\infty)$, we have a canonical morphism $\xi : j_!(H\,\mathrm{F}) \to H \circ j_! \mathrm{F}$ in $\mathrm{Fun}(\mathcal{P}(\Lambda_\infty), \mathcal{C})$ which, evaluated at an object of the form $\Lambda_\infty(\Gamma)$, yields the canonical morphism

$$(H\,\mathrm{F})(\Gamma) \to H(\mathrm{F}(\Gamma))$$

in $\mathcal{C}$ of (4.16). We give the argument for (1). Applying ξ to the square (4.26), we obtain a morphism of pushout squares in $\mathcal{C}$ that is an equivalence on all vertices except for the bottom right vertex

$$(H\,\mathrm{F})(\Gamma_4) \to H(\mathrm{F}(\Gamma_4)).$$

But pushouts of equivalences are equivalences, so that this morphism must be an equivalence as well. The statement of (2) follows similarly. $\square$

LEMMA 4.27. *Let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a localizing $\mathbb{A}^1$-homotopy invariant. The pushout diagrams (4.5), (4.6), and (4.7) (cf. Proposition 4.4), with $X = j : \mathrm{N}(\Lambda_\infty) \to \mathcal{P}(\Lambda_\infty)$, stay pushout diagrams after application of $H \circ j_! \mathrm{F}$.*

Proof. (1) Applying $j_! \mathrm{F}$ to (4.5), with $X = j$, we obtain, by Example 3.4, the pushout diagram

$$\begin{tikzcd} \mathrm{F}^1 \amalg \mathrm{F}^1 \arrow[r] \arrow[d, "f"] & \mathrm{F}^1 \arrow[d, "f'"] \\ \mathrm{F}^2 \arrow[r] & \mathrm{F}^1[t] \end{tikzcd}$$

where f is a 1-Segal map so that, by Proposition 4.20, $H(f)$ is an equivalence. The morphism $H(f')$ is an equivalence, by the argument of [Tab12, Lemma 4.1], since H is $\mathbb{A}^1$-homotopy invariant. Therefore, the square remains a pushout square upon applying H. The statement for (4.6) follows from an argument similar to (2) below, but easier.

(2) Applying $j_! \mathrm{F}$ to (4.7), with $X = j$, we obtain the pushout square

$$\begin{tikzcd} \mathrm{F}^1 \amalg \mathrm{F}^1 \arrow[r] \arrow[d] & \mathrm{F}(\ \ ) \arrow[d] \\ \mathrm{F}(\Gamma') \arrow[r] & \mathrm{F}(\Gamma) \end{tikzcd}$$

which is the top rectangle of the diagram

$$\begin{tikzcd} \mathrm{F}^1 \amalg \mathrm{F}^1 \arrow[r, "f"] \arrow[d] & (\mathrm{F}^1)^{\delta_1} \amalg (\mathrm{F}^1)^{\delta_2} \arrow[r, hook, "g"] \arrow[d] & \mathrm{F}(\ \ ) \arrow[d] \\ \mathrm{F}(\Gamma') \arrow[r, "f'"] \arrow[d] & ((\mathrm{F}(\Gamma'))^{\delta_1})^{\delta_2} \arrow[r, hook, "g'"] \arrow[d] & \mathrm{F}(\Gamma) \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \mathrm{F}(\ \ ) \end{tikzcd}$$

in which all squares are pushout squares, the exterior square is given by (4.8) with $X = \mathrm{F}$. We use the notation from Lemma 4.21 with $|\delta_1| = 1-2r$, $|\delta_2| = 1+2r$, where r is the winding number of the loop in the top right framed graph. The morphisms $H(f)$ and $H(f')$ are equivalences by Lemma 4.21. The morphism g is quasi-fully faithful by explicit verification (cf. Example 3.11), and hence g' is quasi-fully faithful by Lemma 2.4. Therefore, the bottom right square and the right rectangle are exact sequences in $\mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}})$ and thus stay exact after applying H. But this implies that the top right square stays a pushout square after applying H. Hence the top rectangle stays a pushout diagram after applying H as claimed. □

