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RADICAL THEORY FOR GRADED RINGS

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Abstract

In this paper we propose a general setting in which to study the radical theory of group graded rings. If $\mathscr{R}$ is a radical class of associative rings we consider two associated radical classes of graded rings which are denoted by $\mathscr{R}^G$ and $\mathscr{R}_{\text{ref}}$. We show that if $\mathscr{R}$ is special (respectively, normal), then both $\mathscr{R}^G$ and $\mathscr{R}_{\text{ref}}$ are graded special (respectively, graded normal). Also, we discuss a graded version of the ADS theorem and the termination of the Kurosh lower graded radical construction.

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Introduction

Let G be an arbitrary group. An associative ring A is a G-graded ring if the additive group of A is a direct sum of subgroups A_g, $g \in G$, which are such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. If $h, k \in G$, A and B are G-graded rings and $f: A \to B$ is a ring homomorphism, then f is said to have degree (h, k) if $f(A_g) \subseteq B_{hgk}$ for all $g \in G$. Such homomorphisms will be called *graded homomorphisms.* A graded homomorphism may have more than one degree and, in particular, the zero homomorphism has degree (h, k) for all $h, k \in G$. Also note that if f is a graded homomorphism of degree (h, k) and $h \neq k^{-1}$, then $(f(A))^2 = 0$ because if $x, y \in G$, $a \in A_x$ and $b \in A_y$ then $f(a)f(b) = f(ab) \in A_{hxkhyk} \cap A_{hxyk} = 0$.

Throughout this paper we work in the category of G-graded rings and graded homomorphisms. Our purpose is to study radical theory in this

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category. In the next section we apply recent results of Puczylowski to show that the ADS theorem is true for graded radical classes and that the Kurosh lower graded radical construction stops at the first infinite ordinal. The result about the ADS theorem is not new, it appears in [9] and, in the case of abelian groups, was published in [13].

The third section of the paper discusses a way of constructing, for each radical class $\mathscr{R}$ of associative rings, a radical class $\mathscr{R}^G$ of G-graded rings. The map $\mathscr{R} \to \mathscr{R}^G$ embeds the lattice of radical classes of associative rings as a meet subsemilattice of the lattice of radical classes of G-graded rings. If $\mathscr{R}$ is special (respectively, normal) then $\mathscr{R}^G$ is graded special (respectively, graded normal).

Let $\mathscr{R}$ be a radical class of associative rings. The corresponding reflected radical class, $\mathscr{R}_{\text{ref}}$, of G-graded rings is defined in [2] and several concrete reflected radical classes are studied in [2], [3] and [4]. In the fourth section of this paper we show that the map $\mathscr{R} \to \mathscr{R}_{\text{ref}}$ embeds the lattice of radical classes of associative rings as a sublattice of the lattice of radical classes of G-graded rings and that if $\mathscr{R}$ is special (respectively, normal), then $\mathscr{R}_{\text{ref}}$ is graded special (respectively, graded normal).

Although most of the radical classes of graded rings which have been studied in the past (see [5], [6] and [10]) are either of the form $\mathscr{R}^G$ or $\mathscr{R}_{\text{ref}}$, this is not true in general as we point out in the final section of the paper.

Throughout the paper e will denote the identity of G and β will denote the prime radical class. If A is a G-graded ring, A^u will denote the underlying ungraded ring. If B is an associative ring, B^t will denote the corresponding G-graded ring such that $(B^t)_e = B$ and $(B^t)_g = 0$ if $e \neq g \in G$. If I is a subset of a G-graded ring A and $H \subseteq G$, $I_H = \oplus\{I \cap A_h | h \in H\}$. Finally, we refer the reader to [14] for radical theoretic terms which we use but do not define.

Preliminary results

The following definition and technical lemma are required in order to show that the results of [11] apply in our situation.

Definition 1. A G-graded ring A is *trivial* if whenever I is a graded ideal of A and J is a graded ideal of I, then J is an ideal of A.

