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SOME CONGRUENCES FOR 12-COLOURED GENERALIZED FROBENIUS PARTITIONS

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Abstract In his 1984 AMS Memoir, Andrews introduced the family of functions $c\phi_k(n)$, the number of k-coloured generalized Frobenius partitions of n. In 2019, Chan, Wang and Yang systematically studied the arithmetic properties of $\mathrm{C}\Phi_k(q)$ for $2 \leq k \leq 17$ by utilizing the theory of modular forms, where $\mathrm{C}\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. In this paper, we first establish another expression of $\mathrm{C}\Phi_{12}(q)$ with integer coefficients, then prove some congruences modulo small powers of 3 for $c\phi_{12}(n)$ by using some parameterized identities of theta functions due to A. Alaca, S. Alaca and Williams. Finally, we conjecture three families of congruences modulo powers of 3 satisfied by $c\phi_{12}(n)$.

Keywords: congruences; generalized Frobenius partitions; generating functions; integer matrix exact covering systems; parameterized identities

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1. Introduction

Throughout this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following standard notation:

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

In his 1984 AMS Memoir, Andrews [2] defined the notion of a generalized Frobenius partition of n, which is a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

wherein each row, which is of the same length, is arranged in weakly decreasing order with $n = r + \sum_{i=1}^{r}(a_i + b_i)$. Furthermore, Andrews studied a variant of generalized Frobenius partitions whose parts are taken from k copies of the nonnegative integers, which is called k-coloured generalized Frobenius partitions. For any $k \geq 1$, let $c\phi_k(n)$ denote the number of k-coloured generalized Frobenius partitions of n. Among many other things, Andrews [2, Corollary 10.1] proved that for any $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod 5.$$

From then on, many scholars extensively investigated a number of congruence properties for $c\phi_k(n)$ with different moduli. Baruah and Sarmah [3, 4] derived some congruences modulo small powers of 2 for $c\phi_4(n)$ and some congruences modulo small powers of 3 for $c\phi_6(n)$. Congruence properties modulo powers of 5 for $c\phi_3(n)$ and $c\phi_4(n)$ were subsequently considered by Ono [28], Lovejoy [26], Xiong [39], Sellers [31], Xia [38], Hirschhorn and Sellers [21], Chan, Wang and Yang [7], and Wang and Zhang [34]. Congruence properties modulo 7 for $c\phi_4(n)$ were investigated by Lin [25] and Zhang and Wang [41]. Congruence properties of $c\phi_6(n)$ modulo powers of 3 were successively investigated by Xia [37], Hirschhorn [16], Gu, Wang and Xia [14] and the third author [32]. The third author [33] also established congruence properties modulo 5 for $c\phi_8(n)$ and $c\phi_9(n)$. There are other studies on congruence properties for $c\phi_k(n)$; see, for example, [9–13, 22–24, 27, 29, 30, 36].

In 2019, Chan, Wang and Yang [8] systematically investigated the arithmetic properties of $\mathrm{C}\Phi_k(q)$ for $2 \leq k \leq 17$, where $\mathrm{C}\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. In particular, they [8, Equation (6.26)] proved that (some typos have been corrected)

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) = \frac{1}{\Theta_3(q)(q;q)_\infty^{12}}\Big(&-\frac{36207}{160}B_{12,1} + \frac{923091}{4000}B_{12,4} + \frac{35829}{100}B_{12,5} + \frac{891}{4}B_{12,6}\\ &- \frac{1485}{8}B_{12,7} - \frac{143247}{250}B_{12,8} - \frac{891}{4}B_{12,9} - \frac{8109}{160}B_{12,10} - \frac{582717}{16000}B_{12,11}\\ &+ \frac{227691}{200}B_{12,12} + \frac{714249}{8000}B_{12,13} + \frac{8109}{80}B_{12,14} + \frac{33}{8}B_{12,15}\\ &+ \frac{294109}{500}B_{12,16} - \frac{16503}{400}B_{12,17} - \frac{99}{8}B_{12,18} + \frac{10559}{200}B_{12,19}\\ &- \frac{128807}{100}B_{12,20} + \frac{25647}{160}B_{12,21} + \frac{727}{160}B_{12,22}\Big),\end{aligned} \tag{1.1}$$

where the $B_{12,i}$ for $i \in \{1, 4, 5, \ldots, 22\}$ are some functions involving the following two theta functions, given by

$$\Theta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2} = 2q^{1/4}\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty},$$

$$\Theta_3(q) = \sum_{j=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}.$$

