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EDGE PARTITION PROPERTIES OF GRAPHS

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Introduction. Erdös and Hajnal [1] have introduced an edge partition relation for graphs

$$(1) \qquad G \rightarrow (H_1, H_2)$$

which means that whenever the edges of G are separated into two sets, E_1 and E_2, there exists a subgraph G' of G such that G' is isomorphic to H_i and the edges of G' are all in E_i, for $i = 1$ or 2. A class of graphs $\mathscr{K}$ has the *G-R* (*Galvin-Ramsey*) *property* [2] if for each H in $\mathscr{K}$ there exists a G in $\mathscr{K}$ which satisfies $G \rightarrow (H, H)$.

The results in this paper are stated in terms of a partition relation which is stronger than (1). We define

$$(2) \qquad G \rightarrowtail (H_1, H_2)$$

to mean that whenever the edges of G are separated into two sets, E_1 and E_2, there exists an *induced* subgraph G' of G such that G' is isomorphic to H_i and the edges of G are all in E_i, for $i = 1$ or 2. A class of graphs $\mathscr{K}$ has the *strong* G-R property if for each H in $\mathscr{K}$ there exists a G in $\mathscr{K}$ which satisfies $G \rightarrowtail (H, H)$.

Galvin has asked whether the class $\mathscr{K}_3$ of all finite graphs without triangles has the *G-R* property. This question is discussed in [2], where a subclass of $\mathscr{K}_3$ is given which does have the *G-R* property. We give, in Section 1, a different sort of partial answer to Galvin's question: if H_1 is in $\mathscr{K}_3$ and H_2 is either a finite tree or an even circuit C_{2k} or a complete bipartite graph $K_{r,s}$, then (2) holds for some G in $\mathscr{K}_3$. (More generally, this assertion is true if $\mathscr{K}_3$ is replaced by the class $\mathscr{K}_n$ of all finite graphs which have no complete subgraphs with n vertices, $n \geqq 3$.)

In Section 2 we prove that the class of finite comparability graphs [3; 4] has the strong *G-R* property. (Of course it has the *G-R* property, by Ramsey's Theorem, since it contains every finite complete graph.) In contrast, the class of all comparability graphs does not have the strong *G-R* property.

Preliminaries. If G is a graph, we denote the vertex set of G by $|G|$ and the edge set of G by $E(G)$. Thus

$$E(G) = \{\{v, w\} \mid v, w \in |G| \text{ and } v \text{ is joined to } w \text{ in } G\}.$$

A graph H is a subgraph of G if $|H| \subseteq |G|$ and $E(H) \subseteq E(G)$. H is an *induced*

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subgraph if $|H| \subseteq |G|$ and

$$E(H) = \{\{v, w\} \mid v, w \in |H|\} \cap E(G).$$

For each nonempty subset A of $|G|$, let $G|A$ denote the induced subgraph of G with vertex set A. For each v in $|G|$, the induced subgraph of G obtained by removing v is denoted by $G - v$. The cardinality of $|G|$ is denoted by $c(G)$.

A complete graph with n vertices is denoted by K_n. A circuit with n vertices is denoted by C_n. Thus $|C_n| = \{v_1, \ldots, v_n\}$ where $E(C_n)$ consists of the edges $\{v_i, v_{i+1}\}$ for $1 \leqq i < n$ and the edge $\{v_n, v_1\}$. A complete bipartite graph is denoted by $K_{r,s}$. This graph has $r + s$ vertices $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$, where $E(K_{r,s})$ consists of all possible edges of the form $\{v_i, w_j\}$. A tree is a graph with no subgraph isomorphic to a circuit. A graph G is a comparability graph [**3**; **4**] if there is a partial ordering $<_1$ on $|G|$ such that

$$E(G) = \{\{v, w\} \mid v <_1 w\}.$$

In that case we say that G is the graph defined by $<_1$.

The class of finite graphs with no subgraph isomorphic to K_n is denoted by $\mathscr{K}_n$, for $n \geqq 3$.