Lemma 1. *Suppose that B is a G-graded ring, A is a graded ideal of B and I is a graded ideal of A. Further, suppose that whenever J and K are graded ideals of A such that $J \subseteq I \subseteq K$ and $I/J \cong K/I$ is trivial we have $J = I = K$. Then I is an ideal of B.*

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PROOF. Let $x \in B_g$. Define $f: I \to (Ix + I)/I$ by $f(y) = yx + I$ for all $y \in I$. Since I is an ideal of A and A is an ideal of B, $IxA \subseteq I$ and $IAx \subseteq I$. It follows that if $y, z \in I$, then $f(y)f(z) = 0 + I = f(yz)$ and so f is a graded homomorphism of degree (e, g). Also, both $\ker(f)$ and $Ix + I$ are graded ideals of A and since f is onto, $I/\ker(f) \cong (Ix + I)/I$. Moreover, since $(Ix + I)^2 \subseteq I$, $(Ix + I)/I$ is trivial. Thus it follows from our hypothesis that $\ker(f) = I = Ix + I$. From this we see that I is a right ideal of B and a similar argument shows that I is a left ideal.

DEFINITION 2. A nonempty class $\mathscr{R}$ of G-graded rings is a *graded radical class* if

1. $\mathscr{R}$ is closed under graded homomorphic images,
2. if B is a graded ideal of A and both B and A/B are in $\mathscr{R}$, then $A \in \mathscr{R}$,
3. if $I_\lambda : \lambda \in \Lambda$ is an ascending chain of graded ideals of A and $I_\lambda \in \mathscr{R}$ for all $\lambda \in \Lambda$, then $\cup\{I_\lambda : \lambda \in \Lambda\}$ is in $\mathscr{R}$.

If $\mathscr{R}$ is a graded radical class of G-graded rings, then each G-graded ring A has a graded ideal $\mathscr{R}(A) \in \mathscr{R}$ which contains all graded ideals of A which are in $\mathscr{R}$ and is such that $A/\mathscr{R}(A)$ has no nonzero graded ideals in $\mathscr{R}$.

A nonzero class $\mathscr{C}$ of associative rings is *regular* if every nonzero ideal of a ring $A \in \mathscr{C}$ has a nonzero homomorphic image in $\mathscr{C}$. A nonzero class $\mathscr{M}$ of G-graded rings is *graded regular* if every nonzero graded ideal of a graded ring in $\mathscr{M}$ has a nonzero graded homomorphic image in $\mathscr{M}$. In what follows we will omit explicit definitions of graded concepts which are such obvious generalizations of their ungraded counterparts.

In order to apply the results of [11] we must ensure that axioms A1 to A6 of that paper are satisfied in our category of G-graded rings. Checking axioms A1 to A5 is straightforward and Lemma 1 establishes A6, so we can apply the results of [11] the main consequences of which we summarize in the following theorem.

THEOREM 1. (i) (ADS) *If $\mathscr{R}$ is a graded radical class and I is a graded ideal of A, then $\mathscr{R}(I)$ is an ideal of A.*

(ii) *If $\mathscr{M}$ is any class of G-graded rings, then there is a largest graded radical class $\mathscr{U}\mathscr{M}$ such that every ring in $\mathscr{M}$ is $\mathscr{U}\mathscr{M}$ semisimple. When $\mathscr{M}$ is graded regular, $\mathscr{U}\mathscr{M} = \{A | A$ is G-graded and no nonzero graded homomorphic image of A is in $\mathscr{M}\}$.*

(iii) *Graded semisimple classes are graded hereditary and every graded isomorphism closed class of G-graded rings which is graded regular, graded coinductive and graded extension closed is a graded semisimple class.*

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(iv) *The Kurosh lower graded radical construction stops at the first infinite ordinal.*

The graded prime radical of a G-graded ring A, $\beta_G(A)$, is the intersection of all the graded prime ideals of A (see [6]). As in the case of associative rings, it can be shown that the class

$$\beta_G = \{A|A \text{ is } G\text{-graded and } \beta_G(A) = A\}$$

is the lower graded radical class determined by the class of G-graded nilpotent rings. In fact, because our category includes all homomorphisms of degree (e, g), for all $g \in G$, one can show that β_G is the lower graded radical class determined by the zero ring on the infinite cyclic group with trivial grading. A graded radical class $\mathscr{R}$ is *graded supernilpotent* if $\beta_G \subseteq \mathscr{R}$ and $\mathscr{R}$ is graded hereditary.

A nonempty class $\mathscr{K}$ of G-graded rings is *graded* (*weakly*) *special* if (a) $\mathscr{K}$ consists of graded (semi) prime rings, (b) $I \in \mathscr{K}$ whenever I is a nonzero graded ideal of some $A \in \mathscr{K}$ and (c) whenever $B \in \mathscr{K}$ is a graded two-sided essential ideal of a G-graded ring A, then $A \in \mathscr{K}$.