5. Delooping of paracyclic 1-Segal objects

The methods used in this section are inspired by similar techniques used in [Lur09, ch. 6], specifically § 6.1, which develops an ∞-categorical theory of effective groupoid objects. Let $\mathcal{C}$ be a stable ∞-category and let $H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \to \mathcal{C}$ be a localizing $\mathbb{A}^1$-homotopy invariant functor. Given a framed graph Γ, Theorem 4.18 reduces the calculation of $H(F(\Gamma))$ to a state sum of the coparacyclic object

$$HF : \mathrm{N}(\Lambda_\infty) \longrightarrow \mathcal{C},$$

which, by Proposition 4.20, is 1-Segal. We establish a general result in order to evaluate this state sum explicitly.

THEOREM 5.1. *Let $\mathcal{C}$ be a stable ∞-category with colimits and let $X : \mathrm{N}(\Lambda_\infty) \to \mathcal{C}$ be a coparacyclic 1-Segal object with $X^0 \simeq 0$. Let Γ be a framed graph modelling a stable framed marked surface (S, M). Then we have an equivalence in $\mathcal{C}$*

$$X(\Gamma) \simeq \Omega(X^1)(S/M),$$

where the right-hand side is defined using (1.31), and $\Omega = [-1]$ denotes the loop functor on $\mathcal{C}$.

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

The results of § 1.4 reduce the proof of Theorem 5.1 to the following statement.

PROPOSITION 5.2 (Delooping). *Let $\mathcal{C}$ be a stable ∞-category. Let $\mathcal{C}$ be a stable ∞-category with colimits. Let $X : \mathrm{N}(\Lambda_\infty) \to \mathcal{C}$ be a coparacyclic 1-Segal object with $X^0 \simeq 0$. Then there is an equivalence*

$$X \simeq L_{\Omega(X^1)},$$

where the right-hand side denotes the coparacyclic object of (1.33).

Remark 5.3. Let M be a monoid object in an abelian category $\mathcal{A}$ (equipped with the Cartesian monoidal structure so that the tensor product is given by the direct sum). Then it is easy to see that the monoid structure must be given by the unit $0 \longrightarrow M$ and multiplication $M \otimes M \to M$ given by the sum. In particular, M is automatically a group object with inverse given by $-\mathrm{id} : M \to M$. Proposition 5.2 is an ∞-categorical analog of this statement, where the abelian category $\mathcal{A}$ gets replaced by the stable ∞-category $\mathcal{C}^{\mathrm{op}}$. In this context, the statement becomes even nicer since we can describe the group object in terms of the delooping ΩX^1. We also note that the statement of Proposition 5.2 becomes false if we replace the paracyclic category by the cyclic category: there may be various inequivalent cocyclic structures on a cosimplicial 1-Segal object.

We establish some preparatory results for the proof of Proposition 5.2.

PROPOSITION 5.4. *Let $\mathcal{C}$ be a stable ∞-category and let $X : \mathrm{N}(\Delta) \to \mathcal{C}$ be a cosimplicial 1-Segal object. Then $\lim_{\mathrm{N}(\Delta)} X \simeq \Omega X_1$.*

Proof. Let $\mathcal{I}$ be the full subcategory of Δ spanned by the objects [0] and [1]. Let $\mathcal{I}_+$ denote the category obtained from $\mathcal{I}$ by adjoining an initial object. We have a diagram of right Kan extension functors

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{N}(\mathcal{I}), \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathrm{N}(\mathcal{I}_+), \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(\mathrm{N}(\Delta), \mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathrm{N}(\Delta_+), \mathcal{C}) \end{tikzcd} \tag{5.5}$$

which commutes up to equivalence. We show that X is a right Kan extension of $X_{|\mathrm{N}(\mathcal{I})}$. The statement of the proposition then follows immediately from (5.5).