1. Galvin's question about $\mathscr{K}_3$ may be broadened to ask whether $\mathscr{K}_n$ has the G-R property, for $n \geqq 3$. We observe that if $\mathscr{K}_n$ has the G-R property then it must have the strong G-R property. This is a consequence of the fact that each H in $\mathscr{K}_n$ is an induced subgraph of some H' in $\mathscr{K}_n$ which is *maximal*, in the sense that the addition of any new edge to H' would create a subgraph isomorphic to K_n. If G is in $\mathscr{K}_n$ and satisfies $G \rightarrow (H', H')$, then the maximality of H' implies that $G \rightarrowtail (H', H')$, and therefore G satisfies $G \rightarrowtail (H, H)$.

This observation suggests an attempt to analyze those pairs H_1, H_2 in $\mathscr{K}_n$ for which there exists a G in $\mathscr{K}_n$ such that (2) holds. The results which follow each assert that if H_2 is a certain fixed graph and H_1 is an arbitrary member of $\mathscr{K}_n$, then there exists a G in $\mathscr{K}_n$ for which (2) holds. The proofs are inductive and, in each case, depend on the fact that the induction assumption is stated in terms of the stronger partition relation (2).

THEOREM 1. *Let H_2 be a finite graph of one of the following types:* (I) *a tree,* (II) *an even circuit C_{2k}, or* (III) *a complete bipartite graph $K_{r,s}$. If H_1 is in $\mathscr{K}_n$* ($n \geqq 3$), *then there exists a G in $\mathscr{K}_n$ which satisfies $G \rightarrowtail (H_1, H_2)$.*

Proof. Fix $n \geqq 3$. Logically this proof consists of three different inductive arguments, each covering one of the three possible types of graph into which H_2 may fall. In the cases where H_2 is an even circuit or a complete bipartite graph, the arguments depend on the result for the case where H_2 is a tree. However there are many common features of the three proofs, so that we shall present them together. In particular, the three induction steps all use a certain graph construction described below.

In Case (I), where H_2 is a tree, the argument is by induction on the number $c(H_1) + c(H_2)$. In Case (II) H_2 is taken to be a fixed even circuit and the proof is by induction on $c(H_1)$. In Case (III), where H_2 is a complete bipartite graph $K_{r,s}$, the argument is by induction on $c(H_1) + r$.

In all three cases the basis step is trivial. Moreover, if v is an isolated vertex in H_1 and if $G \rightarrowtail (H_1 - v, H_2)$, then $G' \rightarrowtail (H_1, H_2)$, where G' is the result of adding an isolated vertex to G. Therefore in the various induction steps we may suppose that H_1 and H_2 have no isolated vertices.

The three induction steps make use of the following construction. Let G_1 and G_2 be members of $\mathscr{K}_n$, and let $\mathscr{A}$ be the collection of all nonempty subsets A of $|G_2|$ such that no two vertices in A are joined in G_2. For each A in $\mathscr{A}$ let $G_1{}^A$ be a copy of G_1. Let $\mathscr{F}$ be the set of all functions f defined on $\mathscr{A}$ such that for each $A \in \mathscr{A}$, $f(A)$ is a nonempty subset of $|G_1{}^A|$ and $G_1{}^A | f(A)$ is in $\mathscr{K}_{n-1}$. For each f in $\mathscr{F}$ let $G_2{}^f$ be a copy of G_2 and let $\alpha^f : G_2 \rightarrow G_2{}^f$ be an isomorphism. Given $A \in \mathscr{A}$, let

$$A^f = \{\alpha^f(v) | v \in A\}.$$

We assume that the vertex sets of the various graphs $G_1{}^A$ and $G_2{}^f$ are pairwise disjoint. A graph $G = G(G_1, G_2)$ is constructed as follows. The vertex set of G is the union of all the vertex sets $|G_1{}^A|$ for $A \in \mathscr{A}$ and $|G_2{}^f|$ for $f \in \mathscr{F}$. $E(G)$ consists of the edge sets $E(G_1{}^A)$ for $A \in \mathscr{A}$ and $E(G_2{}^f)$ for $f \in \mathscr{F}$, together with all edges $\{v, w\}$ where $v \in A^f$ and $w \in f(A)$ for some $A \in \mathscr{A}$ and $f \in \mathscr{F}$. Note that for $A \in \mathscr{A}$ and $f \in \mathscr{F}$, the graphs $G_1{}^A$ and $G_2{}^f$ are induced subgraphs of G.