THEOREM 2. *A graded radical class $\mathscr{R}$ is graded supernilpotent if and only if $\mathscr{R} = \mathscr{U}\mathscr{M}$ for some graded weakly special class $\mathscr{M}$. In this case, for any G-graded ring A, $\mathscr{R}(A) = \cap\{\ker f | f: A \to B \in \mathscr{M}$ is a surjective graded homomorphism$\}$.*

We omit the proof since it is a straightforward adaptation of the proof of the corresponding theorem for associative rings which is due to Ryabukhin (see, for instance, [14, Theorem 11.5]).

When $\mathscr{M}$ is a graded special class of rings, $\mathscr{U}\mathscr{M}$ is called *graded special*. It follows from Theorem 2 that if $\mathscr{M}$ is graded special and A is $\mathscr{U}\mathscr{M}$ semisimple, then A is isomorphic to a subdirect sum of graded prime rings in $\mathscr{M}$.

A left A-module M is a *graded left A-module* if M is the direct sum of a family of submodules $\{M_g : g \in G\}$ and if $A_h M_g \subseteq M_{hg}$ for all $h, g \in G$. Graded bimodules are defined in a similar way.

A radical class $\mathscr{R}$ of associative rings is *normal* if for every Morita context (A, V, W, B), $V\mathscr{R}(B)W \subseteq \mathscr{R}(A)$ (equivalently, $W\mathscr{R}(A)V \subseteq \mathscr{R}(B)$). The meaning of *graded normal* will be clear once we have defined the notion of a graded Morita context: a Morita context (A, V, W, B), where A and B are G-graded rings and V and W are graded bimodules, is a *graded Morita context* if $V_g W_h \subseteq A_{gh}$ and $W_h V_g \subseteq B_{hg}$ for all $g, h \in G$.

Recall that a radical class $\mathscr{R}$ of associative rings is left strong if whenever L is a left ideal of a ring A and $L \in \mathscr{R}$, then $L \subseteq \mathscr{R}(A)$. Also, $\mathscr{R}$ is principally left hereditary if $a \in A \in \mathscr{R}$ implies that $Aa \in \mathscr{R}$. Sands [12] has shown that $\mathscr{R}$ is normal if and only if $\mathscr{R}$ is left strong and principally left hereditary.

We say that a graded radical class $\mathscr{R}$ is *graded principally left hereditary* if for every $A \in \mathscr{R}$ and every homogeneous element $a \in A$, $Aa \in \mathscr{R}$. Note that in both the graded and ungraded case, Aa may not be a principal left ideal of A if A does not have an identity.

THEOREM 3 (Sands). *A graded radical class $\mathscr{R}$ is graded normal if and only if $\mathscr{R}$ is left graded strong and graded principally left hereditary.*

PROOF. Let A be a G-graded ring which does not have an identity, and let A^1 be the Dorroh extension of A. Then A^1 can be G-graded by $(A^1)_e = \{(a, n) | a \in A_e, n \in \mathbb{Z}\}$ and $(A^1)_g = \{(a, 0) | a \in A_g\}$ if $g \neq e$, and with this grading A is a graded ideal of A^1. Also, if (A, V, W, B) is a graded Morita context, then

$$R = \begin{bmatrix} A & V \\ W & B \end{bmatrix}$$

can be G-graded by defining

$$R_g = \begin{bmatrix} A_g & V_g \\ W_g & B_g \end{bmatrix} \text{ for all } g \in G.$$

With these two remarks in mind it is easy to see how to adjust the arguments in [12] in order to complete the proof.

The next proposition concerns one way in which graded radical classes give rise to radical classes of associative rings.

PROPOSITION 1. *Let $\mathscr{R}$ be a graded radical class of G-graded rings. Then $\mathscr{R}^{-t} = \{A | A^t \in \mathscr{R}\}$ is a radical class of associative rings and if $\mathscr{R}$ is graded hereditary (respectively, contains β_G, is graded supernilpotent, is graded special, is graded normal) then $\mathscr{R}^{-t}$ is hereditary (respectively, contains β, is supernilpotent, is special, is normal).*

PROOF. The proof is straightforward and so we consider, as an example, only the case of normality.

