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Obstructions to $\mathcal{Z}$-Stability for Unital Simple C^*-Algebras

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Abstract. Let $\mathcal{Z}$ be the unital simple nuclear infinite dimensional C^*-algebra which has the same Elliott invariant as $\mathbb{C}$, introduced in [9]. A C^*-algebra is called $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$. In this note we give some necessary conditions for a unital simple C^*-algebra to be $\mathcal{Z}$-stable.

0 Introduction and Summary of Results

The purpose of this short note is to initiate the study a class of C^*-algebras which, we hope, will turn out to be both rich and well-behaved.

In [9], a unital simple nuclear infinite dimensional C^*-algebra $\mathcal{Z}$ is introduced. In many ways, it resembles the algebra $\mathbb{C}$ of complex numbers. In particular, $\mathcal{Z}$ has a unique tracial state, is projectionless, and is KK-equivalent to $\mathbb{C}$. It is also shown that $A \cong A \otimes \mathcal{Z}$ for a large class of simple C^*-algebras A. Note also that $\mathcal{Z}$ was proposed as a C^*-analogue of the hyperfinite factor $\mathcal{R}$ of type II_1.

We call a C^*-algebra A $\mathcal{Z}$-stable, if $A \cong A \otimes \mathcal{Z}$. An interesting question is to characterize $\mathcal{Z}$-stability. See [11] and [5] for solutions to the corresponding question in the theory of von Neumann algebras (that is, the characterization of separable factors $\mathcal{M}$ for which $\mathcal{M} \cong \mathcal{M} \otimes \mathcal{R}$).

In this note, we show some obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras.

To compare A with $A \otimes \mathcal{Z}$, it is natural to compare the known invariants. Let $\iota\colon A \to A \otimes \mathcal{Z}$ be the canonical embedding. It is quite easy to see (*cf.* Lemmas 2.11 and 2.12 of [9]) that ι induces isomorphisms between the Elliott invariants [8] of A and $A \otimes \mathcal{Z}$, except possibly the pre-ordered structures on the K_0 groups. In particular, the induced map $\iota_*\colon K_0(A) \to K_0(A \otimes \mathcal{Z})$ is a group isomorphism, but it might fail to be an isomorphism of pre-ordered groups. In Section 2, we prove the following:

Theorem 1 *Let A be a unital simple C^*-algebra. Then:*

(a) $K_0(A \otimes \mathcal{Z})$ *is weakly unperforated;*
(b) $\iota_*\colon K_0(A) \to K_0(A \otimes \mathcal{Z})$ *is an isomorphism of pre-ordered groups if and only if* $K_0(A)$ *is weakly unperforated.*

Note that there are examples of simple unital approximately homogeneous (hence separable and nuclear) C^*-algebras whose K_0 groups are not weakly unperforated (*cf.* [14]).

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Therefore, Theorem 1 provides a useful necessary condition for unital simple algebras to be $\mathcal{Z}$-stable. What is perhaps more remarkable is that this condition might be sufficient for unital separable infinite-dimensional nuclear simple C^*-algebras. This is a very interesting test case of Elliott's classification program [8].

On the other hand, there are other obstructions for C^*-algebras that are not nuclear. Let $C^*_r(\mathbb{F}_2)$ denote the reduced group C^*-algebra of the free group on two generators. It is well-known that $C^*_r(\mathbb{F}_2)$ is a unital simple C^*-algebra whose K_0 group is weakly unperforated. In Section 3, we show that $C^*_r(\mathbb{F}_2)$ is not $\mathcal{Z}$-stable. For this purpose, we propose an analogue of property Γ [12] for C^*-algebras, and show that:

Theorem 2 *Every unital $\mathcal{Z}$-stable C^*-algebra enjoys property Γ.*

It follows from a classical result in [12] that $C^*_r(\mathbb{F}_2)$ does not have property Γ.