Since $X^0 \simeq 0$, the cosimplicial object X is a right Kan extension of $X_{|\mathrm{N}(\mathcal{I})}$ if and only if, for every $n \geqslant 2$, the maps $X_n \to X_1$ induced by the n surjective maps $p_i : [n] \mapsto [1]$ exhibit X^n as a product

$$X^n \simeq X^1 \times X^1 \times \cdots \times X^1, \tag{5.6}$$

with n factors. This can be verified in the homotopy category $\mathrm{h}\mathcal{C}$ which is additive. Since X is 1-Segal, the maps $[1] \to [n]$ given by the inclusions $q_j : \{j, j+1\} \subset [n]$ exhibit X^n as a coproduct of n copies of X^1. The morphisms $X^1 \to X^1$ induced by $p_i \circ q_j$ are homotopic to the identity on X_1 if $i = j$ and zero if $i \neq j$. This implies the equivalence (5.6). □

Let $i : \Delta \to \Lambda_\infty$ denote the natural inclusion functor.

LEMMA 5.7. *The opposite inclusion functor*

$$\mathrm{N}(i)^{\mathrm{op}} : \mathrm{N}(\Delta)^{\mathrm{op}} \longrightarrow \mathrm{N}(\Lambda_\infty)^{\mathrm{op}}$$

is cofinal.

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

Proof. By [Lur09, 4.1.3.1], it suffices to show that, for every object $\langle m\rangle$ of Λ_∞, the nerve of the category $\Delta/\langle m\rangle$ has a contractible geometric realization. The nerve of the category $\Delta/\langle m\rangle$ can be identified with the simplicial set

$$\mathrm{Hom}_{\Lambda_\infty}(-,\langle m\rangle)_{|\Delta^{\mathrm{op}}}$$

whose geometric realization is homeomorphic to $|\Delta^m| \times \mathbb{R}$ (cf. [FL91]). □

COROLLARY 5.8. *Let $\mathcal{C}$ be a stable ∞-category and let $X : \mathrm{N}(\Lambda_\infty) \to \mathcal{C}$ be a coparacyclic 1-Segal object. Then $\lim_{\mathrm{N}(\Lambda_\infty)} X \simeq \Omega X_1$.*

Proof. Immediate from Proposition 5.4 and Lemma 5.7. □

We denote by Λ_∞^+ the *augmented paracyclic category* obtained from Λ_∞ by adjoining an initial object $\emptyset$. Consider the full subcategory $j : \mathrm{N}(\mathcal{J}) \subset \mathrm{N}(\Lambda_\infty^+)$ spanned by the objects $\emptyset$ and $\langle 0\rangle$ and the map $p : \mathrm{N}(\mathcal{J}) \to \Delta^1$ which maps the unique edge $\emptyset \to \langle 0\rangle$ to $\{0, 1\}$.

PROPOSITION 5.9. *Let $\mathcal{C}$ be a stable ∞-category and let $X : \mathrm{N}(\Lambda_\infty) \to \mathcal{C}$ be a coparacyclic 1-Segal object with $X^0 \simeq 0$. Let X^+ denote the limit cone of X, i.e. the right Kan extension of X along $\mathrm{N}(\Lambda_\infty) \to \mathrm{N}(\Lambda_\infty^+)$. Then there is an equivalence*

$$X \simeq j_!(p^*(\Omega(X^1) \longrightarrow 0))$$

of coparacyclic objects.

Proof. By Proposition 5.8, we have

$$X^+(\emptyset) \simeq \Omega(X^1)$$

and by assumption $X^+(\langle 0\rangle) \simeq 0$ so that we may exhibit X^+ is a left extension of $p^*(\Omega(X^1) \longrightarrow 0)$ along $\mathrm{N}(\mathcal{J}) \to \mathrm{N}(\Lambda_\infty)$. We use the pointwise formula to show that it is a left Kan extension. For the object $\langle 1\rangle$, this amounts to the statement that the square

$$\begin{tikzcd} \Omega(X^1) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X^1 \end{tikzcd}$$

is a pushout square, which holds since $\mathcal{C}$ is stable. For the object $\langle n\rangle$, the pointwise formula reduces to the requirement that the n coface maps $X^1 \to X^n$ exhibit X^n as an n-fold coproduct of copies of X^1. This holds since X is assumed 1-Segal. □