Assume that $\mathscr{R}$ is graded normal and (A, V, W, B) is a Morita context. Then (A^t, V^t, W^t, B^t) is a graded Morita context where V^t and W^t have trivial module gradings. Since $\mathscr{R}$ is graded normal $W^t\mathscr{R}(A^t)V^t \subseteq \mathscr{R}(B^t)$. But since $\mathscr{R}(X^t) = (\mathscr{R}^{-t}(X))^t$ for all associative rings X, it follows that $W\mathscr{R}^{-t}(A)V \subseteq \mathscr{R}^{-t}(B)$ and so $\mathscr{R}^{-t}$ is normal.

Restricted radical classes

Let $\mathscr{R}$ be a radical class of associative rings. The restriction of $\mathscr{R}$ to our category of G-graded rings is $\mathscr{R}^G = \{A | A$ is G-graded and $A^u \in \mathscr{R}\}$. Note that $(\mathscr{R}^G)^{-t} = \mathscr{R}$ and so part of the next proposition follows directly from Proposition 1.

PROPOSITION 2. *Let $\mathscr{R}$ be a nonempty class of associative rings. Then $\mathscr{R}$ is a radical class if and only if $\mathscr{R}^G$ is a graded radical class and in this case we have the following: for any G-graded ring A, $\mathscr{R}^G(A)$ is the unique largest graded ideal I of A such that $I^u \in \mathscr{R}$ and, if $\mathscr{R}$ is hereditary, then $\mathscr{R}^G(A) = (\mathscr{R}(A))_G$. Moreover, $\mathscr{R}$ is hereditary (respectively, contains β, is supernilpotent) if and only if $\mathscr{R}^G$ is graded hereditary (respectively, contains β_G, is graded supernilpotent).*

We omit the straightforward proof of this proposition.

The graded prime radical class, the graded strongly prime radical class of [10] and, when G is finite, the graded Jacobson radical class of [5] (see [6, Theorem 4.4(1)]) are radical classes of this kind, as are the graded Levitzki and graded von Neumann regular radical classes of [2].

We remark that the correspondence $\mathscr{R} \to \mathscr{R}^G$ is one-to-one, order preserving and preserves lattice meets since if $\mathscr{R}_1$ and $\mathscr{R}_2$ are radical classes of associative rings, $\mathscr{R}_1^G \cap \mathscr{R}_2^G = (\mathscr{R}_1 \cap \mathscr{R}_2)^G$. It is not in general, however, a lattice homomorphism as the following simple example shows.

EXAMPLE 1. Let $G = \{e, a\}$ be the cyclic group of two elements and let $A = \mathbb{Z}_2[X]/(X^2)$ be G-graded as in [2, Example 3.10]; that is, $A_e = \{0, 1\}$ and $A_a = \{0, 1+x\}$ where $1 = 1 + (X^2)$ and $x = X + (X^2)$. Clearly A is graded simple.

Let $\mathscr{R}_1 = \beta$ and let $\mathscr{R}_2$ be the radical class of associative rings which consists of all Boolean rings. Since $A^u \notin \mathscr{R}_2$, $\mathscr{R}_1^G(A) = \mathscr{R}_2^G(A) = 0$. However, $A^u \in L(\mathscr{R}_1 \cup \mathscr{R}_2)$ which is the join, $\mathscr{R}_1 \vee \mathscr{R}_2$, of $\mathscr{R}_1$ and $\mathscr{R}_2$ in the lattice of radical classes of associative rings. It follows that $(\mathscr{R}_1 \vee \mathscr{R}_2)^G \supsetneqq \mathscr{R}_1^G \vee \mathscr{R}_2^G$ where the second $\vee$ denotes the join in the lattice of G-graded radical classes.

THEOREM 4. *Let $\mathscr{R}$ be a radical class of associative rings. Then $\mathscr{R}$ is special (respectively, normal) if and only if $\mathscr{R}^G$ is graded special (respectively, graded normal).*

PROOF. First assume that $\mathscr{R}$ is special, say $\mathscr{R} = \mathscr{U}\mathscr{P}$ where $\mathscr{P}$ is a special class of associative rings. Let $\mathscr{K} = \{A | A$ is graded prime and

$\mathscr{R}^G(A) = 0\}$. It is straightforward to check that $\mathscr{K}$ is a graded special class, and since $\mathscr{R}^G(A) = 0$ for all $A \in \mathscr{K}$, $\mathscr{R}^G \subseteq \mathscr{U}\mathscr{K}$.