Inspired by a paper of Rordam [13], in Section 3, we prove a dichotomy on finiteness for unital simple $\mathcal{Z}$-stable C^*-algebras:

Theorem 3 *Let A be a unital simple $\mathcal{Z}$-stable C^*-algebra. Then A is either stably finite or purely infinite.*

It is not clear, however, whether this represents a genuine obstruction: All known examples of unital simple C^*-algebras are either stably finite or purely infinite.

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1 Weak (Un)Perforation on K_0

In this section we prove Theorem 1. To establish notations, we first recall some basic facts from [9].

Notations 1.1 Let A be a unital C^*-algebra and m and n two positive integers.

1°. Let $\mathbf{M}_n$ denote the algebra of all $n \times n$ matrices, with unit $\mathbf{1}_n$ and zero element $\mathbf{0}_n$.

2°. We shall not distinguish between $\mathbf{M}_n(A)$ and $A \otimes \mathbf{M}_n$. In particular, for any $a \in A$, we have the following identification:

$$a \otimes \mathbf{1}_n = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \in \mathbf{M}_n(A).$$

3°. For $x \in \mathbf{M}_m(A)$ and $y \in \mathbf{M}_n(A)$, we write:

$$x \oplus y = \operatorname{diag}(x, y) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in \mathbf{M}_{m+n}(A).$$

4°. Let $Z_{m,n}(A)$ denote the unital C^*-algebra of all continuous functions $f\colon [0,1] \to \mathbf{M}_{mn}(A)$ with

$$f(0) = x \otimes \mathbf{1}_n, \quad \text{for some} \quad x \in \mathbf{M}_m(A),$$

and

$$f(1) = y \otimes \mathbf{1}_m, \quad \text{for some} \quad y \in \mathbf{M}_n(A).$$

When $A = \mathbb{C}$, we shall denote $Z_{m,n}(A)$ simply by $Z_{m,n}$. This is (isomorphic to) the dimension drop algebra denoted by $\mathbf{I}[m, mn, n]$ in [9]. Again, we shall not distinguish between $Z_{m,n}(A)$ and $A \otimes Z_{m,n}$.

If m and n are relatively prime, then $Z_{m,n}$ is called a prime dimension drop algebra.

The algebra $\mathcal{Z}$ constructed in [9] is the only simple C^*-algebra with a unique tracial state which is an inductive limit of prime dimension drop algebras (with unital connecting maps). In fact, it contains any prime dimension drop algebra as a unital subalgebra (*cf.* proof of Proposition 2.7 in [9]). Note also that $\mathcal{Z}$ is a nuclear C^*-algebra, since each prime dimension drop algebra is nuclear. Therefore, there is no ambiguity about $A\otimes\mathcal{Z}$, or $A\otimes Z_{m,n}$ for any C^*-algebra A.

Let A be a unital C^*-algebra, and $\iota\colon A \to A \otimes Z_{m,n}$ the unital embedding given by $\iota(a) = a \otimes 1$ for $a \in A$. The following lemma should be well-known:

Lemma 1.2 *$\iota_*\colon K_0(A) \to K_0(A \otimes Z_{m,n})$ is a group isomorphism.*

Proof We shall construct its inverse. Let $\nu_0\colon Z_{m,n}(A) \to \mathbf{M}_m(A)$ be the evaluation map at 0:

$$\nu_0(f) = x, \quad \text{if} \quad f(0) = x \otimes 1_q.$$

(*Cf.* Section 1.1.4.) Similarly, let $\nu_1\colon Z_{m,n}(A) \to \mathbf{M}_n(A)$ be the evaluation map at 1.

