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ON CONSTRAINED L^2-APPROXIMATION OF COMPLEX FUNCTIONS

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ABSTRACT. A function f analytic in any disc of radius greater than 1 is approximated in the L^2-sense over a class of polynomials which also interpolate f on a subset of the roots of unity. The resulting solution is used to discuss Walsh-type equiconvergence. The main theorem of the paper generalizes certain results of Walsh, Rivlin and Cavaretta *et al.*

1. **Introduction.** Let $A_\rho(\rho > 1)$ denote the class of all functions $f(z)$ analytic in the disc $D_\rho = \{z : |z| < \rho\}$ but not necessarily analytic in $\bar{D}_\rho$. Let $L_{n-1}(z,f)$ denote the Lagrange polynomial of degree $n-1$ which interpolates $f(z)$ at the zeros of $z^n - 1$. This polynomial is, in fact, the optimal solution [9] of the problem

(P1) $\min_{p\in\pi_{n-1}} \sum_{k=0}^{n-1} |f(w^k) - p(w^k)|^2$

where π_{n-1} denotes the class of all polynomials of degree $\leq n-1$ and w is any of the primitive n-th roots of unity. Similarly, if $f \in A_\rho$, $\rho > 1$, then its Taylor's polynomial of degree $n-1$ about the origin which we shall denote by $S_{n-1}(z,f)$ is the optimal solution of the problem

(P2) $\min_{p\in\pi_{n-1}} \int_{|z|=1} |f(z) - p(z)|^2 |dz|$.

It is known and easily verifiable that

$$S_{n-1}(z,f) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)(t^n - z^n)}{(t^n - 1)(t - z)}\,dt, \quad L_{n-1}(z,f) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)(t^n - z^n)}{t^n(t - z)}\,dt,$$

where Γ_R is the circle $|t| = R$, $1 < R < \rho$.

Although the sequences $\{S_n(z,f)\}_{n=1}^\infty$ and $\{L_n(z,f)\}_{n=1}^\infty$, when $f \in A_\rho$, fail to converge everywhere on the boundary of the disc D_ρ, J. L. Walsh proved that [9, p. 153]

$$\lim_{n\to\infty} \big(L_{n-1}(z,f) - S_{n-1}(z,f)\big) = 0, \quad \forall z \in D_{\rho^2}. \tag{1.1}$$

T. J. Rivlin [4] extended the result of Walsh by considering the problem (P1) over a large set of the primitive roots of unity. He noticed that the optimal solution $\tilde{L}_{n-1}(z,f)$ of the problem

(P3) $\min_{p\in\pi_{n-1}} \sum_{k=0}^{qn-1} |f(w^k) - p(w^k)|^2$, $w^{qn} = 1$ $q \geq 1$

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is essentially the $(n-1)$-th degree Taylor's polynomial $S_{n-1}\big(z, L_{qn-1}(z,f)\big)$ of $L_{qn-1}(z,f)$ and that

$$\lim_{n\to\infty}\{\tilde{L}_{n-1}(z,f) - S_{n-1}(z,f)\} = 0, \quad \forall z \in D_{\rho^{1+q}}. \tag{1.2}$$

In [6] A. Sharma and Z. Ziegler further modified the problem (P1) by dividing the set of the zeros of $z^{sqn} - 1$, $s \geq 1$ into two disjoint subsets, namely,

$$\begin{aligned} U_{s,q} &:= \text{ set of the zeros of } (z^{sqn}-1)/(z^{qn}-1) \\ V_q &:= \text{ set of the zeros of } z^{qn}-1 \end{aligned} \tag{1.3}$$

and looked for the optimal solution of the problem

(P4) $\min_{Q\in \mathcal{L}(f,U_{s,q})} \sum_{w\in V_q} |f(w) - Q(w)|^2$,

where $\mathcal{L}(f, U_{s,q})$ denotes the class of all polynomials of degree $\leq nq(s-1)+n-1$ interpolating f at the points of $U_{s,q}$. It turned out that the optimal solution of (P4) could be expressed as

$$L^*_{N+n-1}(z,f) := Q^*(z,f) + W_s(z)S_{n-1}\big(z, L_{qn-1}(z,g)\big) \tag{1.4}$$

where

$$\begin{aligned} N &= nq(s-1) \\ Q^*(z,f) &= \text{ The Lagrange interpolant of degree } N-1 \text{ to } f \text{ on } U_{s,q} \\ g(z) &= s^{-1}[f(z) - Q^*(z,f)] \\ W_s(z) &= (z^{sqn}-1)/(z^{qn}-1). \end{aligned} \tag{1.5}$$