Suppose that A is G-graded and $\mathscr{R}^G(A) = 0$. Let $\mathscr{R}(A) = \cap\{P_\lambda : \lambda \in \Lambda\}$ where A/P_λ is in $\mathscr{P}$ for all $\lambda \in \Lambda$. Each $(P_\lambda)_G$ is graded prime [6, Lemma 5.1] and since $\mathscr{R}$ is hereditary $(\mathscr{R}(A))_G = \mathscr{R}^G(A) = 0$ and hence $\cap\{(P_\lambda)_G : \lambda \in \Lambda\} = 0$. Since $P_\lambda/(P_\lambda)_G$ is an ideal of $A/(P_\lambda)_G$ and $\mathscr{R}(A/P_\lambda) = 0$, we must have

$$\mathscr{R}(A/(P_\lambda)_G) \subseteq P_\lambda/(P_\lambda)_G.$$

Now $\mathscr{R}^G(A/(P_\lambda)_G) \subseteq \mathscr{R}(A/(P_\lambda)_G)$ and so $\mathscr{R}^G(A/(P_\lambda)_G) \subseteq P_\lambda/(P_\lambda)_G$ from which it follows that $\mathscr{R}^G(A/(P_\lambda)_G) = 0$ because 0 is the only graded ideal of $A/(P_\lambda)_G$ which is contained in $P_\lambda/(P_\lambda)_G$. This shows that each factor $A/(P_\lambda)_G$ is in $\mathscr{K}$, so $\mathscr{U}\mathscr{K}(A) = 0$ and $\mathscr{R}^G = \mathscr{U}\mathscr{K}$ as required.

Now assume that $\mathscr{R}$ is normal and let (A, V, W, B) be a graded Morita context. Since $(\mathscr{R}^G(B))^u$ is an ideal of B which is in $\mathscr{R}$ it follows from [8, Corollary 6] that $V^u(\mathscr{R}^G(B))^u W^u = (V\mathscr{R}^G(B)W)^u$ is in $\mathscr{R}$. Now, since $V\mathscr{R}^G(B)W$ is a graded ideal of A, $V\mathscr{R}^G(B)W \subseteq \mathscr{R}^G(A)$. Hence $\mathscr{R}^G$ is graded normal.

The converse follows from Proposition 1 because $(\mathscr{R}^G)^{-t} = \mathscr{R}$.

If $\mathscr{R}$ is a radical class and $\mathscr{R} \cap \beta = \{0\}$ (that is, $\mathscr{R}$ contains no nonzero nilpotent rings), then any nonzero graded homomorphism with domain $A \in \mathscr{R}^G$ must have degree (g, g^{-1}) for some $g \in G$. From this we see that if H is a normal subgroup of G, then

$$\mathscr{R}^H = \{A | A \text{ is } G\text{-graded},\ A = A_H \text{ and } A^u \in \mathscr{R}\}$$

is closed under graded homomorphic images. It is easy to check the other two conditions required to show that $\mathscr{R}^H$ is a graded radical class. Also, from Theorem 1 (iii) it is clear that if $\mathscr{R}$ is a semisimple class, then $\mathscr{R}^H$ is a graded semisimple class.

Now suppose that $\mathscr{R}$ is a radical semisimple class of associative rings (see [7]). From the above remarks we see that for each normal subgroup H of G, $\mathscr{R}^H$ is a graded radical graded semisimple class. Thus we have a large supply of graded radical graded semisimple classes, but the question of how to characterize such classes remains open.

Reflected radical classes

Reflected radical classes were introduced in [2] where graded rings were studied in the category which contained only those homomorphisms of degree

(e, e). The results of that paper, and of [3] and [4], remain valid in our category, but not conversely: for instance, it is shown in [9, Theorem 2.2] that if $G \neq \{e\}$, then there is a radical class in the category of G-graded rings and homomorphisms of degree (e, e) which is not on ADS radical class.

In order to define reflected radical classes we recall the definition of the smash product [1]. If A is G-graded, the smash product $A\#G^*$ is the free left A-module $\oplus\{Ap_g : g \in G\}$ with an associative multiplication which satisfies $ap_g bp_h = ab_{gh^{-1}}p_h$ for all $a, b \in A$, and $g, h \in G$ (here $b_{gh^{-1}}$ is the gh^{-1} component of b). For any class $\mathscr{C}$ of associative rings, the reflected class is $\mathscr{C}_{\text{ref}} = \{A | A \text{ is } G\text{-graded and } A\#G^* \in \mathscr{C}\}$.