Proof of Proposition 5.2. Let $\pi : \Lambda_\infty \to \Lambda$ be the covering functor. Consider $L^\bullet$ from (1.27) as a coparacyclic 1-Segal pointed space via pullback along π (cf. Remark 1.35). Given an object $E \in \mathcal{C}$, the functor $E(-)$ from (1.31) commutes with finite colimits so that the coparacyclic object $E(L^\bullet)$ in $\mathcal{C}$ is 1-Segal. We have $E(L^1) \simeq E(S^1) \simeq E[1]$ so that, by Proposition 5.9, we have

$$L_E \simeq E(L^\bullet) \simeq j_!(p^*(E \longrightarrow 0)).$$

Combining this with another application of Proposition 5.9 to the given coparacyclic 1-Segal object X yields the desired equivalence

$$X \simeq j_!(p^*(\Omega(X^1) \longrightarrow 0)) \simeq L_{\Omega(X^1)}.$$ □

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press

6. Main result and examples

Combining Theorems 4.18 and 5.1, we obtain the following main result of this work.

THEOREM 6.1. *Let $\mathcal{C}$ be a stable ∞-category with colimits and let*

$$H : \mathrm{L}_{\mathrm{mo}}(\mathcal{C}at_{\mathrm{dg}}) \longrightarrow \mathcal{C}$$

be a localizing $\mathbb{A}^1$-homotopy invariant. Let Γ be a framed graph which models a stable framed marked surface (S, M) and let $\mathrm{F}(S, M)$ denote its $\mathbb{Z}$-graded topological Fukaya category. Then there is an equivalence

$$H(\mathrm{F}(S, M)) \simeq \Sigma^{\infty}(S/M) \otimes H(k)[-1].$$

As a specific application, we obtain the following corollary.

COROLLARY 6.2. *Let k be a field of characteristic 0. We have the formulas*

$$\mathrm{HP}_0(\mathrm{F}(S, M)) \cong H_1(S, M; k),$$
$$\mathrm{HP}_1(\mathrm{F}(S, M)) \cong H_2(S, M; k)$$

for periodic cyclic homology over k.

Example 6.3. We compute some explicit examples.

(1) Let $(S, M) = (S^2, \{0, \infty\})$ be a 2-sphere with two marked points. Then, equipping the punctured sphere with the standard framing with winding number 0, we have $\mathrm{F}(S, M) \simeq \mathrm{Perf}(\mathbb{A}^1 \backslash \{0\})$. In agreement with Hochschild–Kostant–Rosenberg, we obtain

$$\mathrm{HP}_0(\mathrm{F}(S, M)) \cong H_1(S, M; k) \cong k,$$
$$\mathrm{HP}_1(\mathrm{F}(S, M)) \cong H_2(S, M; k) \cong k.$$

(2) Let $(S, M) = (T, \{0\})$ be a once marked torus equipped with standard framing. We have a Morita equivalence

$$\mathrm{F}(S, M) \simeq \mathcal{D}^b(\mathrm{coh}(C)),$$

where C denotes a nodal plane cubic. We obtain

$$\mathrm{HP}_0(\mathrm{F}(S, M)) \cong H_1(S, M; k) \cong k^2,$$
$$\mathrm{HP}_1(\mathrm{F}(S, M)) \cong H_2(S, M; k) \cong k.$$

ACKNOWLEDGEMENTS

I would like to thank Chris Brav, Mikhail Kapranov, Jacob Lurie, Pranav Pandit, Paul Seidel, and Nicolo Sibilla for inspiring conversations on the subject of this paper. Further, I am very grateful to an anonymous referee for carefully reading this work and providing useful suggestions.

REFERENCES

Bei78 A. Beilinson, *Coherent sheaves on Pn and problems of linear algebra*, Funct. Anal. Appl. **12** (1978), 214–216.

Bla16 A. Blanc, *Topological K-theory of complex noncommutative spaces*, Compositio Math. **152** (2016), 489–555.

