Can. J. Math., Vol. XXXIII, No. 2, 1981, pp. 412–420

NILPOTENT PARTITION-INDUCING AUTOMORPHISM GROUPS

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1. Introduction. If A is a group acting on a set X and $x \in X$, we denote the stabilizer of x in A by $C_A(x)$ and let $\Gamma(x)$ be the set of elements of X fixed by $C_A(x)$. We shall say the action of A is *partitive* if the distinct subsets $\Gamma(x)$, $x \in X$, partition X. A special example of this phenomenon is the case of a semiregular action (when $C_A(x) = 1$ for every $x \in X$ so the induced partition is a trivial one). Our concern here is with the case that A is a group of automorphisms of a finite group G and $X = G^\#$, the set of non-identity elements of G. We shall prove that if A is nilpotent, then except in a very restricted situation, partitivity implies semiregularity.

THEOREM. *Suppose G is a finite group and A is a nilpotent group of automorphisms of G whose action on $G^\#$ is partitive. Then $O_{2'}(A)$ is cyclic and semiregular on $G^\#$. Moreover, if A is not semiregular on $G^\#$ (and, in particular, if $O_2(A)$ is neither cyclic nor generalized quaternion), then for some Mersenne or Fermat prime p, G is an elementary abelian p-group on which $O_2(A)$ acts irreducibly and one of the following holds:*

(a) *G is of type (p, p) and, if S is a Sylow 2-subgroup of* Aut (G) *($\cong GL_2(p)$) containing $O_2(A)$, then A contains every involution of S.*

(b) *G is of type $(3, 3, 3, 3)$ and $O_2(A)$ is non-abelian of order 2^n, $5 \leqq n \leqq 9$.*

As will be noted in the conclusion of this paper, statement (a) of the theorem represents a complete characterization of the partitive non-semiregular nilpotent automorphism groups which arise when G is elementary of rank two. Presumably, somewhat more can be said in the rank four, exponent three case.

It is easily seen that a half-transitive action is partitive so, in a purely group theoretic context, the theorem can be regarded as a direct extension of Theorem II of [**2**]. Our original motivation for studying partitive automorphism groups, however, was some work of C. J. Maxson and K. C. Smith on a certain class of near-rings. The object of their study was the near-ring $C(A, G)$ of identity-preserving maps from a finite group G to itself which commute with the action of a group A of automorphisms (addition in $C(A, G)$ being defined pointwise using the group

Received August 21, 1979.

operation in G and multiplication being defined by composition). In the case that G is a vector space, this is the non-linear analogue of the ring of A-endomorphisms of G. The two results of Maxson and Smith which are relevant here are, first, that $C(A, G)$ is simple if and only if all point stabilizers $C_A(x)$, $x \in G^\#$, are conjugate in A and, second, that $C(A, G)$ is semisimple if and only if A acts partitively on $G^\#$ [**3**]. In this context then, the significance of our theorem is that, when A is nilpotent, semi-simplicity of the near-ring $C(A, G)$ usually implies simplicity. More precisely, we have

COROLLARY. *Suppose G is a finite group and A is a nilpotent group of automorphisms of G such that neither of the two exceptional situations described in the theorem applies. If the near-ring $C(A, G)$ is semisimple, then it is simple.*

This generalizes a fact previously noticed by Maxson and Smith in the case that A is abelian.

The semiregularity conclusion of the theorem is stronger than the conjugacy condition required for the corollary, but only slightly so. In fact, using the theorem of Isaacs and Passman mentioned above, it can be easily verified that the Sylow 2-subgroups of $GL_2(p)$, p a Mersenne prime, are the only examples of nilpotent non-semiregular automorphism groups which contain a single conjugacy class of point stabilizers.

Although there are a number of deep results available on partitioned groups, the argument we shall present here employs only fairly standard group theoretic machinery (most of which may be found in the earlier chapters of [**1**]). It should be acknowledged, however, that while we have included for the reader's convenience a detailed account in Section 3 of the special case that G is an irreducible A-module, much of the analysis in this situation is adapted more or less directly from arguments used in [**2**].

The author wishes to thank Professor K. Smith for his encouragement and for many helpful and motivating conversations on the near-ring theoretic aspects of the problem.

2. Preliminaries. Henceforth, we shall assume G is a finite group and A is a group of automorphisms of G whose action on $G^\#$ is partitive. For the moment, we do not need to assume A is nilpotent.

First, we state a property of the point stabilizers which is actually equivalent to partitivity. The proof is easy and is omitted.