Proposition 3. *Let $\mathscr{R}$ be a radical class of associative rings. Then $\mathscr{R}_{\text{ref}}$ is a graded radical class and $\mathscr{R}_{\text{ref}}$ is graded hereditary (respectively, contains β_G, is graded supernilpotent) if and only if $\mathscr{R}$ is hereditary (respectively, contains β, is supernilpotent). Moreover, for any G-graded ring A, $\mathscr{R}_{\text{ref}}(A)\#G^* = \mathscr{R}(A\#G^*)$.*

Proof. Suppose A and B are G-graded rings and $f: A \to B$ is a surjective graded homomorphism of degree (g_1, g_2). Consider the additive map $f^*: A\#G^* \to B\#G^*$ which satisfies $f^*(a_g p_h) = f(a_g)p_{g_1 h g_2}$ for $g, h \in G$ and $a_g \in A_g$. To see that f^* is multiplicative we need to show that $f^*((a_g p_h)(a_\lambda p_\gamma)) = f^*(a_g p_h) f^*(a_\lambda p_\gamma)$ and this amounts to showing that

$$(*) \qquad f(a_g (a_\lambda)_{h\gamma^{-1}}) = f(a_g)(f(a_\lambda))_{g_1 h\gamma^{-1} g_1^{-1}}.$$

Since $(f(A))^2 = 0$ if $g_1 \neq g_2^{-1}$ we can assume that $g_1 = g_2^{-1}$. If $f(a_g(a_\lambda)_{h\gamma^{-1}}) \neq 0$, then $(a_\lambda)_{h\gamma^{-1}} \neq 0$ and so $h\gamma^{-1} = \lambda$. Also if $f(a_g)(f(a_\lambda))_{g_1 h\gamma^{-1} g_1^{-1}} \neq 0$, then $(f(a_\lambda))_{g_1 h\gamma^{-1} g_1^{-1}} \neq 0$ and so $g_1 \lambda g_2 = g_1 h\gamma^{-1} g_1^{-1}$ from which it follows, since $g_1 = g_2^{-1}$, that $h\gamma^{-1} = \lambda$. Hence, if either side of the equation $(*)$ is nonzero we have $g_1 = g_2^{-1}$ and $h\gamma^{-1} = \lambda$. In this case it is clear that $(*)$ is true.

The foregoing implies that $\mathscr{R}_{\text{ref}}$ is closed under graded homomorphic images and it is straightforward to complete the verification that $\mathscr{R}_{\text{ref}}$ is a graded radical class.

It is clear that if $\mathscr{R}$ is hereditary (respectively, $\mathscr{R} \supseteq \beta$), then $\mathscr{R}_{\text{ref}}$ is graded hereditary (respectively, contains β_G). In view of Proposition 1, the converse will follow once we show that $(\mathscr{R}_{\text{ref}})^{-t} = \mathscr{R}$. Suppose $A \in (\mathscr{R}_{\text{ref}})^{-t}$. Then $A^t\#G^* \in \mathscr{R}$ and since $A^t\#G^*$ is isomorphic to a direct sum of copies of A, $A \in \mathscr{R}$. On the other hand, $A \in \mathscr{R}$ implies $A^t\#G^* \in \mathscr{R}$ since $\mathscr{R}$ is closed under direct sums and so $A \in (\mathscr{R}_{\text{ref}})^{-t}$.

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The last statement in the proposition is [2, Proposition 1.3].

PROPOSITION 4. *The map* $\mathscr{R} \to \mathscr{R}_{\text{ref}}$ *is an embedding of the lattice of radical classes of associative rings as a sublattice of the lattice of graded radical classes.*

PROOF. Let $\mathscr{U}$ and $\mathscr{V}$ be radical classes of associative rings.