LEMMA 1. *If G_1 and G_2 are in $\mathscr{K}_n$, then $G = G(G_1, G_2)$ is also in $\mathscr{K}_n$.*

Proof of the Lemma. Suppose otherwise, so that there exist vertices $v_1, \ldots, v_n$ each two of which are joined in G. Each induced subgraph $G_1{}^A$ or $G_2{}^f$ of G is in $\mathscr{K}_n$. Thus there exists $A \in \mathscr{A}$ and $f \in \mathscr{F}$ so that one of the edges $\{v_i, v_j\}$ is of the form $\{v, w\}$ where $v \in A^f$ and $w \in f(A)$. Then each v_i must lie in $A^f \cup f(A)$. Since no two members of A^f are joined in $G_2{}^f$ it follows that if $v_i \neq v$, then $v_i \in f(A)$. But this implies $G_2{}^f | f(A)$ is not in $\mathscr{K}_{n-1}$, which is a contradiction.

Now we outline the proof of the rest of Theorem 1. In each of the three induction steps below, H_1 is a given member of $\mathscr{K}_n$ and H_2 is a special graph of the type being considered. A vertex v is chosen from H_1 and the set of vertices joined to v in H_1 is denoted by V. A certain graph H_3 is constructed, depending on H_2 and the case being considered, and a nonempty subset B of $|H_3|$ is defined which has the property that no two vertices in B are joined in H_3.

The induction assumption in each case assures the existence of a graph G_1 in $\mathscr{K}_n$ which satisfies $G_1 \rightarrowtail (H_1 - v, H_2)$. There also must exist a graph G_2 in $\mathscr{K}_n$ which satisfies $G_2 \rightarrowtail (H_1, H_3)$. (In Case (I) this is assured by the

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induction assumption; in Cases (II) and (III) it is assured by the induction assumption or by Case (I).) The graph $G = G(G_1, G_2)$, which is in $\mathscr{K}_n$ by Lemma 1, is shown to satisfy $G \rightarrowtail (H_1, H_2)$, thus completing the induction. This last step is proved by contradiction. Thus we assume that E_1, E_2 is a partition of $E(G)$ such that no induced subgraph of G exists which is isomorphic to H_1 and has all its edges in E_1 or is isomorphic to H_2 and has all its edges in E_2. Therefore, for each A in $\mathscr{A}$ there is an induced subgraph G_A of $G_1{}^A$ such that G_A is isomorphic to $H_1 - v$ and $E(G_A) \subseteq E_1$. For each such A let $f(A)$ be the subset of G_A corresponding to V under the given isomorphism between G_A and $H_1 - v$. Since H_1 is in $\mathscr{K}_n$, $G_A | f(A)$ must be in $\mathscr{K}_{n-1}$. Thus the function f is an element of $\mathscr{F}$. There is an induced subgraph H' of $G_2{}^f$ such that H' is isomorphic to H_3 and $E(H') \subseteq E_2$. Let B' be the subset of $|H'|$ which corresponds to the set B under the given isomorphism between H' and H_3. Then no two vertices in B' are joined in $G_2{}^f$, so there exists an A in $\mathscr{A}$ for which $B' = A^f$.

Keeping this notation, we now turn to the details of each case.

Case (I). In this case H_2 is a finite tree. Choose a vertex w in H_2 which is joined to exactly one vertex, u. (This is possible since H_2 is a finite tree and may be assumed to have no isolated vertices.) In this case H_3 is taken to be $H_2 - w$ and B is $\{u\}$.

Let u' be the vertex of H' for which $B' = \{u'\} = A^f$. Then $E(G)$ contains every edge between u' and elements of $f(A)$. If all these edges are in E_1, then the induced subgraph of G with vertex set $|G_A| \cup \{u'\}$ is isomorphic to H_1 and has all its edges in E_1. Otherwise there exists v' in $f(A)$ such that $\{v', u'\}$ is in E_2. In that case the induced subgraph of G with vertex set $|H'| \cup \{v'\}$ is isomorphic to H_2 and has all its edges in E_2. Therefore we have shown $G \rightarrowtail (H_1, H_2)$, as asserted.