LEMMA 2.1. *If x and $y \in G^\#$ such that $C_A(x) \leqq C_A(y)$, then $C_A(x) = C_A(y)$. In particular, if $x^n \neq 1$, then $C_A(x) = C_A(x^n)$ and $\Gamma(x) = \Gamma(x^n)$.*

It is worth noting as an immediate consequence of Lemma 2.1 that, if

$A \neq 1$, then every point stabilizer $C_A(x)$, $x \in G^\#$, is both proper and non-trivial.

LEMMA 2.2. *Suppose H and K are non-trivial subgroups of G with $H \cap K = 1$. If $x \in H^\#$ and $y \in K^\#$ such that*

$$C_A(xy) \leqq N_A(H) \cap N_A(K),$$

then $C_A(x) = C_A(y)$.

Proof. If $\alpha \in C_A(xy)$, then $x^\alpha y^\alpha = xy$ so $x^{-\alpha}x = y^\alpha y^{-1} \in H \cap K = 1$. It follows that

$$C_A(xy) = C_A(x) \cap C_A(y)$$

so, by Lemma 2.1,

$$C_A(x) = C_A(xy) = C_A(y).$$

3. The module case. We shall first deal with the case that G is an elementary abelian p-group (i.e., a $GF(p)$-module) with A acting irreducibly. As previously indicated, this part of the argument closely parallels the analysis in [**2**] with some minor modifications necessitated by the less numerical nature of the partitivity hypothesis.

LEMMA 3.1. *The theorem is true if G is an irreducible module for A over the prime field $GF(p)$.*

Proof. Suppose this is false and let the pair (A, G) be a counterexample with $|A|$ minimal.

If $O_{2'}(A)$ is non-cyclic, then by Theorem 5.4.10 of [**1**], it contains a non-cyclic abelian normal subgroup D. Since

$$A = O_{2'}(A)C_A(O_{2'}(A)),$$

G is homogeneous as an $O_{2'}(A)$-module so, if H is an irreducible $O_{2'}(A)$-submodule of G, H is faithful. But if K is an irreducible D-submodule of H, K is not faithful (since D is non-cyclic). Hence, H is not homogeneous as a D-module, so the inertia group I of K does not contain $O_{2'}(A)$. Let B be a maximal subgroup of A which contains $O_2(A)I$, so $|A:B| = q > 2$ for some prime q. G is not homogeneous as a B-module (else all A-conjugates of K appear as constituents of any irreducible B-submodule of G, contradicting $|B:I| < |A:I|$) so

$$G = G_1 \oplus G_2 \oplus \ldots \oplus G_q,$$

where the G_i's are non-isomorphic homogeneous B-submodules. If $x \in G_1^\#$ and $y \in G_2^\#$, then $(xy)^A \neq (xy)^B$ (since $(xy)^B \subseteq G_1 \oplus G_2 \neq G$) so we must have $C_A(xy) \leqq B$. It follows from Lemma 2.2 that $C_A(x) = C_A(y)$. Repeating this argument for all pairs G_i, G_j, $i \neq j$, we find that

https://doi.org/10.4153/CJM-1981-036-2 Published online by Cambridge University Press

all non-identity elements of all the G_i's have the same stabilizer in A, so A is semiregular on $G^\#$, a contradiction.

Hence, $O_{2'}(A)$ is cyclic so $O_{2'}(A) \leqq Z(A)$. Then $C_G(\alpha)$ is A-invariant for every $\alpha \in O_{2'}(A)$ so, by the irreducibility of G, $C_G(\alpha) = 1$ for all $\alpha \in O_{2'}(A)^\#$. It follows that $O_{2'}(A)$ is semiregular on $G^\#$.

Now we have $C_A(x) \leqq O_2(A)$ for every $x \in G^\#$ so $O_2(A)$ is partitive on $G^\#$. Also, $C_G(O_2(A)) = 1$ so $p \neq 2$. Hence, by Maschke's theorem, G is completely reducible as an $O_2(A)$-module. But if $G = G_1 \oplus G_2$ where G_1 and G_2 are non-trivial $O_2(A)$-submodules, then Lemma 2.2 yields that all elements of $G^\#$ have the same stabilizer in $O_2(A)$. This implies $O_2(A)$ (and hence A) is semiregular on $G^\#$. Therefore, G is irreducible as an $O_2(A)$-module. By the minimality of $|A|$, we may now assume A is a 2-group.