Boc11 R. Bocklandt, *Noncommutative mirror symmetry for punctured surfaces*, Preprint (2011), arXiv:1111.3392.

DK12 T. Dyckerhoff and M. Kapranov, *Higher Segal spaces I*, Preprint (2012), arXiv:1212.3563.

DK13 T. Dyckerhoff and M. Kapranov, *Triangulated surfaces in triangulated categories*, e-Print (2013), arXiv:1306.2545.

DK15 T. Dyckerhoff and M. Kapranov, *Crossed simplicial groups and structured surfaces*, in *Stacks and categories in geometry, topology, and algebra*, Contemporary Mathematics, vol. 643 (American Mathematical Society, Providence, RI, 2015), 37–110, doi:10.1090/conm/643/12896.

FL91 Z. Fiedorowicz and J.-L. Loday, *Crossed simplicial groups and their associated homology*, Trans. Amer. Math. Soc. **326** (1991), 57–87.

HKK14 F. Haiden, L. Katzarkov and M. Kontsevich, *Stability in Fukaya categories of surfaces*, Preprint (2014), arXiv:1409.8611.

Kel98 B. Keller, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra **123** (1998), 223–273.

Kon09 M. Kontsevich, *Symplectic geometry of homological algebra*, available at the author's webpage (2009), http://www.ihes.fr/∼maxim/TEXTS/Symplectic_AT2009.pdf.

Lur09 J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170 (Princeton University Press, Princeton, NJ, 2009).

Lur11 J. Lurie, *Higher algebra*, Preprint (2011), http://www.math.harvard.edu/∼lurie/papers/HA.pdf.

Lur14 J. Lurie, *Rotation invariance in algebraic K-theory*, Preprint (2014), http://www.math.harvard.edu/∼lurie/papers/Waldhaus.pdf.

Nad13 D. Nadler, *Arboreal singularities*, Preprint (2013), arXiv:1309.4122.

Nad14 D. Nadler, *Fukaya categories as categorical morse homology*, SIGMA Symmetry Integrability Geom. Methods Appl. **10** (2014), 18–47.

Nad15 D. Nadler, *Cyclic symmetries of A_{n}-quiver representations*, Adv. Math. **269** (2015), 346–363.

PS16 J. Pascaleff and N. Sibilla, *Topological Fukaya category and mirror symmetry for punctured surfaces*, Preprint (2016), arXiv:1604.06448.

Rez01 C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), 973–1007 (electronic).

Seg74 G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

Sei04 P. Seidel, *Exact Lagrangian submanifolds in T^*S^n and the graded Kronecker quiver*, in *Different faces of geometry*, International Mathematical Series (New York), vol. 3 (Kluwer/Plenum, New York, 2004), 349–364; MR 2103000 (2005h:53153).

STZ14 N. Sibilla, D. Treumann and E. Zaslow, *Ribbon graphs and mirror symmetry*, Selecta Math. (N.S.) **20** (2014), 979–1002.

Tab05 G. Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris **340** (2005), 15–19.

Tab12 G. Tabuada, *The fundamental theorem via derived Morita invariance, localization, and $\mathbb{A}^1$-homotopy invariance*, J. K-Theory **9** (2012), 407–420, doi:10.1017/is011004009jkt155.

Toë07 B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. **167** (2007), 615–667.

TT90 R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, in *The Grothendieck Festschrift, Vol. III*, Progress in Mathematics, vol. 88 (Birkhäuser Boston, Boston, MA, 1990), 247–435; MR 1106918 (92f:19001).

TV07 B. Toën and M. Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. Éc. Norm. Supér. (4) **40** (2007), 387–444, doi:10.1016/j.ansens.2007.05.001.

TVdB15 G. Tabuada and M. Van den Bergh, *The Gysin triangle via localization and A1-homotopy invariance*, Preprint (2015), arXiv:1510.04677.

Tobias Dyckerhoff dyckerho@math.uni-bonn.de
Hausdorff Center for Mathematics, Endenicher Allee 62, 53115 Bonn, Germany

https://doi.org/10.1112/S0010437X17007205 Published online by Cambridge University Press