Furthermore, since m and n are relatively prime, there exist integers α and β such that

$$\alpha \cdot m + \beta \cdot n = 1. \tag{1}$$

Then it is straightforward to check that $\alpha \cdot (\nu_0)_* + \beta \cdot (\nu_1)_*$ is the inverse to ι_*. ■

Abusing notations, we also denote by ι the canonical embedding of A into $A \otimes \mathcal{Z}$. From the construction of $\mathcal{Z}$ and Lemma 1.2, it follows that:

***Corollary 1.3* (*cf.* Lemma 2.9 in [9])** *$\iota_*\colon K_0(A) \to K_0(A \otimes \mathcal{Z})$ is a* group *isomorphism.*

Of course, this also follows from the Kunneth Theorem.

It is natural to ask whether ι_* is actually an isomorphism of *pre-ordered groups*. The following result answers this question when A is simple.

Theorem 1.4 *Let A be a unital simple C^*-algebra and $\iota\colon A \to A \otimes \mathcal{Z}$ the canonical embedding. Suppose that $g \in K_0(A)$. Then $\iota_*(g) > 0$ if and only if $n \cdot g > 0$ for some integer $n > 0$.*

Proof If A is not stably finite, then $K_0^+(A) = K_0(A)$ (*cf.* [7]). In this case, the conclusion follows immediately from Corollary 1.3.

For the rest of this proof, we assume that A is stably finite. In this case, $K_0(A)$ is an ordered group (*cf.* Section 6.3.3 in [1]). In particular, if $0 < g_0 \in K_0(A)$ and $0 < g_1 \in K_0(A)$, then $g_0 + g_1 > 0$.

Suppose that $\iota_*(g) > 0$. By the construction of $\mathcal{Z}$, there exists a prime dimension drop algebra $Z_{m,n}$ such that $(\iota_{m,n})_*(g) \geq 0$, where $\iota_{m,n} \colon A \to A \otimes Z_{m,n}$ is the canonical embedding. Using the notations in the proof of Lemma 1.2, we have:

$$m \cdot g = (\nu_0)_*[(\iota_{m,n})_*(g)] \geq 0,$$

and

$$n \cdot g = (\nu_1)_*[(\iota_{m,n})_*(g)] \geq 0.$$

We claim that either $m \cdot g \neq 0$ or $n \cdot g \neq 0$. Indeed, if they were both zero, then by (1), $g = 0$, which would contradicts with the hypothesis that $\iota_*(g) > 0$. Therefore, either $m \cdot g > 0$, or $n \cdot g > 0$. This proves the "only if" part of the theorem.

We now turn to the "if" part. Suppose that $n \cdot g > 0$ for some $n > 0$. Since $K_0(A)$ is a simple ordered group, there exists an integer n_0 such that $n \cdot g > 0$ for all $n > n_0$. Let m and n be a pair of relatively prime integers larger than n_0. Then there are projections $e \in \mathbf{M}_{jm}(A)$ and $f \in \mathbf{M}_{jn}(A)$, where $j > 0$ is an integer, such that:

$$m \cdot g = [e], \quad \text{and} \quad n \cdot g = [f].$$

Consider these two projections e and f. We have $n \cdot [e] = mn \cdot g = m \cdot [f]$. By Theorem 3.1.4 of [2], $e \otimes \mathbf{1}_{kn}$ and $f \otimes \mathbf{1}_{km}$ are equivalent for all sufficiently large integers k. Choose one such integer k such that km and n remain relatively prime. Increasing j if necessary, we assume that $e \otimes \mathbf{1}_{kn}$ and $f \otimes \mathbf{1}_{km}$ are homotopic in $\mathbf{M}_{jkmn}(A)$. That is, there exists a continuous path E_t of projections in $M_{jkmn}(A)$ such that:

$$E_0 = e \otimes 1_{kn}, \quad \text{and} \quad E_1 = f \otimes 1_{km}.$$

In other words, E is a (nonzero) projection in $\mathbf{M}_j(A) \otimes Z_{km,n}$. It is easy to verify (using Lemma 1.2) that $(\iota_{km,n})_*(g) = [E]$, where $\iota_{km,n} \colon A \to A \otimes Z_{km,n}$ is the canonical embedding.