Now since A is not semiregular on $G^\#$, it is not cyclic or generalized quaternion. Thus, if A contains no non-cyclic abelian normal subgroup, it is dihedral or semihedral by Theorem 5.4.10 of [**1**]. The argument in this situation is drawn directly from Lemmas 6 and 7 of [**2**]. First, A contains in either case a cyclic maximal subgroup which, because of the irreducibility of G, is semiregular on $G^\#$. It follows that $|C_A(x)| \leqq 2$ for all $x \in G^\#$ so, by the remark following Lemma 2.1, $|C_A(x)| = 2$ for every $x \in G^\#$. If σ is the central involution in A, the irreducibility of G also forces $C_G(\sigma)$ to be trivial, so $x^\sigma = x^{-1}$ for all $x \in G^\#$. Thus, the components $\Gamma(x)$ of the partition induced by A are precisely the centralizers $C_G(\alpha)$ of the non-central involutions in A. Now for any non-central involution α,

$$G = C_G(\alpha) \oplus C_G(\alpha\sigma)$$

since, relative to α, $C_G(\alpha\sigma)$ is the eigenspace for the eigenvalue -1. If β is any non-central involution such that $\beta \neq \alpha \neq \beta\sigma$, it follows that either

$$|C_G(\alpha)| \geqq |G|^{1/2} \quad \text{or} \quad |C_G(\alpha\sigma)| \geqq |G|^{1/2}$$

and similarly,

$$|C_G(\beta)| \geqq |G|^{1/2} \quad \text{or} \quad |C_G(\beta\sigma)| \geqq |G|^{1/2}.$$

Since, as we have shown, any two of these four centralizers intersect trivially, we conclude that all centralizers in G of non-central involutions in A (and hence, all components of the partition) have order $|G|^{1/2}$. Now the number of non-central involutions in a dihedral or semidihedral 2-group is 2^n for some n (where the corresponding group order is 2^{n+1} or 2^{n+2} respectively), so if $|G| = p^{2m}$, we have

$$2^n(p^m - 1) = p^{2m} - 1.$$

Therefore, $2^n = p^m + 1$. Now m is odd (else $2^n = p^m + 1 \equiv 2 \pmod 4$), implying $n = 1$), so

$$2^n = (p + 1)(p^{m-1} - \ldots + 1).$$

But the second factor contains an odd number of terms and hence is odd. We conclude that $m = 1$ so $p = 2^n - 1$, a Mersenne prime, and $|G| = p^2$. In this case, a Sylow 2-subgroup of $GL_2(p)$ is semidihedral of order 2^{n+2} so statement (a) of the theorem holds, a contradiction.

Therefore, we may assume A contains a non-cyclic abelian normal subgroup D. Arguing as before, we conclude that D is contained in some maximal subgroup B of A such that G decomposes non-trivially as a direct sum $G_1 \oplus G_2$ of B-submodules. Furthermore, any element of $A \backslash B$ interchanges G_1 and G_2.

Let $x \in G_1^{\#}$. We claim $G_2 \nleqq \Gamma(x)$. For if $G_2 \leqq \Gamma(x)$, then choosing $\alpha \in A \backslash B$, we have $G_1 = G_2^{\alpha} \leqq \Gamma(x^{\alpha})$. Since $G = G_1 \oplus G_2$ and $\Gamma(x) \cap \Gamma(x^{\alpha}) = 1$ (else $\Gamma(x) = \Gamma(x^{\alpha})$ contains $G_1 \oplus G_2 = G$ and A is semiregular on $G^{\#}$), we then have $x \in \Gamma(x) = G_2$, a contradiction.

Let $x, y \in G_1^{\#}$ so by the preceding paragraph,

$$\Gamma(x) \cap G_2 \neq G_2 \neq \Gamma(y) \cap G_2.$$

Since no group can be the union of two of its proper subgroups, we may choose an element z in G_2 which is not in $\Gamma(x) \cup \Gamma(y)$. Then $C_A(xz) \nleqq B$, for otherwise Lemma 2.2 implies that $C_A(x) = C_A(z)$ so $z \in \Gamma(x)$, a contradiction. Hence, $A = C_A(xz)B$ and similarly, $A = C_A(yz)B$. Let $\alpha \in A \backslash B$ so $\alpha\beta \in C_A(xz)$ for some $\beta \in B$. Then

$$xz = (xz)^{\alpha\beta} = x^{\alpha\beta}z^{\alpha\beta}$$

so, since $\alpha\beta$ interchanges G_1 and G_2, $x = z^{\alpha\beta}$. Similarly, we conclude that $y = z^{\alpha\gamma}$ for some $\gamma \in B$. But then $\beta^{-1}\gamma$ maps x to y so we have proven that B is transitive on $G_1^{\#}$.

