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Correction of Proofs in "Purely Infinite Simple C^*-algebras Arising from Free Product Constructions" and a Subsequent Paper

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Abstract. The proofs of Theorem 2.2 of K. J. Dykema and M. Rørdam, *Purely infinite simple C^*-algebras arising from free product constructions*, Canad. J. Math. **50** (1998), 323–341 and of Theorem 3.1 of K J. Dykema, *Purely infinite simple C^*-algebras arising from free product constructions, II*, Math. Scand. **90** (2002), 73–86 are corrected.

1 Introduction

There was an error in the statement and application of [2, Theorem 2.1(i)]. The word "outer" that appears there should be "multiplier outer," *i.e.,* outer relative to the multiplier algebra of A, instead of relative to the unitization of A. (The term "outer" alone may be ambiguous in the nonunital setting, and this error arose from confusion about this.)

This led to deficiencies in the proofs of [2, Theorem 2.2] and of [1, Theorem 3.1]. In this note, we correct these deficiencies. In particular, [2, Lemma 2.3] showed that a certain automorphism of a nonunital C^*-algebra $\overline{A}$ is outer relative to the unitization of the C^*-algebra, and here we show it is multiplier outer. The proof of [1, Theorem 3.1] shows that a certain automorphism of a nonunital C^*-algebra $\overline{\mathfrak{A}_{(-\infty,\infty)}}$ is outer relative to the unitization of the C^*-algebra, and here we show that it is multiplier outer.

2 Outerness of Automorphisms

We will need the following lemma.

Lemma 2.1 *Let $(\mathfrak{A}, \phi) = (A, \phi_A) * (B, \phi_B)$, where ϕ_A and ϕ_B are faithful states on A and B respectively. Let $b \in B$ such that $\|b\| = 1$ and $\|\widehat{b}\|_2 \leq \epsilon < 1/6$. Then for all $a \in A$, we have $\|a - b\| > 1/3$.*

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Proof Assume that $\eta := \|a-b\| \le 1/3$. Then $|\phi(a)-\phi(b)| \le \eta$, and so

$$|\phi(a)| \le \eta + |\phi(b)| \le \eta + \|\widehat{b}\|_2 \le \eta + \epsilon.$$

Since $\pi_A\colon A \to B(H_A)$ is faithful, there is $d \in A$ such that $\|\widehat{d}\|_2 = 1$ and $\|\widehat{ad}\|_2 > \|a\| - \epsilon$. We compute

$$\begin{aligned}(a-b)\widehat{d} &= \widehat{ad} - \phi_B(b)\widehat{d} - (b - \phi_B(b)1_B)\widehat{d} \\ &= \big(\widehat{ad} - \phi_B(b)\widehat{d}\big) + \big(-\phi_A(d)(\widehat{b} - \phi_B(b)\xi)\big) \\ &\quad + \big(-(\widehat{b} - \phi_B(b)\xi) \otimes (\widehat{d} - \phi_A(d)\xi)\big) \\ &\in H_A \oplus H_B^o \oplus (H_B^o \otimes H_A^o).\end{aligned}$$

Since H_A, H_B^o and $H_B^o \otimes H_A^o$ are pairwise orthogonal, we get

$$\|(a-b)\widehat{d}\|_2 \ge \|\widehat{ad} - \phi_B(b)\widehat{d}\|_2,$$

and thus

$$\|(a-b)\widehat{d}\|_2 \ge \|\widehat{ad}\|_2 - |\phi_B(b)| \ge \|\widehat{ad}\|_2 - \|\widehat{b}\|_2 > \|a\| - 2\epsilon.$$

Hence $\|a-b\| > \|a\| - 2\epsilon$. Since $\|a\| \ge \|b\| - \|a-b\| = 1-\eta$, we get

$$\|a-b\| > \|a\| - 2\epsilon \ge 1 - \eta - 2\epsilon \ge 1 - 1/3 - 1/3 = 1/3,$$

obtaining that $\|a-b\| > 1/3$, a contradiction. ■

As in [2], we will consider the following situation. Let A be a unital C^*-algebra and let $\sigma\colon A \to A$ be an injective endomorphism. Let $\overline{A}$ be the inductive limit of the sequence

$$A \xrightarrow{\sigma} A \xrightarrow{\sigma} A \xrightarrow{\sigma} \cdots,$$

and let $\mu_n\colon A \to \overline{A}$ be the corresponding $*$-homomorphisms that satisfy $\mu_{n+1} \circ \sigma = \mu_n$ and $\overline{A} = \overline{\cup_{n=1}^{\infty} \mu_n(A)}$. Let α be the automorphism on $\overline{A}$ defined by $\alpha(\mu_n(a)) = \mu_n(\sigma(a))$. Recall that a *corner endomorphism* of A is an injective endomorphism $\sigma\colon A \to A$ such that $\sigma(A) = \sigma(1)A\sigma(1)$.

Lemma 2.2 *Let m be a positive integer, and let σ be a corner endomorphism of a unital C^*-algebra A. With the above notation, we have that if α^m is multiplier inner, then there is an isometry s in A such that $\sigma^m(a) = sas^*$ for all $a \in A$.*

Proof Set $f_n = \mu_n(1)$ and $q = \sigma(1)$. Then $\mu_1(A) = f_1\overline{A}f_1$, by the fact that $\sigma(A) = qAq$. This again entails that $f_1\mathcal{M}(\overline{A})f_1 = \mu_1(A)$, where $\mathcal{M}(\overline{A})$ is the multiplier algebra of $\overline{A}$.