As we pointed out before, $\mathcal{Z}$ contains any prime dimension drop algebra as a unital subalgebra. Let $\phi \colon Z_{km,n} \to \mathcal{Z}$ be a unital embedding, and $\varphi = \mathrm{id}_A \otimes \phi$. Then the following diagram commutes:

$$\begin{array}{ccc} A & = & A \\ \downarrow{\scriptstyle \iota_{km,n}} & & \downarrow{\scriptstyle \iota} \\ A \otimes Z_{km,n} & \xrightarrow{\varphi} & A \otimes \mathcal{Z}. \end{array}$$

In particular, $\iota_*(g) = \varphi_*([E]) \geq 0$. Therefore, $\iota_*(g) > 0$ (since $g \neq 0$, and ι_* is injective). This proves the "if" part. ■

Theorem 1 is a direct consequence of Theorem 1.4.

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Proof of Theorem 1 Recall that a pre-ordered group G is weakly unperforated, if for any $g \in G$, $g > 0$ whenever there exists an $n \in \mathbb{N}$ such that $n \cdot g > 0$.

Let A be a unital simple C^*-algebra.

Let $g \in K_0(A \otimes \mathcal{Z})$. Suppose that $n \cdot g > 0$ for some integer $n > 0$. Then by Corollary 1.3 and Theorem 1.4, there is an integer $m > 0$ such that $m \cdot \iota_*^{-1}(n \cdot g) > 0$. That is, $mn \cdot \iota_*^{-1}(g) > 0$. Again, by Theorem 1.4, we have $g = \iota_*\big(\Phi_*^{-1}(g)\big) > 0$. This establishes Theorem 1(a).

Theorem 1(b) is now easy to see: The "if" part follows from Theorem 1.4 while the "only if" part follows Theorem 1(a). ∎

Theorem 1 gives a necessary condition for a unital simple C^*-algebra to be $\mathcal{Z}$-stable. By [14], this condition is not vacuous even for separable nuclear C^*-algebras. On the other hand, it could be also sufficient for such algebras. More precisely, if A is a unital simple C^*-algebra and $K_0(A)$ is weakly unperforated, then by Theorem 1 (and Lemmas 2.11 and 2.12 of [9]), $\iota\colon A \to A \otimes \mathcal{Z}$ induces an isomorphism of the Elliott invariants [8]. This raises the following question:

Question 1.5 Let A be a unital simple nuclear C^*-algebra, separable and infinite dimensional. If $K_0(A)$ is weakly unperforated, is A $\mathcal{Z}$-stable?

2 Property Γ

In this section, we propose a C^*-algebra analogue of property Γ [12], and prove that $\mathcal{Z}$-stability implies property Γ. As a consequence, the reduced group C^*-algebra $C_r^*(\mathbb{F}_2)$ is not $\mathcal{Z}$-stable, even though its K_0 group is weakly unperforated.

Recall, from [12], that a type II_1 factor $\mathcal{M}$ is said to have property Γ, if for any given finite set F in $\mathcal{M}$ and $\epsilon > 0$, there exists a unitary $u \in \mathcal{M}$ satisfying the following:

$$\tau(u) = 0; \quad \text{and} \quad \|ua - au\|_\tau < \epsilon, \quad \forall a \in F,$$

where τ is the unique tracial state on $\mathcal{M}$ and $\|a\|_\tau^2 = \tau(a^*a)$.

Inspired by this, we introduce the following:

Definition 2.1 Let A be a unital (separable) C^*-algebra. A will be said to have property Γ, if for any finite set $F \subseteq A$ and any $\epsilon > 0$, there is a unitary $u \in A$ such that:

$$\tau(u) = 0 \quad \text{for all traces } \tau \text{ on } A,$$

and

$$\|ua - au\| < \epsilon, \quad \text{for all } a \in F.$$

In particular, if A does not have any tracial state, then it has property Γ. But the interest of this section lies in unital simple C^*-algebras with a unique tracial state.