Sharma and Ziegler established the following extension of (1.1), which is the inspiration of our work:

THEOREM A ([6]). *Let $f(z) \in A_\rho(\rho > 1)$ and let $N = nq(s-1)$ where $s > 1$ and q are fixed positive integers. If $L^*_{N+n-1}(z,f) \in \mathcal{L}(f, U_{s,q})$ solves the minimization problem (P4) and if $S_{N+n-1}(z,f)$ is the $(N+n-1)$-th degree Taylor's polynomial of $f(z)$ about the origin then*

$$\lim_{n\to\infty}\{S_{N+n-1}(z,f) - L^*_{N+n-1}(z,f)\} = 0, \quad \forall z \in D_{\rho^{1+1/(s-1)}} \tag{1.6}$$

The notable point in the above result is that the region of convergence $D_{\rho^{1+1/(s-1)}}$ (*cf.* (1.6)) is independent of the integer q. This anomaly is due to the fact that the optimal processes $S_{N+n-1}(z,f)$ and $L^*_{N+n-1}(z,f)$ are not alike as we observe in (1.1) and (1.2). An appropriate replacement of $S_{N+n-1}(z,f)$ which is the object of this paper, emerges from the following problem:

(P5) Let $s \geq 1$ and $q \geq 1$ be fixed integers. For a given $f \in A_\rho$, $\rho > 1$, find the polynomial $S^*_{N+n-1}(z,f) \in \mathcal{L}(f, U_{s,q})$ which solves the problem

$$\min_{p\in \mathcal{L}(f,U_{s,q})} \int_{|z|=1} |f(z) - p(z)|^2 |dz|.$$

The solution of this problem which is a continuous analogue of the problem (P4) is determined in the next section. The Section 3 is devoted to study the region of convergence of the sequence

$$\{L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f)\}_{n=1}^{\infty} \tag{1.7}$$

which arises from the solutions of the minimization problems (P4) and (P5). We show that the sequence (1.7) tends to zero, as $n \longrightarrow \infty$ in

$$|z| < \rho^{1+q/((s-1)q+1)},$$

thereby extending the results of Walsh [9, p. 153] and Rivlin [4, Theorem 1].

In Section 4, we describe an extension of our main result (Theorem 2.1) and state certain problems related to it.

2. **Preliminaries and solution of (P5).** Let B denote the unit circle $|z| = 1$ in the complex plane. Recall that $f \in L^2(B)$ if $\int_{|z|=1} |f(z)|^2 |dz| < \infty$. Also, $L^2(B)$ is an inner product space with inner product

$$\langle f, g\rangle = \int_{|z|=1} f(z)\overline{g(z)}|dz|. \tag{2.1}$$

The class of polynomials given by

$$W_s\pi_{n-1} := \{W_s(z)p(z) : p \in \pi_{n-1}\},$$

where $W_s(z)$ is defined in (1.5), may be regarded as a subspace of $L^2(B)$. Since

$$W_s(z) = \sum_{j=0}^{s-1} z^{jnq},$$

it is easy to note that for any non-negative integers k and ν

$$\langle z^k W_s(z), z^\nu W_s(z)\rangle = \begin{cases} 0 & \text{if } k \neq \nu \\ 2\pi s & \text{if } k = \nu. \end{cases}$$

Thus, the polynomials

$$b_\nu(z) := z^\nu W_s(z)/\sqrt{2\pi s}, \quad \nu = 0, 1, \ldots, n-1, \tag{2.2}$$

form an orthonormal basis of $W_s\pi_{n-1}$. Now we can describe the solution of (P5) in the following theorem:

THEOREM 2.1. *The optimal solution of the problem (P5) can be expressed as*

$$S^*_{N+n-1}(z,f) = Q^*(z,f) + \frac{W_s(z)}{\sqrt{2\pi s}} \sum_{j=0}^{n-1} a_j^* z^j \tag{2.3}$$

where $Q^*(z,f)$ and $g(z)$ are defined in (1.5) and

$$a_j^* := s\langle g, b_j\rangle, \quad j = 0, 1, \ldots, n-1. \tag{2.4}$$

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PROOF. First we note that any polynomial $p \in \mathcal{L}(f, U_{s,q})$ can be written as

$$p(z) = Q^*(z,f) + W_s(z)t(z) \tag{2.5}$$

for some $t \in \pi_{n-1}$. Thus, the problem (P5) is equivalent to the problem

$$\min_{t\in\pi_{n-1}} \int_{|z|=1} |sg(z) - W_s(z)t(z)|^2 |dz|. \tag{2.6}$$

Since $\{b_k(z) : k = 0, 1, \ldots, n-1\}$ forms an orthonormal basis of $W_s\pi_{n-1}$ (*cf.* (2.2)),we have another equivalent form of (P5) as follows:

$$\min_{\substack{a_k\in C\\ k=0,1,\ldots,n-1}} \int_{|z|=1} |sg(z) - \sum_{k=0}^{n-1} a_k b_k(z)|^2 |dz|. \tag{2.7}$$