Case (II). Here H_2 is an even circuit C_{2k}, $k \geqq 2$. Let $p > 0$ be the number of elements in V. H_3 is a certain finite tree, in this case, with distinguished vertices $v_0, v_1, \ldots, v_{p+1}$. Let H_3 have other vertices and edges so that v_0 is joined to each other v_i by a path of exactly $k - 1$ edges. The set B is defined to be $\{v_1, v_2, \ldots, v_{p+1}\}$. Since H_3 is a tree, the existence of G_2 is assured by Case (I).

Now let $v_1', \ldots, v_{p+1}'$ be the elements of B', so that $A^f = \{v_1', \ldots, v_{p+1}'\}$. Then $E(G)$ contains all the edges between the vertices v_i' and the elements of $f(A)$. Suppose that for some $1 \leqq i \leqq p + 1$, every edge of the form $\{v_i', u\}$, where u is in $f(A)$, is an element of E_1. Then the induced subgraph of G with vertex set $|G_A| \cup \{v_i'\}$ is isomorphic to H_1 and has all its edges in E_1. Otherwise, for each i $(1 \leqq i \leqq p + 1)$ there is an element u_i of $f(A)$ so that $\{v_i', u_i\}$ is in E_2. Since $f(A)$ has p elements there must exist i, j $(1 \leqq i < j \leqq p + 1)$ such that $u_i = u_j$. Let G' be the induced subgraph of G whose vertices are u_i, v_i', v_j' and all the vertices on the unique path in H' from v_i' to v_j'. Then G' is a circuit with $2k$ vertices and $E(G') \subseteq E_2$. This completes the argument that $G \rightarrowtail (H_1, H_2)$ in this case.

Case (III). In this case H_2 is a complete bipartite graph $K_{r,s}$. If $r = 1$, then H_2 is a tree and Case (I) applies. Thus we assume $r > 1$. Again let $p > 0$ be the cardinality of V. Here we let H_3 be $K_{r-1,ps}$; let

$$|H_3| = \{v_1, \ldots, v_{r-1}, w_1, \ldots, w_{ps}\}$$

where $E(H_3)$ consists of all possible edges of the form $\{v_i, w_j\}$. Let $B = \{w_1, \ldots, w_{ps}\}$. The induction in this case is on the quantity $c(H_1) + r$, so the existence of G_2 is assured by the induction assumption.

As in Case (II), if there is a vertex in B' whose edges with elements of $f(A)$ are all in E_1, then we obtain an induced subgraph of G which is isomorphic to H_1 and has all its edges in E_1. Otherwise, since $f(A)$ has p elements and B' has ps elements, there is a vertex u in $f(A)$ and s vertices $v_1', \ldots, v_s'$ in B' such that $\{u, v_i'\}$ is in E_2 for $i = 1, \ldots, s$. Let G' be the induced subgraph of G whose vertices are $u, v_1', \ldots, v_s'$ and the $r - 1$ vertices in $|H'| \sim |B'|$. Then G' is a complete bipartite graph $K_{r,s}$ and $E(G') \subseteq E_2$. This completes the proof of Theorem 1.

The simplest open problem of the type in Theorem 1 is the following: if H_1 is in $\mathscr{K}_3$ and H_2 is the 5-circuit C_5, does there exist a G in $\mathscr{K}_3$ such that $G \rightarrowtail (H_1, H_2)$? Also, given finite graphs H_1 and H_2, one may ask whether any graph G exists which satisfies $G \rightarrowtail (H_1, H_2)$. In other words, does the class of all finite graphs have the strong G-R property?

2. Erdos and Hajnal [**2**] have made the observation that if G satisfies $G \rightarrow (G, G)$, then the class of finite subgraphs of G has the G-R property. This is the principal method by which classes of finite graphs with the G-R property are obtained in [**2**]. Actually, many of the infinite graphs they construct satisfy the stronger condition $G \rightarrowtail (G, G)$. In that case the class of finite *induced* subgraphs of G, which we will denote by $\mathscr{K}(G)$, has the strong G-R property.

In general it is difficult to give an intrinsic characterization of $\mathscr{K}(G)$. Therefore this method is not likely to be sufficient to decide whether a class like $\mathscr{K}_n$ or the class of all finite graphs has the strong G-R property. However, as we show next, the class of finite comparability graphs is the union of certain subclasses of the form $\mathscr{K}(G)$ where G satisfies $G \rightarrowtail (G, G)$. As a consequence we have the following result.