We recall a basic property of $\mathcal{Z}$:

Theorem 2.2 **(Theorem 4 of [9])** $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$. *More generally,* $\mathcal{Z} \otimes \mathcal{Z} \otimes \cdots \cong \mathcal{Z}$.

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Note that $\mathcal{Z}^{\otimes\infty}$ is the limit of the sequence $(\mathcal{Z}^{\otimes n}, \iota_n)$, where

$$\mathcal{Z}^{\otimes n} = \underbrace{\mathcal{Z} \otimes \cdots \otimes \mathcal{Z}}_{n \text{ times}},$$

and $\iota_n \colon \mathcal{Z}^{\otimes n} \to \mathcal{Z}^{\otimes(n+1)}$ is again the canonical embedding:

$$\iota_n(a) = a \otimes 1, \quad a \in \mathcal{Z}^{\otimes n}.$$

Proposition 2.3 *$\mathcal{Z}$ has property Γ.*

Proof By Theorem 2.2, it suffices to find a unitary $u \in \mathcal{Z}$ such that $\tau(u) = 0$, where τ is the unique tracial state on $\mathcal{Z}$. Since $\mathcal{Z}$ contains a unital copy of $Z_{2,3}$, it is enough to find a unitary $u \in Z_{2,3}$ such that $t(u) = 0$ for all tracial state t on $Z_{2,3}$.

For this purpose, we construct a continuous path u of unitaries in $\mathbf{M}_6$ with

$$u_0 = \operatorname{diag}(1, -1, 1, -1, 1, -1), \quad \text{and}$$
$$u_1 = \operatorname{diag}(1, z^2, z^4, 1, z^2, z^4), \quad \text{where} \quad z = e^{2\pi i/6}.$$

This can be done as follows: From u_0, we keep the first and forth diagonal entries fixed, rotate the second and fifth diagonal entries clockwise by $2\pi/6$ in a synchronized way so that the sum of these two entries remains 0, and rotate the remaining two diagonal entries clockwise by $2\pi/3$ in a similar way. After this operation, we get

$$u_{1/2} = \operatorname{diag}(1, z^2, z^4, -1, z^5, z).$$

Note that $-1 + z^5 + z = 0$. To connect $u_{1/2}$ to u_1, we rotate the last three diagonal entries of $u_{1/2}$ by π in a similar way.

By construction, $u \in Z_{2,3}$, u is unitary, and $\operatorname{tr}(u_x) = 0$ for each $x \in [0, 1]$, where tr is the (normalized) trace on $\mathbf{M}_6$. It follows (*cf.* Lemma 2.5 of [9]) that $t(u) = 0$ for all tracial state t on $Z_{2,3}$. This completes the proof. ■

Theorem 2 follows immediately from Proposition 2.3, since any tracial state on $A \otimes \mathcal{Z}$, if exists, must be of the form $t \otimes \tau$ for some tracial state t on A, where τ is the unique tracial state on $\mathcal{Z}$ (*cf.* Lemma 2.11 of [9]).

We now turn to the reduced group C^*-algebra $C^*_r(\mathbb{F}_2)$. It is well-known that $C^*_r(\mathbb{F}_2)$ is simple, and $K_0\big(C^*_r(\mathbb{F}_2)\big) \cong \mathbb{Z}$ is weakly unperforated. On the other hand, $C^*_r(\mathbb{F}_2)$ does not have property Γ, since the corresponding group von Neumann algebra does not have property Γ [12]. Therefore, $C^*_r(\mathbb{F}_2)$ is not $\mathcal{Z}$-stable.

3 A Dichotomy on Finiteness

The goal of this section is to establish Theorem 3, which follows immediately from the following theorem:

Theorem 3.1 *Let A be a simple C^*-algebra. If A is not stably finite, then $A \otimes \mathcal{Z}$ is purely infinite.*

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The proof of Theorem 3.1 will be divided into several lemmas. We start with the following well-known facts:

Lemma 3.2

(1) Let $A \neq 0$ be a simple C^*-algebra. If A is infinite, then there is an embedding $\psi\colon \mathcal{O}_2 \to A$.
(2) Any two nonunital nonzero endomorphisms of $\mathcal{O}_2$ are homotopic.