The optimal values a_k^*, $k = 0, 1, \ldots, n-1$, for this problem, due to an application of Riesz-Fischer Theorem [9], are given by

$$a_k^* = \langle sg, b_k\rangle, \quad k = 0, 1, \ldots, n-1. \tag{2.8}$$

Now (2.5) and (2.8) provide us the optimal solution of (P5) as claimed in (2.3).

3. **Convergence problem.** This section deals with the convergence problem of the sequence (1.7). Our main result is as follows:

THEOREM 3.1. *Let $s \geq 1$ and $q \geq 1$ be fixed integers. If $f \in A_\rho$, $\rho > 1$, then (cf. (1.7))*

$$\lim_{n\to\infty} \{L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f)\} = 0, \quad \forall z \in D_\lambda \tag{3.1}$$

where $\lambda := \rho^{1+q/((s-1)q+1)}$ and $N := nq(s-1)$. The convergence in (3.1) is uniform and geometric in any compact subset of the region D_λ. More precisely, for any τ with $\rho < \tau < \infty$, there holds

$$\overline{\lim_{n\to\infty}} \{\max_{|z|\leq\tau} |L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f)|\}^{\frac{1}{n}} \leq \frac{\tau^{(s-1)q+1}}{\rho^{sq+1}}. \tag{3.2}$$

Moreover, the result is sharp in the sense that (3.1) fails for every z satisfying $|z| = \lambda$ for an $f \in A_\rho$.

PROOF. First we note that (*cf.* (1.4) and (2.3))

$$L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f) = W_s(z)\Big\{S_{n-1}\big(z, L_{qn-1}(z,g)\big) - \frac{1}{\sqrt{2\pi s}} \sum_{k=0}^{n-1} a_k^* z^k\Big\}.$$

We have

$$L_{qn-1}(z,g) = A(z) - z^{qn}B(z)$$

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where

$$A(z) := \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(t)t^{qn}}{(t^{qn}-1)(t-z)}\,dt, \quad B(z) := \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(t)}{(t^{qn}-1)(t-z)}\,dt$$

are holomorphic in $|z| < R$. Hence

$$S_{n-1}\big(z, L_{qn-1}(z,g)\big) = S_{n-1}(z,A).$$

It is clear that $A(z) = \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu$ where

$$\alpha_\nu := \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(t)t^{qn-\nu-1}}{t^{qn}-1}\,dt, \quad (\nu = 0,1,2,\ldots).$$

Hence

$$\begin{aligned} S_{n-1}\big(z, L_{qn-1}(z,g)\big) = \sum_{\nu=0}^{n-1} \alpha_\nu z^\nu &= \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(t)t^{qn-n}}{t^{qn}-1}\Big(\sum_{\nu=0}^{n-1} t^{n-1-\nu}z^\nu\Big)\,dt \\ &= \frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(t)t^{qn-n}(t^n-z^n)}{(t^{qn}-1)(t-z)}\,dt \end{aligned}$$

where Γ_R is a circle $|t| = R$ with $1 < R < \rho$. In view of the relations (2.4), (2.2) and (2.1) it turns out that

$$\sum_{k=0}^{n-1} a_k^* z^k = \frac{\sqrt{s}}{\sqrt{2\pi} i}\int_{\Gamma_R} \frac{g(t)W_s(t)(t^n-z^n)}{t^{(s-1)qn+n}(t-z)}\,dt.$$

Now we can write

$$L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f) = \frac{W_s(z)}{2\pi i}\int_{\Gamma_R} g(t)k(t,z)\,dt$$

where

$$k(t,z) = \frac{t^{qn}(t^n-z^n)}{t^{sqn+n}(t^{qn}-1)(t-z)}.$$

Since $W_s(z) = \sum_{k=0}^{s-1} z^{kqn}$, an analysis of the kernel $k(t,z)$ shows that

$$|L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f)|^{\frac{1}{n}} \le M^{\frac{1}{n}} \frac{|z|^{(s-1)q}}{R^{sq+1}} \cdot \max(R,|z|).$$

for all sufficiently large values of n. Here M is a constant independent of n. Thus for $|z| = \tau \ge \rho$, we have

$$\text{(3.3)} \qquad \varlimsup_{n\to\infty}\Big\{\max_{|z|=\tau} |L^*_{N+n-1}(z,f) - S^*_{N+n-1}(z,f)|\Big\}^{\frac{1}{n}} \le \frac{\tau^{(s-1)q+1}}{R^{sq+1}}.$$

But, as the left side of (3.3) is independent of the choice of R with $1 < R < \rho$, we arrive at the desired conclusion (3.1).