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It is easy to see that the coefficients of many terms in (1.1) are not integers. Therefore, a natural question is whether there is another expression with integral coefficients for $\mathrm{C}\Phi_{12}(q)$. The first purpose of this paper is to establish the following expression for $\mathrm{C}\Phi_{12}(q)$. For the sake of convenience, denote

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \qquad \text{and} \qquad E(q^k) := (q^k;q^k)_\infty. \tag{1.2}$$

Theorem 1.1.

$$\begin{aligned}\mathrm{C}\Phi_{12}(q) = \frac{1}{E(q)^{12}}\Bigg\{ & a(q)^4\left(\frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2} + 8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}\right)\\ & + 108qa(q)^2\frac{E(q^3)^6}{E(q)^2}\left(\frac{E(q^4)E(q^6)^4E(q^8)}{E(q^2)^2E(q^{24})} + q\frac{E(q^2)^3E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^6)E(q^8)}\right)\\ & + 216q^2a(q)\frac{E(q^3)^9}{E(q)^3}\left(\frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2} + 2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}\right)\\ & + 486q^2\frac{E(q^3)^{12}}{E(q)^4}\left(\frac{E(q^4)^2E(q^6)^{11}E(q^{24})}{E(q^2)E(q^3)^4E(q^8)E(q^{12})^5} + 4q\frac{E(q^8)E(q^{12})^6}{E(q^4)E(q^6)^2E(q^{24})}\right)\Bigg\}.\end{aligned} \tag{1.3}$$

By utilizing a general congruence relation [8, Theorem 5.3], Chan et al. [8, Equations (6.28) and (6.29)] derived that for any $n \geq 0$,

$$\begin{aligned} c\phi_{12}(3n+1) &\equiv 0 \pmod{9},\\ c\phi_{12}(3n+2) &\equiv 0 \pmod{9}. \end{aligned} \tag{1.4}$$

The other purpose of this paper is to prove the following congruences modulo 27 and 81 enjoyed by $c\phi_{12}(n)$.

Theorem 1.2. *For any $n \geq 0$,*

$$c\phi_{12}(3n+2) \equiv 0 \pmod{27}, \tag{1.5}$$

$$c\phi_{12}(9n+5) \equiv 0 \pmod{81}, \tag{1.6}$$

$$c\phi_{12}(9n+8) \equiv 0 \pmod{81}. \tag{1.7}$$

Remark 1.3. Obviously, (1.5) is a stronger form of (1.4). By computation, one sees that $c\phi_{12}(2) = 4644 \not\equiv 0 \pmod{81}$. From this perspective, the modulus in (1.5) is best possible. So does (1.6) and (1.7).

Actually, (1.5)–(1.7) appear to be just the tip of the iceberg. With the help of a computer, we pose the following three families of conjectural congruences modulo powers of 3 satisfied by $c\phi_{12}(n)$.

Conjecture 1.4. For any $n \geq 0$ and $\alpha \geq 0$,

$$c\phi_{12}\left(3^{2\alpha+1}n + \frac{3^{2\alpha+1}+1}{2}\right) \equiv 0 \pmod{3^{3\alpha+3}},$$
$$c\phi_{12}\left(3^{2\alpha+2}n + \frac{3^{2\alpha+2}+1}{2}\right) \equiv 0 \pmod{3^{3\alpha+4}},$$
$$c\phi_{12}\left(3^{2\alpha+2}n + \frac{5\times 3^{2\alpha+1}+1}{2}\right) \equiv 0 \pmod{3^{3\alpha+4}}.$$

The rest of this paper is organized as follows. In § 2, we collect some necessary lemmas which will be utilized to prove the main results later. The proofs of Theorems 1.1 and 1.2 are presented in § 3.

2. Some preliminary results

To prove (1.5)–(1.7), we first collect some necessary identities.

Lemma 2.1.

$$E(q)^4 = \frac{E(q^4)^{10}}{E(q^2)^2E(q^8)^4} - 4q\frac{E(q^2)^2E(q^8)^4}{E(q^4)^2}, \tag{2.1}$$

$$\frac{1}{E(q)^4} = \frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}}. \tag{2.2}$$

Proof. The identities (2.1) and (2.2) are (2.9) and (2.10) in [40], respectively. □

Lemma 2.2.

$$\frac{E(q^2)^5}{E(q)^2E(q^4)^2} = \frac{E(q^{18})^5}{E(q^9)^2E(q^{36})^2} + 2q\frac{E(q^6)^2E(q^9)E(q^{36})}{E(q^3)E(q^{12})E(q^{18})}, \tag{2.3}$$

$$\frac{E(q^2)^2}{E(q)} = \frac{E(q^6)E(q^9)^2}{E(q^3)E(q^{18})} + q\frac{E(q^{18})^2}{E(q^9)}. \tag{2.4}$$