From the last part of the proof of Proposition 3 we see that if $\mathscr{U}_{\text{ref}} = \mathscr{V}_{\text{ref}}$ then $\mathscr{U} = \mathscr{V}$ because in general we have $\mathscr{R} = (\mathscr{R}_{\text{ref}})^{-t}$. Also, it is clear that $\mathscr{U}_{\text{ref}} \cap \mathscr{V}_{\text{ref}} = (\mathscr{U} \cap \mathscr{V})_{\text{ref}}$. Moreover, since $\mathscr{U} \vee \mathscr{V}$ is the lower radical class determined by $\mathscr{U} \cup \mathscr{V}$ and $\mathscr{U}_{\text{ref}} \vee \mathscr{V}_{\text{ref}}$ is the lower graded radical class determined by $\mathscr{U}_{\text{ref}} \cup \mathscr{V}_{\text{ref}}$, we see that $(\mathscr{U} \vee \mathscr{V})_{\text{ref}} \supseteq \mathscr{U}_{\text{ref}} \vee \mathscr{V}_{\text{ref}}$.

Let A be a G-graded ring in $(\mathscr{U} \vee \mathscr{V})_{\text{ref}}$. Then $A\#G^* \in \mathscr{U} \vee \mathscr{V}$ and so either $\mathscr{U}(A\#G^*) \neq 0$ or $\mathscr{V}(A\#G^*) \neq 0$. This means that A is not both $\mathscr{U}_{\text{ref}}$ semisimple and $\mathscr{V}_{\text{ref}}$ semisimple; so A is not $\mathscr{U}_{\text{ref}} \vee \mathscr{V}_{\text{ref}}$ semisimple. From this it follows that $(\mathscr{U} \vee \mathscr{V})_{\text{ref}} \subseteq \mathscr{U}_{\text{ref}} \vee \mathscr{V}_{\text{ref}}$ and the proof is complete.

THEOREM 5. *Let* $\mathscr{R}$ *be a radical class of associative rings. Then* $\mathscr{R}_{\text{ref}}$ *is graded special (respectively, graded normal) if and only if* $\mathscr{R}$ *is special (respectively, normal).*

PROOF. First assume that $\mathscr{R}$ is special, say $\mathscr{R} = \mathscr{U}\mathscr{P}$ where $\mathscr{P}$ is a special class of associative rings. Let $\mathscr{K} = \{A | A$ is graded prime and $\mathscr{R}(A\#G^*) = 0\}$. It is straightforward to check that $\mathscr{K}$ is a graded special class, and since $\mathscr{R}_{\text{ref}}(A) = 0$ for all $A \in \mathscr{K}$, $\mathscr{R}_{\text{ref}} \subseteq \mathscr{U}\mathscr{K}$.

Suppose that A is G-graded and $\mathscr{R}_{\text{ref}}(A) = 0$. Then $\mathscr{R}(A\#G^*) = 0$ and so there are prime ideals $P_\lambda : \lambda \in \Lambda$ of $A\#G^*$ such that $\cap\{P_\lambda : \lambda \in \Lambda\} = 0$ and $(A\#G^*)/P_\lambda \in \mathscr{P}$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ define $P_\lambda^\downarrow = \{a \in A | ap_g \in P_\lambda$ for all $g \in G\}$. As in [2], the $P_\lambda^\downarrow$ are graded prime ideals of A and since $P_\lambda^\downarrow \# G^* \subseteq P_\lambda$ for all $\lambda \in \Lambda$, $\cap\{P_\lambda^\downarrow : \lambda \in \Lambda\} = 0$. For each $\lambda \in \Lambda$,

$$\begin{aligned}\mathscr{R}_{\text{ref}}(A/P_\lambda^\downarrow)\#G^* &= \mathscr{R}((A/P_\lambda^\downarrow)\#G^*)\\ &\cong \mathscr{R}((A\#G^*)/(P_\lambda^\downarrow\#G^*)) \subseteq P_\lambda/(P_\lambda^\downarrow\#G^*).\end{aligned}$$

From this it follows that if $a + P_\lambda^\downarrow$ is in $\mathscr{R}_{\text{ref}}(A/P_\lambda^\downarrow)$, then $ap_g \in P_\lambda$ for all $g \in G$. Hence $\mathscr{R}_{\text{ref}}(A/P_\lambda^\downarrow) = 0$ and we see that $(A/P_\lambda^\downarrow) \in \mathscr{K}$. Since $\cap P_\lambda^\downarrow = 0$, $\mathscr{U}\mathscr{K}(A) = 0$ and so $\mathscr{R} = \mathscr{U}\mathscr{K}$ as required.