As for part (ii), we need only consider $f(z) = (\rho - z)^{-1}$ and carry out a straightforward computation to see that (3.1) fails for this function when $z \in \bar{D}_\lambda$.

REMARK 3.1. Theorem 3.1 provides a generalization of Rivlin's result [4, Theorem 1]. For this, choose $s = 1$ in (3.1). If, in addition, we select $q = 1$ in (3.1), we obtain Walsh's result [9, p. 153].

4. **An extension of main result and related problems.** To motivate a generalization of Theorem 3.1, consider $f(z) = \sum_{j=i}^{\infty} a_j z^j$ in A_ρ and set

$$S_{m,\ell}(z,f) = \sum_{j=0}^{\ell-1} \sum_{k=0}^{m} a_{k+j(m+1)} z^k \quad \ell = 1, 2, \ldots, \tag{4.1}$$

so that $S_{m,\ell}(z,f) \in \pi_m$ for each $\ell \geq 1$. The following known result gives Walsh's Theorem (*cf.* (1.1)) as a special case $\ell = 1$:

THEOREM B ([1]). *For each $f \in A_\rho$ and for each positive integer ℓ the sequence*

$$\{L_m(z,f) - S_{m,\ell}(z,f)\}_{m=1}^{\infty}$$

converges to zero as $m \longrightarrow \infty$ for all $|z| < \rho^{\ell+1}$.

To deduce an analogue of Theorem B for Theorem 3.1, consider any three positive integers s, q, ℓ and set

$$S^*_{N+n-1,\ell}(z,f) := Q^*(z,f) + W_s(z) \sum_{j=0}^{\ell-1} \tilde{p}_{n-1,j}(z,f) \tag{4.2}$$

where $Q^*(z,f)$ is defined in (1.5) and (*cf.* (2.4))

$$\tilde{p}_{n-1,j}(,f) := \sum_{k=0}^{n-1} a^*_{k+jsqn} z^k \quad j = 0, 1, \ldots, l-1.$$

It may be noted that $S^*_{N+n-1,1}(z,f) = S^*_{N+n-1}(z,f)$. Now we state the following result which generalizes Theorem 3.1 as well as Theorem B.

THEOREM 4.1. *For each $f \in A_\rho$ and for any positive integers s, q, ℓ, let the polynomials $L^*_{N+n-1}(z,f)$ and $S^*_{N+n-1,\ell}(z,f)$ be defined as in (1.4) and (4.2). Then the sequence*

$$\{L^*_{N+n-1}(z,f) - S^*_{N+n-1,\ell}(z,f)\}_{n=1}^{\infty} \tag{4.3}$$

converges to zero as $n \longrightarrow \infty$ for all $z \in D_\delta$ where $\delta := \rho^{1+\ell q/((s-1)q+1)}$. The convergence is also uniform and geometric on any closed subset of D_δ. Moreover, the result is sharp in the sense of Theorem 3.1.

We omit the proof of this theorem as it follows along the lines of the proof of Theorem 3.1.

It may be noted that Theorem 4.1 provides Theorem 3.1 as the special case $\ell = 1$ whereas the choices of $s = 1$ and $q = 1$ lead us to retrieve Theorem B.

We conclude this section with the following problems:

1. It is an open question to determine the behavior of the sequence (4.3) in the region $|z| \geq \rho^{1+\ell q/((s-1)q+1)}$.
2. Suppose that $f(z)$ is known to be analytic on $|z| < 1$ and continuous on $|z| = 1$. For $\rho \geq 1$ and fixed positive integers ℓ, s, q, assume that the sequence (4.3) is

uniformly bounded in every closed subdomain of $|z| < \rho^{1+\ell q/((s-1)q+1)}$. Does this imply that f is analytic in $|z| < \rho$?

It may be interesting to note that the problem 1 has been tackled by Saff and Varga [5], Totik [8] and Ivanov and Sharma [3] for the sequence (4.1). On the other hand, Szabados, Ivanov and Sharma have answered the second problem for the sequence (4.1) in [7] and [2].

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