Proof. The identities (2.3) and (2.4) follow from Corollary (i) and (ii) on page 49 of Berndt's book [5], respectively. □

Lemma 2.3.

$$\frac{E(q^2)}{E(q)^2} = \frac{E(q^6)^4E(q^9)^6}{E(q^3)^8E(q^{18})^3} + 2q\frac{E(q^6)^3E(q^9)^3}{E(q^3)^7} + 4q^2\frac{E(q^6)^2E(q^{18})^3}{E(q^3)^6}, \tag{2.5}$$

$$\frac{E(q)}{E(q^2)^2} = \frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}, \tag{2.6}$$

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$$\frac{E(q^2)}{E(q)E(q^4)} = \frac{E(q^{18})^9}{E(q^3)^2E(q^9)^3E(q^{12})^2E(q^{36})^3} + q\frac{E(q^6)^2E(q^{18})^3}{E(q^3)^3E(q^{12})^3} + q^2\frac{E(q^6)^4E(q^9)^3E(q^{36})^3}{E(q^3)^4E(q^{12})^4E(q^{18})^3}. \tag{2.7}$$

Proof. The identity (2.5) was derived by Hirschhorn and Sellers [19, Theorem 1.1]. The identity (2.6) is equivalent to Lemma 2.2 due to Hirschhorn and Sellers [20]. Moreover, replacing q by $-q$ in (2.6) and utilizing the fact

$$E(-q) = \frac{E(q^2)^3}{E(q)E(q^4)},$$

upon simplification, we obtain (2.7). □

Lemma 2.4. *If $a(q)$ is defined by (1.2), then*

$$a(q) = a(q^3) + 6q\frac{E(q^9)^3}{E(q^3)} \tag{2.8}$$

and

$$\frac{1}{E(q)^3} = \frac{E(q^9)^3}{E(q^3)^{12}}\big(a(q^3)^2E(q^3)^2 + 3qa(q^3)E(q^3)E(q^9)^3 + 9q^2E(q^9)^6\big). \tag{2.9}$$

Proof. The identity (2.8) was established by Hirschhorn, Garvan and Borwein [18, Equation (1.3)]. The identity (2.9) was proved by Wang [35, Equation (2.28)]. □

Hirschhorn et al. [18, Equation (1.5)] also proved that

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \sum_{n=1}^{\infty}\frac{q^{3n-1}}{1-q^{3n-1}}\right),$$

from which we find that

$$a(q) \equiv 1 \pmod 3 \qquad \text{and} \qquad a(q)^3 \equiv 1 \pmod 9. \tag{2.10}$$

According to the binomial theorem, one can easily establish the following congruence, which will be used frequently in the sequel.

Lemma 2.5. *For any $k \geq 1$,*

$$E(q^k)^3 \equiv E(q^{3k}) \pmod 3. \tag{2.11}$$

3. Proofs of the main results

To prove Theorem 1.1, we require the following two related lemmas.

Lemma 3.1.

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3} r_i^2+3\sum_{1\le i<j\le 3} r_ir_j} = \frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2} + 8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2}, \tag{3.1}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3} r_i^2+3\sum_{1\le i<j\le 3} r_ir_j+r_1+r_2+2r_3} = \frac{E(q^4)E(q^6)^4E(q^8)}{E(q^2)^2E(q^{24})} + q\frac{E(q^2)^3E(q^3)^2E(q^{12})E(q^{24})}{E(q)^2E(q^6)E(q^8)}, \tag{3.2}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3} r_i^2+3\sum_{1\le i<j\le 3} r_ir_j+3r_1+3r_2+2r_3} = \frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2} + 2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}, \tag{3.3}$$

$$\sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{3\sum_{i=1}^{3} r_i^2+3r_1r_2-3r_1r_3-3r_2r_3+2r_1+2r_2} = \frac{E(q^4)^2E(q^6)^{11}E(q^{24})}{E(q^2)E(q^3)^4E(q^8)E(q^{12})^5} + 4q\frac{E(q^8)E(q^{12})^6}{E(q^4)E(q^6)^2E(q^{24})}. \tag{3.4}$$

Proof. The main ingredient in proofs of (3.1)–(3.4) is the integer matrix exact covering systems, developed by Cao [6]. Similar treatments were used for deriving the generating functions of four- and six-coloured generalized Frobenius partitions; see [3, 4] for a detailed account of such applications.

We only present the proof of (3.1), and the remaining cases can be demonstrated in a similar manner.