Now assume that $\mathscr{R}$ is normal. Since $\mathscr{R}$ is left strong it follows easily that $\mathscr{R}_{\text{ref}}$ is left graded strong. We now verify that $\mathscr{R}_{\text{ref}}$ is graded principally left hereditary.

Let a be a homogeneous element in A where $A \in \mathscr{R}_{\text{ref}}$. Then $Aap_h = (A\#G^*)ap_h$ and so because $A\#G^* \in \mathscr{R}$ and $\mathscr{R}$ is principally left hereditary, $Aap_h \in \mathscr{R}$. Also, since $\mathscr{R}$ is left strong and Aap_h is a left ideal of $Aa\#G^*$, $Aap_h \subseteq \mathscr{R}(Aa\#G^*)$. Now, $Aa\#G^* = \sum\{Aap_h : h \in G\}$ and hence $Aa\#G^* \in \mathscr{R}$. Thus $Aa \in \mathscr{R}_{\text{ref}}$ and so $\mathscr{R}_{\text{ref}}$ is graded principally left hereditary.

It now follows from Theorem 3 that $\mathscr{R}_{\text{ref}}$ is graded normal.

Since $(\mathscr{R}_{\text{ref}})^{-t} = \mathscr{R}$, the converse is a consequence of Proposition 1.

Postscript

It follows from Theorem 4 that the graded strongly prime, the graded prime and the graded Levitzki radical classes are all graded special, and that the last two of these radical classes are graded normal. Since the graded Jacobson radical class is J_{ref} (see [2, Proposition 2.2]), where J is the Jacobson radical class, it follows from Theorem 5 that the graded Jacobson radical class is a graded special, graded normal radical class.

However, it is easy to find examples of graded radical classes to which Theorems 4 and 5 do not apply. Consider, for instance, the upper graded radical class $\mathscr{R} = \mathscr{U}\mathscr{M}$ where $\mathscr{M}$ is the class of all G-graded rings with trivial grading. Clearly $\mathscr{R}^{-t} = \{0\}$ and $\mathscr{R} \neq \{0\}$, so $\mathscr{R}$ is not a restricted or a reflected radical class. In conclusion, we consider one further example of such a radical class: the graded Brown-McCoy radical class $\mathscr{G}_G$ of [2].

For any G-graded ring A, $\mathscr{G}_G(A) = \cap\{I|I$ is a graded ideal of A and A/I is a graded simple ring with identity$\}$. Let $\mathscr{K} = \{A|A$ is a graded simple ring with identity$\}$. Then $\mathscr{K}$ is a graded special class and so $\mathscr{U}\mathscr{K}$ is a graded special radical class. Clearly $\mathscr{U}\mathscr{K}(A) \subseteq \mathscr{G}_G(A)$ and Sulinski's proof for associative rings (see, for example, [14, Theorem 39.2]) generalizes to the graded case to show that $\mathscr{G}_G(A) \subseteq \mathscr{U}\mathscr{K}(A)$. Thus $\mathscr{G}_G = \mathscr{U}\mathscr{K}$ is a graded special radical class. Also, since $(\mathscr{G}_G)^{-t}$ is the Brown-McCoy radical class $\mathscr{G}$ which is not normal, it follows from Proposition 1 that $\mathscr{G}_G$ is not graded normal.

When G is finite, $\mathscr{G}_G = \mathscr{G}_{\text{ref}}$ [2, Theorem 3.6] and so $\mathscr{G}_G$ is a reflected graded radical class. However, in general $\mathscr{G}_G$ is not a restricted or a reflected radical class.

PROPOSITION 5. *If G is the group of integers, then $\mathscr{G}_G$ is not a restricted or a reflected radical class.*

PROOF. Let $\mathscr{R}$ be a radical class of associative rings. Since $(\mathscr{R}^G)^{-t} = \mathscr{R}$ and $(\mathscr{G}_G)^{-t} = \mathscr{G}$ it follows that $\mathscr{G}_G \neq \mathscr{R}^G$ because $\mathscr{G}_G \neq \mathscr{G}^G$ (this is established in [2, Proposition 3.5] using an example of a $\mathbb{Z}$-graded ring from [5, example following Lemma 12]). Similarly, since $(\mathscr{R}_{\text{ref}})^{-t} = \mathscr{R}$ we see that $\mathscr{G}_G \neq \mathscr{R}_{\text{ref}}$ because $\mathscr{G}_G \neq \mathscr{G}_{\text{ref}}$ [2, Theorem 3.6].

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