If $x \in G_1^{\#}$ and $|B:C_B(x)| = 2^n$, the transitivity of B implies that

$$2^n = p^m - 1 = (p - 1)(p^{m-1} + \ldots + 1),$$

where $|G_1| = p^m$. If m is odd, the second factor is odd so $m = 1$ and p is Fermat. If $m = 2k$ is even, then

$$2^n = (p^k - 1)(p^k + 1)$$

so $p^k - 1$ and $p^k + 1$ are both powers of 2. This can obviously occur only if $p^k - 1 = 2$, so $p = 3$ and $m = 2$. Now $C_A(x) \leqq B$, else $A = C_A(x)B$ so $x^A = x^B \subseteq G_1$, contradicting the irreducibility of G. Thus, in the case that $p = 3$ and $m = 2$, we have $|B:C_B(x)| = 8$ so $|A| \geqq 32 = 2^5$ since $C_B(x) = C_A(x) \neq 1$. Since G is then elementary abelian of rank 4 and since a Sylow 2-subgroup of $GL_4(3)$ has order 2^9, statement (b) of the theorem holds. Hence, we may assume that p is Fermat, $m = 1$ and $|G| = p^2$.

Suppose S is a Sylow 2-subgroup of Aut (G) containing A, and let σ be any involution in S. If σ fixes no element of $G^{\#}$, then $x^{\sigma} = x^{-1}$ for every x in $G^{\#}$. The irreducibility of G then forces σ to be the unique central involution of A. Suppose now that $\sigma \in C_S(x)$ for some $x \in G^{\#}$. Since $p - 1 = 2^n$, the Sylow 2-subgroups of the stabilizer in $GL_2(p)$ of any non-zero vector are isomorphic to the group of matrices

$$\{\text{diag } (1, c)\colon c \in GF(p)^{\times}\}$$

and hence, are cyclic. Therefore, $C_S(x)$ is cyclic so, since $C_A(x) \neq 1$, σ must be in A. Thus, statement (a) of the theorem holds, a contradiction. This completes the proof of Lemma 3.1.

4. Proof of the theorem. We assume from now on that the pair (A, G) is a counterexample to the theorem with $|A| + |G|$ minimal. In particular, for any A-invariant subgroup H of G, either $A/C_A(H)$ acts semiregularly on $H^{\#}$ (so $C_A(x) = C_A(H) \trianglelefteq A$ for all $x \in H^{\#}$) or one of the exceptional situations described in statements (a) and (b) of the theorem holds. In the latter case, we shall need only the fact that A acts irreducibly on H.

This part of the argument does not parallel that of [**2**]. Part of the difficulty here seems to be that, while it is a trivial observation that each Sylow subgroup of a half-transitive nilpotent automorphism group is necessarily half-transitive (so there is no loss in assuming, as Isaacs and Passman did, that A is a p-group), the analogous result for a partitive action seems less transparent. Indeed, a simple demonstration of this fact would enable the present proof to be shortened considerably.

Let $\pi = \pi(G)$ be the set of prime divisors of $|G|$. We proceed in stages.

(4.1) *G contains a non-trivial proper A-invariant subgroup.*

Proof. Suppose this is false. If $p \in \pi$ is a divisor of $|A|$, let P be a Sylow p-subgroup of A, so $P \trianglelefteq A$. If R is a Sylow p-subgroup of the semidirect product GA containing P, then P normalizes the Sylow subgroup $R \cap G$ of G, so $C_G(P)$ is a non-trivial proper A-invariant subgroup of G. Hence, $(|A|, |G|) = 1$. In this case, G has A-invariant Sylow subgroups (by Theorem 6.2.2 of [**1**]) so G must be a p-group for some prime p. The Frattini subgroup of G is proper, hence trivial, so G is elementary abelian. Therefore, G is an irreducible module for A over $GF(p)$, contradicting Lemma 3.1.

(4.2) *G contains a unique maximal A-invariant subgroup $U \neq 1$. Moreover, if A does not act irreducibly on it, then $U = \Gamma(x)$ for every $x \in U^{\#}$.*

Proof. By (4.1), G certainly contains a non-trivial maximal A-invariant subgroup U and, if A does not act irreducibly on U, then the inductive hypothesis implies that $C_A(x) = C_A(U)$ for all $x \in U^\#$. But then

$$U \leqq C_G(C_A(U)) = C_G(C_A(x)) = \Gamma(x)$$

for every x in $U^\#$, so the maximality of U and the fact that point stabilizers in A are all non-trivial force $U = \Gamma(x)$ for each x in $U^\#$.