First, we adopt the matrix

$$B_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then, the congruences $B_1\mathbf{r} \equiv 0 \pmod 2$ satisfy that

$$\begin{cases} -r_1 + r_2 \equiv 0 & \pmod 2, \\ r_1 + r_3 \equiv 0 & \pmod 2, \\ r_1 - r_3 \equiv 0 & \pmod 2. \end{cases}$$

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Then, the above congruences contain two solutions. Namely, $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ modulo 2. Therefore, we get the following integer matrix exact covering systems

$$\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}=\begin{pmatrix}-1&1&0\\1&0&1\\1&0&-1\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix},$$

$$\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}=\begin{pmatrix}-1&1&0\\1&0&1\\1&0&-1\end{pmatrix}\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Using the above integer matrix exact covering systems, we obtain that

$$\begin{aligned}\sum_{r_1,r_2,r_3=-\infty}^{\infty} & q^{3\sum_{i=1}^{3}r_i^2+3\sum_{1\le i<j\le 3}r_ir_j}\\ &=\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{6n_1^2+3n_2^2+3n_3^2}+\sum_{n_1,n_2,n_3=-\infty}^{\infty}q^{6n_1^2+6n_1+3n_2^2+3n_2+3n_3^2+3n_3+3}\\ &=\frac{E(q^6)^8E(q^{12})}{E(q^3)^4E(q^{24})^2}+8q^3\frac{E(q^{12})^3E(q^{24})^2}{E(q^6)^2},\end{aligned}$$

which is nothing but (3.1). For (3.2) and (3.3), we also adopt the matrix B_1 and utilizing a similar strategy. However, for (3.4), we need the following matrix

$$B_2=\begin{pmatrix}1&0&1\\1&0&-1\\1&1&0\end{pmatrix}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *The constant term of* $\left(\sum_{r=-\infty}^{\infty}a^rq^{r^2}\right)^{12}$ *is*

$$\begin{aligned}\mathrm{CT}_a & \left(\left(\sum_{r=-\infty}^{\infty}a^rq^{r^2}\right)^{12}\right)\\ &=a(q^2)^4\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j}\\ &\quad+108qa(q^2)^2\frac{E(q^6)^6}{E(q^2)^2}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j+2r_1+2r_2+4r_3}\end{aligned}$$

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$$+216q^4a(q^2)\frac{E(q^6)^9}{E(q^3)^3}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j+6r_1+6r_2+4r_3}$$
$$+486q^4\frac{E(q^6)^{12}}{E(q^2)^4}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6r_1r_2-6r_1r_3-6r_2r_3+4r_1+4r_2}. \tag{3.5}$$

Proof. Hirschhorn [15] proved the following identity

$$\left(\sum_{r=-\infty}^{\infty}a^rq^{r^2}\right)^3=a(q^2)\sum_{r=-\infty}^{\infty}a^{3r}q^{3r^2}$$
$$+3q\frac{E(q^6)^3}{E(q^2)}\left(a\sum_{r=-\infty}^{\infty}a^{3r}q^{3r^2+2r}+a^{-1}\sum_{r=-\infty}^{\infty}a^{-3r}q^{3r^2+2r}\right), \tag{3.6}$$

where $a(q)$ is defined as in (1.2). With the help of (3.6), we obtain that

$$\mathrm{CT}_a\left(\left(\sum_{r=-\infty}^{\infty}a^rq^{r^2}\right)^{12}\right)$$
$$=a(q^2)^4\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j}$$
$$+108q^2a(q^2)^2\frac{E(q^6)^6}{E(q^2)^2}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j+2r_1+2r_2+4r_3}$$
$$+216q^4a(q^2)\frac{E(q^6)^9}{E(q^3)^3}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6\sum_{1\le i<j\le 3}r_ir_j+6r_1+6r_2+4r_3}$$
$$+486q^4\frac{E(q^6)^{12}}{E(q^2)^4}\sum_{r_1,r_2,r_3=-\infty}^{\infty}q^{6\sum_{i=1}^{3}r_i^2+6r_1r_2-6r_1r_3-6r_2r_3+4r_1+4r_2},$$

which is nothing but (3.5).

We therefore complete the proof of Lemma 3.2. □

Now, it is time to prove Theorem 1.1.