Proof Part (1) follows from the proof of Lemma 4.1 of [7] (see also Theorem 1.4 and Proposition 1.5 of [7]). Part (2) is a very special case of Lemma 2.9 of [10] (though it might have been known earlier). ■

Lemma 3.3 *Let A be a simple C^*-algebra. If A is not stably finite, then $A \otimes \mathcal{Z}$ contains an infinite projection.*

Proof It suffices to show that $A \otimes Z_{m,n}$ is infinite for some prime dimension drop algebra $Z_{m,n}$. The basic strategy is to take m, n large enough so that there are nonunital embeddings of $\mathcal{O}_2$ into $M_m(A)$ and $M_n(A)$, respectively. In fact, such embeddings can be chosen so that they are homotopic as embeddings of $\mathcal{O}_2$ in $M_{mn}(A)$. Such a homotopy provides an embedding of $\mathcal{O}_2$ into $A \otimes Z_{m,n}$, making the latter infinite.

Since A is not stably finite, we can choose an integer $m > 0$ such that $\mathbf{M}_m(A)$ is infinite, we then choose an integer $n > 3m$ so that m and n are relatively prime. We shall show that $A \otimes Z_{m,n}$ is infinite.

By Lemma 3.2 (1), there exists an embedding $\psi\colon \mathcal{O}_2 \to \mathbf{M}_m(A)$.

We define two embeddings ψ_0, ψ_1 of $\mathcal{O}_2$ into $\mathbf{M}_m(A)$ and $\mathbf{M}_n(A)$, respectively, as follows. Choose a nonunital nonzero endomorphism λ of $\mathcal{O}_2$. Define $\psi_0 = \psi \circ \lambda$, and $\psi_1 = \psi \oplus \mathbf{0}_{n-m}$ (*cf.* Section 1.1).

Let $\iota_0\colon \mathbf{M}_m(A) \to \mathbf{M}_{mn}(A)$ be the canonical embedding given by

$$\iota_0(x) = x \otimes 1_n, \quad x \in M_m(A).$$

Define $\iota_1\colon \mathbf{M}_n(A) \to \mathbf{M}_{mn}(A)$ accordingly. Let $\Psi_0 = \iota_0 \circ \psi_0$ and $\Psi_1 = \iota_1 \circ \psi_1$. We now show that Ψ_0 is homotopic to Ψ_1 (as embeddings of $\mathcal{O}_2$ in $\mathbf{M}_{mn}(A)$).

Let $B = \mathbf{M}_n\big(\psi(\mathcal{O}_2)\big) \subseteq \mathbf{M}_{mn}(A)$. It is well-known that $M_k(\mathcal{O}_2) \cong \mathcal{O}_2$ for any positive integer k. Therefore, $B \cong \mathcal{O}_2$. It follows from the definition that $\Psi_0(\mathcal{O}_2) \subseteq B$.

The image of Ψ_1 is not in B, but this can be fixed easily. For any integer j, $1 \leq j \leq m$, let k_j be the smallest integer such that $j \cdot n \leq k_j \cdot m$. Let $u_{j,t}$ be a continuous path in $\mathbf{S}U_n$ such that $u_{j,0} = I_n$ and

$$u_{j,1} = \begin{bmatrix} 0 & 0 & 1_m & 0 \\ 0 & 1_{(k_j \cdot m - j \cdot n)} & 0 & 0 \\ -1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{[(j+1)\cdot n - (k_j+2)\cdot m]} \end{bmatrix}.$$