THEOREM 2. *The class of finite comparability graphs has the strong G-R property.*

Proof. Let N be the set of positive integers. For each $A \subseteq N$ and each $k \geqq 1$, let $[A]^k$ denote the collection of finite subsets of A which have cardinality k. We will always write an element $a = \{a_1, a_2, \ldots, a_k\}$ of $[N]^k$ so that $a_1 < a_2 < \ldots < a_k$. Define a partial ordering $<_k$ on $[N]^{2k}$ by

$$a <_k b \Leftrightarrow \text{for each } j = 1, \ldots, k, \quad a_{2j-1} < b_{2j-1} < b_{2j} < a_{2j}.$$

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Let G_k be the comparability graph defined on the vertex set $[N]^{2k}$ by the partial ordering $<_k$. We will show that $G_k \rightarrowtail (G_k, G_k)$ for each $k \geqq 1$ and that each finite comparability graph is isomorphic to an induced subgraph of some G_k. From these facts will follow the Theorem, as discussed above.

Let E_1, E_2 be a partition of the edge set of G_k. Define a bijection from $E(G_k)$ onto $[N]^{4k}$ by letting the edge $\{a, b\}$ of G_k correspond to the set

$$\{a_1, b_1, b_2, a_2, \ldots, a_{2k-1}, b_{2k-1}, b_{2k}, a_{2k}\}.$$

This determines an induced partition of $[N]^{4k}$. Applying Ramsey's Theorem yields an infinite subset A of N (and $i = 1$ or 2) such that if a and b are in $[A]^{2k}$ and $a <_k b$, then $\{a, b\}$ is in E_i. But the induced subgraph of G_k with vertex set $[A]^{2k}$ is obviously isomorphic to G_k. Therefore G_k satisfies $G_k \rightarrowtail (G_k, G_k)$.

Now let G be an arbitrary finite comparability graph. Suppose G is defined by the partial ordering $\ll$ on $|G|$. Let $n = c(G)$ and let $\alpha_1, \ldots, \alpha_k$ be a list of all bijections α from $|G|$ onto $\{1, \ldots, n\}$ which satisfy

$$v \ll w \Rightarrow \alpha(v) < \alpha(w), \text{ for all } v, w \in |G|.$$

Each such bijection α corresponds to an extension of $\ll$ to a total ordering on $|G|$. Therefore, it follows that for each $v, w \in |G|$

$$v \ll w \Leftrightarrow \alpha_j(v) < \alpha_j(w) \text{ for all } j = 1, \ldots, k.$$

Now we define a mapping F of $|G|$ into $[N]^{2k}$. Given v in $|G|$, let

$$F(v) = \{a_1, a_2, \ldots, a_{2k}\},$$

where

$$a_{2j-1} = 2n(j-1) + \alpha_j(v) \text{ and}$$
$$a_{2j} = 2nj + 1 - \alpha_j(v) \text{ for each } j = 1, \ldots, k.$$

It is easy to check that for each v, w in $|G|$, $v \ll w$ is equivalent to $F(v) <_k F(w)$. Therefore F is an embedding of the graph G onto an induced subgraph of G_k. This shows that every finite comparability graph is an element of $\mathscr{K}(G_k)$ for some k, completing the proof.

The class of *all* comparability graphs does not have the strong *G-R* property, as is implied by the following observations.

(i) *If G and H are comparability graphs and $G \rightarrowtail (H, H)$, then there is a well-founded partial ordering which defines H.*

Proof. Let $<_1$ be a partial ordering on $|G|$ which defines G and let $<_2$ be a well-ordering of $|G|$. Define

$$E_1 = \{\{v, w\} \mid v <_1 w \text{ and } v <_2 w\}$$
$$E_2 = \{\{v, w\} \mid v <_1 w \text{ and } w <_2 v\}.$$

Then E_1, E_2 is a partition of $E(G)$, so that there is an induced subgraph G' of

G such that G' is isomorphic to H and $E(G') \subseteq E_1$ or $\subseteq E_2$. We may assume that $E(G') \subseteq E_1$, changing from $<_1$ to its converse partial ordering if necessary. Then the restriction of $<_1$ to $|G'|$ is a well-founded partial ordering and it defines G', since G' is an induced subgraph of G.