Proof of Theorem 1.1. In view of (3.5), we deduce that

$$\mathrm{CT}_a\left(\left(\sum_{r=-\infty}^{\infty}a^rq^{r^2}\right)^{12}\right)$$
$$=\sum_{\substack{m_1+m_2+\cdots+m_{12}=0\\ m_1,m_2,\ldots,m_{12}=-\infty}}^{\infty}q^{\sum_{i=1}^{12}m_i^2}$$

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$$
\begin{aligned}
&= \sum_{m_1,\ldots,m_{11}=-\infty}^{\infty} q^{2\sum_{i=1}^{11} m_i^2 + 2\sum_{1\le i<j\le 11} m_i m_j} \\
&= a(q^2)^4 \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2 + 6\sum_{1\le i<j\le 3} r_i r_j} \\
&\quad + 108q^2 a(q^2)^2 \frac{E(q^6)^6}{E(q^2)^2} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2 + 6\sum_{1\le i<j\le 3} r_i r_j + 2r_1 + 2r_2 + 4r_3} \\
&\quad + 216q^4 a(q^2) \frac{E(q^6)^9}{E(q^2)^3} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2 + 6\sum_{1\le i<j\le 3} r_i r_j + 6r_1 + 6r_2 + 4r_3} \\
&\quad + 486q^4 \frac{E(q^6)^{12}}{E(q^2)^4} \sum_{r_1,r_2,r_3=-\infty}^{\infty} q^{6\sum_{i=1}^{3} r_i^2 + 6r_1r_2 - 6r_1r_3 - 6r_2r_3 + 4r_1 + 4r_2}. \qquad (3.7)
\end{aligned}
$$

Moreover, Andrews [2, Theorem 5.2] established the following expression for $\mathrm{C}\Phi_k(q)$, namely,

$$
\mathrm{C}\Phi_k(q) = \frac{1}{E(q)^k} \sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty} q^{\sum_{i=1}^{k-1} m_i^2 + \sum_{1\le i<j\le k-1} m_i m_j}. \qquad (3.8)
$$

The identity (1.3) follows from (3.1)–(3.4), (3.7) and (3.8).

This finishes the proof of Theorem 1.1. □

Next, we are in a position to prove Theorem 1.2.

In what follows, all congruences are modulo 81 unless otherwise specified.

Proof of Theorem 1.2. According to (2.10) and (2.11), we find that

$$
\begin{aligned}
\mathrm{C}\Phi_{12}(q) \equiv{}& \frac{a(q)^4}{E(q)^{12}} \left(\frac{E(q^6)^8 E(q^{12})}{E(q^3)^4 E(q^{24})^2} + 8q^3 \frac{E(q^{12})^3 E(q^{24})^2}{E(q^6)^2} \right) \\
&+ 27q \frac{E(q^3)^2}{E(q)^2} \left(\frac{E(q^4)E(q^6)^4}{E(q^2)^2 E(q^8)^2} + q \frac{E(q^3)^2 E(q^{12}) E(q^{24})}{E(q)^2 E(q^8)} \right) \\
&+ 54q^2 E(q^3)^4 \left(\frac{E(q^2)E(q^4)E(q^6)^5 E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2} + 2 \frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})} \right). \qquad (3.9)
\end{aligned}
$$

Next, we consider the following three auxiliary functions, defined by

$$
\sum_{n=0}^{\infty} g_1(n) q^n := \frac{a(q)^4}{E(q)^{12}} \left(\frac{E(q^6)^8 E(q^{12})}{E(q^3)^4 E(q^{24})^2} + 8q^3 \frac{E(q^{12})^3 E(q^{24})^2}{E(q^6)^2} \right), \qquad (3.10)
$$

$$
\sum_{n=0}^{\infty} g_2(n) q^n := 27q \frac{E(q^3)^2}{E(q)^2} \left(\frac{E(q^4)E(q^6)^4}{E(q^2)^2 E(q^8)^2} + q \frac{E(q^3)^2 E(q^{12}) E(q^{24})}{E(q)^2 E(q^8)} \right), \qquad (3.11)
$$

$$\sum_{n=0}^{\infty} g_3(n)q^n := 54q^2E(q^3)^4\left(\frac{E(q^2)E(q^4)E(q^6)^5E(q^{24})}{E(q)E(q^3)E(q^8)E(q^{12})^2}+2\frac{E(q^6)E(q^8)E(q^{12})^3}{E(q^2)E(q^{24})}\right). \quad (3.12)$$

Substituting (2.8) and (2.9) into (3.10), extracting all the terms of the form q^{3n+2}, after simplification, we deduce that

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n \equiv 27a(q)^{10}\frac{E(q^2)^8E(q^3)^{18}E(q^4)}{E(q)^{46}E(q^8)^2}+54qa(q)^{10}\frac{E(q^3)^{18}E(q^4)^3E(q^8)^2}{E(q)^{42}E(q^2)^2}.$$

Thanks to (2.10) and (2.11),

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n \equiv 27E(q^3)^4\left(\frac{E(q^2)^8E(q^4)}{E(q)^4E(q^8)^2}+2q\frac{E(q^4)^3E(q^8)^2}{E(q^2)^2}\right). \quad (3.13)$$

The congruence (1.5) follows from (3.9) and (3.13) immediately.