Assume that α^m is multiplier inner, and let $u \in \mathcal{M}(\overline{A})$ be a unitary such that $uxu^* = \alpha^m(x)$ for all $x \in \overline{A}$. Put $v = uf_1$. Then

$$v^*v = f_1, \qquad vv^* = uf_1u^* = \alpha^m(f_1) = \alpha^m\big(\mu_1(1)\big) = \mu_1\big(\sigma^m(1)\big) \le \mu_1(1) = f_1.$$

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This shows that v belongs to $f_1\mathcal{M}(\overline{A})f_1 = \mu_1(A)$, and so there exists s in A (necessarily an isometry) such that $v = \mu_1(s)$. Since

$$\mu_1(sxs^*) = v\mu_1(x)v^* = u\mu_1(x)u^* = \alpha^m(\mu_1(x)) = \mu_1(\sigma^m(x)),$$

we see that $sxs^* = \sigma^m(x)$ for all $x \in A$. This shows that σ^m is implemented by an isometry of A. ■

With this lemma at hand, one can prove that the endomorphism σ on $\mathfrak{A}_{(-\infty,\infty)}$ considered in [1] satisfies that the corresponding automorphism α on $\overline{\mathfrak{A}}_{(-\infty,\infty)}$ is multiplier outer, which is what is needed to apply [2, Theorem 2.1(ii)], as follows.

Lemma 2.3 *For all $m \geq 1$, α^m is multiplier outer in $\overline{\mathfrak{A}}_{(-\infty,\infty)}$, where $\overline{\mathfrak{A}}_{(-\infty,\infty)}$ denotes the inductive limit* $\lim(\mathfrak{A}_{(-\infty,\infty)}\xrightarrow{\sigma}\mathfrak{A}_{(-\infty,\infty)}\xrightarrow{\sigma}\cdots)$

Proof First, note that σ is a corner endomorphism, since, given $a \in p_1\mathfrak{A}_{(-\infty,\infty)}p_1$, we have $a = \sigma(w^*aw) \in \sigma(\mathfrak{A}_{(-\infty,\infty)})$.

Suppose that α^m is multiplier inner. Then by Lemma 2.2 there is an isometry s in $\mathfrak{A}_{(-\infty,\infty)}$ such that $\sigma^m(x) = sxs^*$ for all $x \in \mathfrak{A}_{(-\infty,\infty)}$. Since $\mathfrak{A}_{(-\infty,\infty)} = \varinjlim \mathfrak{A}_{(-\infty,n]}$, there is n_0 and $s' \in \mathfrak{A}_{(-\infty,n_0]}$ such that $\|s'\| \leq 1$ and $\|s - s'\| < 1/6$. Observe that

$$\begin{aligned}\|p_{n_0+m+1} - s'p_{n_0+1}(s')^*\| &= \|\sigma^m(p_{n_0+1}) - s'p_{n_0+1}(s')^*\| \\ &= \|sp_{n_0+1}s^* - s'p_{n_0+1}(s')^*\| < 1/3,\end{aligned}$$

so that, using that $p_{n_0+m+1} \leq p_{n_0+1}$ we get that

$$\|p_{n_0+m+1} - p_{n_0+1}s'p_{n_0+1}(s')^*p_{n_0+1}\| \leq 1/3.$$

Since $\lim \phi(p_k) = 0$, we may take n_0 big enough so that $\phi(p_{n_0+m+1}) < 1/36$. By [1, Claim 3.7], we have that $p_{n_0+1}\mathfrak{A}_{(-\infty,n_0]}p_{n_0+1}$ and $\{p_{n_0+m+1}\}$ are ϕ-free, and thus the above inequalities contradict Lemma 2.1. This proves the claim. ■

Another instance of the application of [2, Theorem 2.1(ii)] occurs in [2, Theorem 2.2]. The following lemma, which replaces [2, Lemma 2.3], corrects the proof of [2, Theorem 2.2].

Lemma 2.4 *Let B be a unital C^*-algebra that contains a non-trivial and proper projection p. Set*

$$A = \bigotimes_{j=1}^{\infty} B,$$

and let σ be the injective endomorphism on A defined by $\sigma(a) = p \otimes a$ (σ acts as the shift). Then, for every positive integer m, the automorphism α^m of $\overline{A}$ is multiplier outer.

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Proof Observe that σ is a corner endomorphism. Suppose that α^m is multiplier inner for some $m \geq 1$. Then by Lemma 2.2, there is an isometry s in A such that $\sigma^m(x) = sxs^*$ for all $x \in A$.

Consider the asymptotically central sequence $\{q_n\}$ of projections in A given by

$$q_n = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p \otimes 1 \otimes \cdots ,$$

(with p in the n-th factor). Then $\sigma^m(q_n) = q_1 q_2 \cdots q_m q_{n+m}$ for all m, whereas

$$\lim_{n\to\infty} \|sq_n s^* - q_n ss^*\| = 0, \qquad q_n ss^* = q_1 q_2 \cdots q_m q_n,$$

a contradiction. ∎

References

[1] K J. Dykema, *Purely infinite simple C^*-algebras arising from free product constructions. II.* Math. Scand. **90**(2002), no. 1, 73–86.

[2] K. J. Dykema and M. Rørdam, *Purely infinite simple C^*-algebras arising from free product constructions.* Canad. J. Math. **50**(1998), no. 2, 323–341. http://dx.doi.org/10.4153/CJM-1998-017-x

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