Let $u_t = u_{1,t} \cdot u_{2,t} \cdot \cdots \cdot u_{m,t}$. It is easy to see that $u_t^* \Psi_1 u_t$ is a homotopy of non-unital embeddings of $\mathcal{O}_2$ into $\mathbf{M}_{mn}(A)$. In fact, $u_1^* \cdot \Psi_1 \cdot u_1$ is a non-unital embedding of $\mathcal{O}_2$ into

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B, hence is homotopic to Ψ_0 by Lemma 3.2(2). In conclusion, we have a homotopy Ψ_t of non-unital embedding of $\mathcal{O}_2$ into $M_{mn}(A)$ connecting Ψ_0 and Ψ_1. This gives rise to an embedding of $\mathcal{O}_2$ in $\mathcal{Z}_{m,n}(A)$. Therefore, $A \otimes \mathcal{Z}_{m,n}$ contains an infinite projection, and so does $A \otimes \mathcal{Z}$. ■

Corollary 3.4 *Let A be a simple C^*-algebra which is not stably finite. Then:*

(1) If $0 \neq p \in A$ is a projection, then $p \otimes 1_{\mathcal{Z}} \in A \otimes \mathcal{Z}$ is an infinite projection; and
(2) Every projection in $A \otimes \mathcal{Z}$ is infinite.

Note that in a simple C^*-algebra which is not stably finite, any nonzero hereditary subalgebra remains simple and not stably finite. This follows from a basic argument due to Cuntz [6].

Proof (1) This is a immediate consequence of Lemma 3.3 (applied to the algebra pAp).

(2) Let q be a projection in $A \otimes \mathcal{Z}$. By Theorem 2.2, $A \otimes \mathcal{Z} \cong \lim(A \otimes \mathcal{Z}^{\otimes n}, \iota_n)$. Using this isomorphism, we might assume, without loss of generality, that $q \in A \otimes \mathcal{Z}^{\otimes n}$ for some $n > 0$ (recall that close projections are unitarily equivalent). Then by part (1) of this lemma, $q \otimes 1_{\mathcal{Z}}$ is an infinite projection in $A \otimes \mathcal{Z}^{\otimes(n+1)}$. This completes the proof. ■

Proof of Theorem 3.1 This is similar to the proof of Corollary 3.4. Let B be any nonzero hereditary subalgebra of $A \otimes \mathcal{Z}$. Then B is simple and not stably finite. By Lemma 3.3, $B \otimes \mathcal{Z}$ contains an infinite projection.

Again, by Theorem 2.2, $A \otimes \mathcal{Z} \cong \lim(A \otimes \mathcal{Z}^{\otimes n}, \iota_n)$. It follows from the previous paragraph that for any n, and any nonzero hereditary subalgebra B of $A \otimes \mathcal{Z}^{\otimes n}$, $\iota_n(B)$ contains a nonzero projection. Therefore, by Lemma 1.8(a) of [3], any nonzero hereditary subalgebra of the limit algebra $A \otimes \mathcal{Z}$ contains a nonzero projection, which, by Corollary 3.4(2), is necessarily infinite. This completes the proof. ■

Remark 3.5 One question arises from the above proof: Let A be a simple $\mathcal{Z}$-stable C^*-algebra. If B is a hereditary subalgebra of A, is B $\mathcal{Z}$-stable?

In [13], it is proved that if A is a unital simple C^*-algebra and $\mathcal{U}$ is a UHF algebra, then either the invertibles are dense in $A \otimes \mathcal{U}$ (hence $A \otimes \mathcal{U}$ is stably finite), or $A \otimes \mathcal{U}$ is purely infinite. Note that $A \otimes \mathcal{U}$ is $\mathcal{Z}$-stable, since $\mathcal{U}$ is $\mathcal{Z}$-stable (Theorem 5 of [9]). This motivates Theorem 3. It is also natural to ask the following:

Question 3.6 Let A be a unital simple $\mathcal{Z}$-stable C^*-algebra. If A is finite, are the invertibles dense in A.

The answer should be affirmative, but the proof has eluded us so far.

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