Moreover, it follows from (3.13) that

$$\sum_{n=0}^{\infty} g_1(3n+2)q^n \equiv 27E(q^3)^4\left(\frac{E(q^2)^{10}}{E(q)^4E(q^4)^4}\cdot\frac{E(q^4)^5}{E(q^2)^2E(q^8)^2}+2q\frac{E(q^4)^4}{E(q^2)^2}\cdot\frac{E(q^8)^2}{E(q^4)}\right). \quad (3.14)$$

Substituting (2.3) and (2.4) into (3.14), after some tedious but straightforward calculations, we deduce that

$$\sum_{n=0}^{\infty} g_1(9n+5)q^n \equiv 27\left\{\frac{E(q^2)^8E(q^4)^{11}}{E(q^8)^6}-\frac{E(q^2)^{10}E(q^4)}{E(q^8)^2}\cdot E(q)^4 - qE(q^4)^3E(q^8)^2\cdot\left(E(q)^4\right)^2+q^2E(q^2)^2E(q^4)E(q^8)^6\cdot E(q)^4\right\} \quad (3.15)$$

and

$$\sum_{n=0}^{\infty} g_1(9n+8)q^n \equiv 27\left\{\frac{E(q^4)^{19}}{E(q^2)^8E(q^8)^6}\cdot\left(E(q)^4\right)^2-\frac{E(q^2)^{32}E(q^8)^2}{E(q^4)^{13}}\cdot\left(\frac{1}{E(q)^4}\right)^2 - q\frac{E(q^4)^{17}}{E(q^2)^6E(q^8)^2}\cdot E(q)^4-q\frac{E(q^2)^{10}E(q^8)^6}{E(q^4)^7}\cdot E(q)^4\right\}. \quad (3.16)$$

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Substituting (2.1) and (2.2) into (31.5) and (3.16), upon simplification, we further obtain that

$$\sum_{n=0}^{\infty} g_1(9n+5)q^n \equiv 27\left(-q\frac{E(q^4)^{23}}{E(q^2)^4E(q^8)^6} + q\frac{E(q^2)^{12}E(q^8)^2}{E(q^4)} + q^3\frac{E(q^2)^4E(q^8)^{10}}{E(q^4)}\right), \tag{3.17}$$

$$\sum_{n=0}^{\infty} g_1(9n+8)q^n \equiv 27\left(\frac{E(q^4)^{39}}{E(q^2)^{12}E(q^8)^{14}} - \frac{E(q^2)^4E(q^4)^{15}}{E(q^8)^6} - q^2\frac{E(q^4)^{15}E(q^8)^2}{E(q^2)^4}\right). \tag{3.18}$$

Now, we recall Horschhorn's version of parameterized identities (see [17, Chapter 35, Equations (35.1.1)–(35.1.6)]), whose idea comes from [1].

$$E(q) = s^{1/2}t^{1/24}(1-2qt)^{1/2}(1+qt)^{1/8}(1+2qt)^{1/6}(1+4qt)^{1/8}, \tag{3.19}$$

$$E(q^2) = s^{1/2}t^{1/12}(1-2qt)^{1/4}(1+qt)^{1/4}(1+2qt)^{1/12}(1+4qt)^{1/4}, \tag{3.20}$$

$$E(q^3) = s^{1/2}t^{1/8}(1-2qt)^{1/6}(1+qt)^{1/24}(1+2qt)^{1/2}(1+4qt)^{1/24}, \tag{3.21}$$

$$E(q^4) = s^{1/2}t^{1/6}(1-2qt)^{1/8}(1+qt)^{1/2}(1+2qt)^{1/24}(1+4qt)^{1/8}, \tag{3.22}$$

$$E(q^6) = s^{1/2}t^{1/4}(1-2qt)^{1/12}(1+qt)^{1/12}(1+2qt)^{1/4}(1+4qt)^{1/12}, \tag{3.23}$$

$$E(q^{12}) = s^{1/2}t^{1/2}(1-2qt)^{1/24}(1+qt)^{1/6}(1+2qt)^{1/8}(1+4qt)^{1/24}, \tag{3.24}$$

where

$$s := s(q) = \frac{E(q)^2E(q^4)^2E(q^6)^{15}}{E(q^2)^5E(q^3)^6E(q^{12})^6} \qquad \text{and} \qquad t := t(q) = \frac{E(q^2)^3E(q^3)^3E(q^{12})^6}{E(q)E(q^4)^2E(q^6)^9}.$$