It remains to show that U is unique. But if V is any other maximal A-invariant subgroup of G, the preceding argument applies equally to V so either A acts irreducibly on V or $V = \Gamma(y)$ for every $y \in V^\#$. In either case, $U \cap V = 1$. Lemma 2.2 then implies that $C_A(x) = C_A(y)$ if $x \in U^\#$ and $y \in V^\#$, whence follows the contradiction $U = \Gamma(x) = \Gamma(y) = V$. Hence, U is unique as claimed.

(4.3) *If* $1 \neq B \trianglelefteq A$, *then* $C_G(B) = 1$ *or* U.

Proof. Since $B \neq 1$, the uniqueness of U implies that $C_G(B) \leqq U$. In the case that A acts irreducibly on U, the conclusion is now immediate, so by (4.2), we may assume $U = \Gamma(x)$ for all x in $U^\#$. Now if $C_G(B) \neq 1$, let $x \in C_G(B)^\#$. Then $B \leqq C_A(x)$ so

$$U = \Gamma(x) = C_G(C_A(x)) \leqq C_G(B).$$

Hence, $C_G(B) = U$.

(4.4) $O_\pi(A) \leqq C_A(U)$.

Proof. If $p \in \pi$ and P is a Sylow p-subgroup of A, then as we showed in the proof of (4.1), $C_G(P)$ is non-trivial and A-invariant. Therefore, (4.3) implies $U \leqq C_G(P)$ and (4.4) follows.

(4.5) *G is a p-group for some prime p.*

Proof. Since A is non-trivial and partitive on $G^\#$, $C_G(A)$ is trivial, so (4.4) implies $O_{\pi'}(A) \neq 1$. From (4.3) and (4.4), we conclude that $C_G(O_{\pi'}(A)) = 1$. Hence, if $p \in \pi$, Theorem 6.2.2 of [**1**] yields that G contains a unique $O_{\pi'}(A)$-invariant Sylow p-subgroup R. But if $\alpha \in A$, then R^α is also $O_{\pi'}(A)$-invariant so $R^\alpha = R$. It follows that G has A-invariant Sylow subgroups for each of its prime divisors. The uniqueness of U then implies that G is a p-group.

(4.6) $C_A(x) \leqq O_{p'}(A)$ *for every* $x \in G \backslash U$.

Proof. Since G is a p-group, $[G, O_p(A)] \neq G$ so, by the uniqueness of U, $[G, O_p(A)] \leqq U$. Then (4.4) implies

$$[G, O_p(A), O_p(A)] = 1$$

so Theorem 2.2.3 of [**1**] yields that $O_p(A)$ is abelian. Then $O_p(A) \leqq Z(A)$ so $C_G(\alpha)$ is A-invariant for all $\alpha \in O_p(A)$. Since by (4.4), $U \leqq C_G(O_p(A))$, the maximality of U implies $C_G(\alpha) = U$ for every $\alpha \in O_p(A)^\#$. Thus,

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$O_p(A) \cap C_A(x) = 1$ for every $x \in G \backslash U$ and (4.6) follows.

(4.7) $U \leqq Z(G)$.

Proof. Since, by (4.5), G is a p-group, Theorem 2.3.4 of [**1**] implies that $U \trianglelefteq G$, hence $U \cap Z(G) \neq 1$, so the result is obvious if A acts irreducibly on U. Therefore, by the inductive hypothesis, we may assume $A/C_A(U)$ is semiregular on $U^\#$. Now suppose $U \not\leqq Z(G)$ so, by the uniqueness of U, $C_G(U) \leqq U$. Using the fact that $U \trianglelefteq G$, we then have

$$[G, U, C_A(U)] = 1 = [U, C_A(U), G],$$

so Theorem 2.2.3 of [**1**] yields

$$[G, C_A(U)] \leqq C_G(U) \leqq U.$$

By Corollary 5.3.3 of [**1**], $C_A(U)$ is a p-group so, by (4.4), $C_A(U) = O_p(A)$. We conclude that $O_{p'}(A) \cong A/C_A(U)$ acts semiregularly on $U^\#$. It follows from Theorem 3.3.3 of [**1**] that $O_{p'}(A)$ contains no non-cyclic abelian subgroups, so all elements of prime order in $O_{p'}(A)$ lie in $Z(A)$. Then, if $\alpha \in O_{p'}(A)^\#$ has prime order, $C_G(\alpha)$ is A-invariant so, since $C_A(U) = O_p(A)$, (4.3) implies $C_G(\alpha) = 1$. Therefore, we must have $C_G(\alpha) = 1$ for every α in $O_{p'}(A)^\#$ so, by (4.6), $C_A(x) = 1$ for every $x \in G \backslash U$. This contradicts the fact that A is partitive but not semi-regular on $G^\#$, so (4.7) is proven.