(ii) *Every partial ordering* $\ll$ *which defines the comparability graph in Figure* 1 *satisfies*

$$u \ll v \ll w \quad \text{or} \quad w \ll v \ll u$$

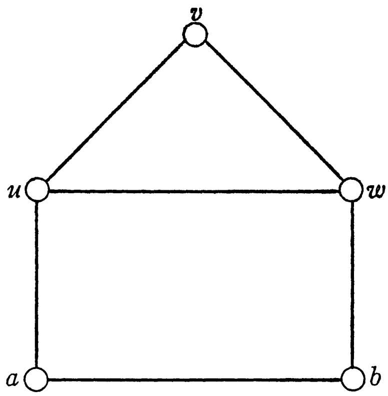

FIGURE 1

(iii) *If H is a comparability graph and $\ll$ is a partial ordering which defines H, then there is a comparability graph H' such that H is an induced subgraph of H' and the restriction to $|H|$ of any partial ordering which defines H' is equal to $\ll$ or to its converse.*

Proof. We may assume that there exist v_0 and v_1 in $|H|$ such that for any other v in $|H|$, $v_0 \ll v \ll v_1$. For each v in $|H|$, add two new vertices v' and v'' to $|H|$. Extend $\ll$ to this new set as follows.

(a) For each v in $|H|$, let $v' \ll v$, $v \ll v''$ and $v' \ll v''$.

(b) For each u, v in $|H|$ with $u \ll v$, let $u' \ll v$, $u \ll v''$, $u' \ll v''$ and $v' \ll u''$.
It is easy to check that this extension of $\ll$ is a partial ordering. Let H' be the comparability graph which it defines. Obviously H is an induced subgraph of H'.

Now let $<_1$ be any partial ordering which defines H'. We may assume that $v_0 <_1 v_1$, passing to the converse partial ordering of $<_1$ if necessary. For u, v, w in $|H|$, if $u \ll v \ll w$ then there is an induced subgraph of H' like that in Figure 1, with a equal to u'' and b equal to w'. Therefore, by (ii), either $u <_1 v <_1 w$ or $w <_1 v <_1 u$. Applying this when $u = v_0$ and $w = v_1$ yields

$$v_0 <_1 v <_1 v_1$$

for any other v in $|H|$. But if u, v are in H and $u \ll v$, then $u <_1 v$ (since $v_1 <_1 v <_1 u$ is impossible). Since the restriction of $<_1$ to $|H|$ defines H, it follows that this restriction must equal $\ll$.

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Now let $\ll$ be any partial ordering such that neither $\ll$ nor its converse is well-founded, and let H be the graph it defines. Let H' be the comparability graph constructed from H as in (iii). Then there is no well-founded partial ordering which defines H'. By (i) it follows that for no comparability graph G does $G \rightarrowtail (H', H')$ hold. Thus we have proved:

THEOREM 3. *The class of all comparability graphs does not have the strong G-R property.*

This argument leaves open the possibility that the class of comparability graphs definable by well-founded partial orderings does have the strong G-R property.

Remark. The proof of Theorem 2 actually proved the following, apparently stronger fact: *given a finite partial ordering* $<_1$ *there is another finite partial ordering* $<_2$ *such that if* E_1, E_2 *is a partition of the set*

$$\{\{x, y\} \mid x <_2 y\}$$

then there is a subset A of the domain of $<_2$ *for which*

$$\{\{x, y\} \mid x, y \in A \text{ and } x <_2 y\}$$

is contained in E_1 *or* E_2 *and the restriction of* $<_2$ *to A is isomorphic to* $<_1$. Indeed, this fact can be inferred easily from Theorem 2, using observation (iii) above: Given $<_1$ let $\ll$ be a finite partial ordering with restrictions isomorphic to $<_1$ and to the converse of $<_1$. Let H' be constructed as in (iii) from the graph H defined by $\ll$, and let G be a comparability graph which satisfies $G \rightarrowtail (H', H')$. Then any partial ordering $<_2$ which defines G will have the property expressed above.

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