It follows immediately from the parameterized identities (3.19)–(3.24) that

$$\left(-\frac{E(q^2)^{23}}{E(q)^4E(q^4)^6} + \frac{E(q)^{12}E(q^4)^2}{E(q^2)} + q\frac{E(q)^4E(q^4)^{10}}{E(q^2)}\right)\cdot\frac{E(q)^3E(q^4)^2E(q^6)^{12}}{E(q^2)^7E(q^3)^5E(q^{12})^4}$$
$$= -15qs^7t^2(1-2qt)^3(1+qt)^5(1+2qt)(1+4qt) \equiv 0 \pmod 3$$

and

$$\left(\frac{E(q^2)^{39}}{E(q)^{12}E(q^4)^{14}} - \frac{E(q)^4E(q^2)^{15}}{E(q^4)^6} - q\frac{E(q^2)^{15}E(q^4)^2}{E(q)^4}\right)\cdot\frac{E(q)^3E(q^4)^3E(q^6)^{15}}{E(q^2)^8E(q^3)^5E(q^{12})^7}$$
$$= 15qs^7t(1-2qt)^2(1+qt)^4(1+2qt)(1+4qt)^3 \equiv 0 \pmod 3).$$

Since

$$\frac{E(q)^3E(q^4)^2E(q^6)^{12}}{E(q^2)^7E(q^3)^5E(q^{12})^4} \qquad \text{and} \qquad \frac{E(q)^3E(q^4)^3E(q^6)^{15}}{E(q^2)^8E(q^3)^5E(q^{12})^7}$$

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are invertible in the ring $\mathbb{Z}/3\mathbb{Z}[[q]]$, we deduce that

$$-\frac{E(q^2)^{23}}{E(q)^4E(q^4)^6}+\frac{E(q)^{12}E(q^4)^2}{E(q^2)}+q\frac{E(q)^4E(q^4)^{10}}{E(q^2)}\equiv 0 \pmod 3, \tag{3.25}$$

$$\frac{E(q^2)^{39}}{E(q)^{12}E(q^4)^{14}}-\frac{E(q)^4E(q^2)^{15}}{E(q^4)^6}-q\frac{E(q^2)^{15}E(q^4)^2}{E(q)^4}\equiv 0 \pmod 3. \tag{3.26}$$

The congruences (3.17) and (3.18), together with (3.25) and (3.26), imply that for any $n\geq 0$,

$$g_1(9n+5)\equiv g_1(9n+8)\equiv 0. \tag{3.27}$$

Similarly, from (3.11), we find that

$$\sum_{n=0}^{\infty} g_2(n)q^n \equiv 27\Big(qE(q^3)E(q^6)^4\cdot\frac{E(q)}{E(q^2)^2}\cdot\frac{E(q^4)}{E(q^8)^2} + q^2E(q^3)^3E(q^{12})E(q^{24})\cdot\frac{E(q^2)}{E(q)E(q^4)}\cdot\frac{E(q^4)}{E(q^2)E(q^8)}\Big).$$

With the help of (2.6) and (2.7), we further obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} g_2(3n+2)q^n \equiv 27\Big(&-\frac{E(q)^4E(q^2)^6E(q^4)^{11}}{E(q^8)^6}+\frac{E(q^2)^{16}E(q^4)^{17}}{E(q)^8E(q^8)^{10}}\\ &+q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2}-q\frac{E(q)^{12}E(q^4)^3E(q^8)^2}{E(q^2)^2}\\ &+q^2E(q)^8E(q^4)E(q^8)^6+q^3\frac{E(q^2)^{14}E(q^8)^{10}}{E(q)^4E(q^4)^5}\Big).\end{aligned} \tag{3.28}$$

Plugging (2.1) and (2.2) into (3.28), after simplification, we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} g_2(3n+2)q^n \equiv 27\Big(&\frac{E(q^4)^{45}}{E(q^2)^{12}E(q^8)^{18}}-\frac{E(q^2)^4E(q^4)^{21}}{E(q^8)^{10}}+q\frac{E(q^4)^{33}}{E(q^2)^8E(q^8)^{10}}\\ &-q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2}-q^2\frac{E(q^4)^{21}}{E(q^2)^4E(q^8)^2}-q^3E(q^4)^9E(q^8)^6\Big).\end{aligned} \tag{3.29}$$