(4.8) *G has exponent p.*

Proof. Suppose G contains a non-identity element x of order not p. If u is any element of order p in U, then by (4.7), $(xu)^p = x^p \neq 1$, so from Lemma 2.1, $\Gamma(xu) = \Gamma(x)$. Then $x, xu \in \Gamma(x)$ so $u \in \Gamma(x)$. This shows that $\Gamma(x)$ contains all elements of order p in U and hence, again by Lemma 2.1, $\Gamma(x)$ contains U. $\Gamma(x)$ must then be A-invariant so by the maximality of U, $\Gamma(x) = U$ and in particular, $x \in U$. Now let $y \in G \backslash U$. The preceding argument implies $y^p = 1$ so since $x \in U \leqq Z(G)$, $(xy)^p = x^p \neq 1$. As before, this yields $y \in \Gamma(x)$, which is a contradiction since $\Gamma(x) = U$ and $y \in G \backslash U$.

(4.9) *No counterexample to the theorem exists.*

Proof. If $p \neq 2$, we first use a "trick" apparently due to Baer. Define a new binary operation "$*$" on the underlying set of G by

$$x*y = xy[y, x]^{1/2}$$

(where $[y, x]^{1/2}$ denotes the unique element of G whose square is $[y, x] = y^{-1}x^{-1}yx$). Now since G is a p-group, $G' \neq G$ and the uniqueness of U forces $G' \leqq U$, so (4.7) implies that G has nilpotence class at most two. Using this fact, it is routine to check that $*$ is an abelian group

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operation. Furthermore, if we denote this new group by G^*, then G^* also has exponent p (by (4.8)) and admits A as a group of automorphisms. The action of A on $(G^*)^\#$ remains, of course, partitive.

If $p = 2$, then (4.8) implies G is itself abelian, so we may let $G^* = G$. In either case, we are reduced to a situation in which A acts partitively on the non-identity elements of a module G^* over $GF(p)$. By Maschke's theorem, there exists an $O_{p'}(A)$-submodule V of G^* such that $G^* = U \oplus V$. Since, by (4.6), $C_A(x) \leqq O_{p'}(A)$ for every $x \in G \backslash U$, Lemma 2.2 implies that if $x \in V^\#$ and $y \in U^\#$, then $C_A(x) = C_A(y)$. We conclude that all non-identity elements of G^* have the same stabilizer in A, so A is semiregular on $(G^*)^\# = G^\#$, a contradiction. Thus, the theorem is proved.

5. Concluding remarks. Suppose p is a Mersenne or Fermat prime and S is a Sylow 2-subgroup of $GL_2(p)$. Let $T = \Omega_1(S)$, the subgroup generated by all involutions in S. In the Mersenne case, S is semidihedral and in the Fermat case, it is the wreath product of a cyclic group with the group of order 2. In either situation, it may be checked that T acts half-transitively but not semiregularly on the non-identity elements of the natural two-dimensional $GF(p)$-module G. Therefore, if $T \leqq A \leqq GL_2(p)$, then for any $x \in G^\#$,

$$\langle x \rangle \leqq C_G(C_A(x)) \leqq C_G(C_T(x)) \neq G$$

so, in fact, $C_G(C_A(x)) = \langle x \rangle$. It follows that A acts partitively on $G^\#$. Hence, statement (a) of the theorem just proved is, in some sense, a complete characterization of this particular exceptional case. A concise and complete description of the possible exceptions when G is four-dimensional over $GF(3)$ seems less obvious. We only observe here that there do exist examples which are not included in Isaacs' and Passman's list (i.e., which are partitive but not half-transitive). In fact, the full Sylow 2-subgroup of $GL_4(3)$ (which is a wreath product of the semidihedral group of order 16 with the group of order 2) is such an example.

Finally, note that the automorphism group of the symmetric group S_3 acts partitively on $S_3^\#$, so it is not immediately apparent what the partitive analogue of Theorem I of [**2**] should be.

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