According to the parameterized identities (3.19)–(3.24), we find that

$$\begin{aligned}&\left(\frac{E(q^2)^{45}}{E(q)^{12}E(q^4)^{18}}-\frac{E(q)^4E(q^2)^{21}}{E(q^4)^{10}}-q\frac{E(q^2)^{21}}{E(q)^4E(q^4)^2}\right)\cdot\frac{E(q)^3E(q^4)^4E(q^6)^{13}}{E(q^2)^8E(q^3)^5E(q^{12})^6}\\ &\quad= 15qs^8t(1-2qt)^3(1+qt)^4(1+2qt)(1+4qt)^4\equiv 0 \pmod 3\end{aligned}$$

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and

$$\left(\frac{E(q^2)^{33}}{E(q)^8E(q^4)^{10}} - \frac{E(q)^8E(q^2)^9}{E(q^4)^2} - qE(q^2)^9E(q^4)^6\right)\cdot\frac{E(q)E(q^6)^7}{E(q^2)^2E(q^3)^3E(q^{12})^2}$$
$$= 15qs^8t^2(1-2qt)^3(1+qt)^5(1+2qt)(1+4qt)^3 \equiv 0 \pmod 3.$$

Since

$$\frac{E(q)^3E(q^4)^2E(q^6)^{13}}{E(q^2)^8E(q^3)^5E(q^{12})^6} \quad \text{and} \quad \frac{E(q)E(q^6)^7}{E(q^2)^2E(q^3)^3E(q^{12})^2}$$

are invertible in the ring $\mathbb{Z}/3\mathbb{Z}[[q]]$, we obtain that

$$\frac{E(q^2)^{45}}{E(q)^{12}E(q^4)^{18}} - \frac{E(q)^4E(q^2)^{21}}{E(q^4)^{10}} - q\frac{E(q^2)^{21}}{E(q)^4E(q^4)^2} \equiv 0 \pmod 3, \tag{3.30}$$

$$\frac{E(q^2)^{33}}{E(q)^8E(q^4)^{10}} - \frac{E(q)^8E(q^2)^9}{E(q^4)^2} - qE(q^2)^9E(q^4)^6 \equiv 0 \pmod 3. \tag{3.31}$$

According to (3.29)–(3.31), we find that for any $n \geq 0$,

$$g_2(3n+2) \equiv 0. \tag{3.32}$$

Finally, from (3.12), we find that

$$\sum_{n=0}^{\infty} g_3(n)q^n = -27\left(q^2\frac{E(q^3)^3E(q^6)^5E(q^{24})}{E(q^{12})^2}\cdot\frac{E(q^2)^2}{E(q)}\cdot\frac{E(q^4)}{E(q^2)E(q^8)}\right.$$
$$\left. - q^2\frac{E(q^3)^4E(q^6)E(q^{12})^3}{E(q^{24})}\cdot\frac{E(q^4)^2}{E(q^2)}\cdot\frac{E(q^8)}{E(q^4)^2}\right).$$

Thanks to (2.4), (2.5), (2.7) and (2.11), we further arrive at

$$\sum_{n=0}^{\infty} g_3(3n+2)q^n \equiv 27\left(\frac{E(q)^4E(q^2)^6E(q^4)^{11}}{E(q^8)^6} - \frac{E(q)^8E(q^4)^{25}}{E(q^2)^8E(q^8)^{10}}\right.$$
$$\left. - q\frac{E(q^2)^8E(q^4)^9}{E(q^8)^2} - q^2\frac{E(q)^4E(q^4)^{11}E(q^8)^2}{E(q^2)^2}\right). \tag{3.33}$$

Substituting (2.1) into (3.33), upon simplification, we obtain that

$$\sum_{n=0}^{\infty} g_3(3n+2)q^n \equiv 27\left(-\frac{E(q^4)^{45}}{E(q^2)^{12}E(q^8)^{18}} + \frac{E(q^2)^4E(q^4)^{21}}{E(q^8)^{10}} + q^2\frac{E(q^4)^{21}}{E(q^2)^4E(q^8)^2}\right)$$
$$+ 27q\left(-\frac{E(q^4)^{33}}{E(q^2)^8E(q^8)^{10}} + \frac{E(q^2)^8E(q^4)^9}{E(q^8)^2} + q^2E(q^4)^9E(q^8)^6\right).$$

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According to (3.30) and (3.31), we conclude that for any $n \geq 0$,

$$g_3(3n+2) \equiv 0. \tag{3.34}$$

The congruences (1.6) and (1.7) follow from (3.9)–(3.12), (3.27), (3.32) and (3.34). We therefore complete the proof of Theorem 1.2. □

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