21

Majorana neutrinos

Majorana fields were introduced in Section 6.6. If neutrino fields are Majorana, then there is no distinction to be made between neutrinos and antineutrinos. As explained in Section 6.7, the smallness of neutrino masses makes the differences between Dirac and Majorana neutrinos difficult to discern experimentally.

In this chapter we elaborate on the theory of Majorana neutrinos and show how they can be accommodated within the Standard Model. Finally we describe experiments on 'double β decay' that may determine the nature of neutrinos.

21.1 Majorana neutrino fields

We shall denote left-handed and right-handed Majorana neutrino fields by $\nu_{\mathrm{L}}(x)$ and $\nu_{\mathrm{R}}(x)$. From (6.28 and 6.29), making the identifications

$$b_{\mathbf{p}+} = d_{\mathbf{p}+}, \;\; b_{\mathbf{p}-} = d_{\mathbf{p}-}$$

we have for a Majorana neutrino field carrying mass m

$$\begin{aligned}\nu_{\mathrm{L}} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sqrt{\frac{m}{2E_p}} \; &\Big[\big(b_{\mathbf{p}+}\mathrm{e}^{-\theta/2}\,|+\rangle\big) + b_{\mathbf{p}-}\mathrm{e}^{\theta/2}\,|-\rangle\;\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)} \\ &+ \big(b^*_{\mathbf{p}+}\mathrm{e}^{\theta/2}\,|-\rangle - b^*_{\mathbf{p}-}\mathrm{e}^{-\theta/2}\,|+\rangle\big)\,\mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r}+Et)}\Big],\end{aligned} \tag{21.1}$$

$$\begin{aligned}\nu_{\mathrm{R}} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \sqrt{\frac{m}{2E_p}} \; &\Big[\big(b_{\mathbf{p}+}\mathrm{e}^{\theta/2}\,|+\rangle\big) + b_{\mathbf{p}-}\mathrm{e}^{-\theta/2}\,|-\rangle\;\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)} \\ &+ \big(-b^*_{\mathbf{p}+}\mathrm{e}^{-\theta/2}\,|-\rangle + b^*_{\mathbf{p}-}\mathrm{e}^{\theta/2}\,|+\rangle\big)\,\mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r}+Et)}\Big].\end{aligned} \tag{21.2}$$

The fields $\nu_{\mathrm{L}}(x)$ and $\nu_{\mathrm{R}}(x)$ are not independent. It is easily shown, using Problem 6.5, that

$$\big(\mathrm{i}\sigma^2\big)\,|-\rangle^* = |+\rangle\,, \qquad \big(\mathrm{i}\sigma^2\big)\,|+\rangle^* = -\,|-\rangle\,,$$

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and then that

$$\nu_{\mathrm{R}} = (\mathrm{i}\sigma^2)\nu_{\mathrm{L}}^* \quad \text{and} \quad \nu_{\mathrm{L}} = -(\mathrm{i}\sigma^2)\nu_{\mathrm{R}}^*. \tag{21.3}$$

Thus either field may be derived from the other. As a consequence, only left-handed Majorana fields or only right-handed Majorana fields need appear in any theory.

The charge conjugate field ν_{L}^c was defined in (7.11b) by

$$\nu_{\mathrm{L}}^c = -(\mathrm{i}\sigma^2)\nu_{\mathrm{R}}^*.$$

But by the results above $-(\mathrm{i}\sigma^2)\nu_{\mathrm{R}}^* = \nu_{\mathrm{L}}$, so that

$$\nu_{\mathrm{L}}^c = \nu_{\mathrm{L}}. \tag{21.4}$$

Thus the charge conjugate of a Majorana field is identical to the field. There is no room in the theory of Majorana neutrinos for a distinguishable antineutrino. For a given momentum, there are two basic particle states, which we may take to be one with helicity $+1/2$, the other with helicity $-1/2$. (In these respects, Majorana neutrinos are somewhat similar to photons, but with photons having helicities ± 1).

21.2 Majorana Lagrangian density

The Majorana field is constructed from solutions of the Dirac equation. We saw in Section 5.2 that the Lagrangian density for a free Dirac particle of mass m is

$$\mathcal{L}^{\mathrm{Dirac}} = \mathrm{i}\psi_{\mathrm{L}}^\dagger \tilde{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} + \mathrm{i}\psi_{\mathrm{R}}^\dagger \sigma^\mu \partial_\mu \psi_{\mathrm{R}} - m\left(\psi_{\mathrm{L}}^\dagger \psi_{\mathrm{R}} + \psi_{\mathrm{R}}^\dagger \psi_{\mathrm{L}}\right).$$

In the case of a Majorana field, ν_{R} is determined by ν_{L}, and given by (21.3) above. We choose to work with ν_{L}, and therefore take the Majorana Lagrangian density to be

$$\mathcal{L}^{\mathrm{M}} = \frac{1}{2}\Big[\mathrm{i}\nu^\dagger \tilde{\sigma}^\mu \partial_\mu \nu + \mathrm{i}(\mathrm{i}\sigma^2\nu^*)^\dagger \sigma^\mu \partial_\mu (\mathrm{i}\sigma^2\nu^*) - m\left\{\nu^\dagger(\mathrm{i}\sigma^2)\nu^* + \nu^{\mathrm{T}}(-\mathrm{i}\sigma^2)\nu\right\}\Big],$$

where $\nu = \nu_{\mathrm{L}}$. For the remainder of this chapter we shall drop the subscript L, for clarity of notation. ν is a two component left-handed neutrino field. We have introduced a factor of $1/2$ to compensate for double counting.

The second dynamical term in $\mathcal{L}^{\mathrm{M}}$ is equivalent to the first (Problem 21.1), so that the Lagrangian density may be written

$$\mathcal{L}^{\mathrm{M}} = \mathrm{i}\nu^\dagger \tilde{\sigma}^\mu \partial_\mu \nu - \frac{m}{2}\left\{\nu^\dagger\left(\mathrm{i}\sigma^2\right)\nu^* + \nu^{\mathrm{T}}\left(-\mathrm{i}\sigma^2\right)\nu\right\}. \tag{21.5}$$

It is interesting and important to note that, with finite mass m and with the Majorana constraints, we lose the $U(1)$ symmetry that gave neutrino number

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conservation in the Dirac case (Section 7.1). We shall see that with Majorana neutrinos the overall lepton number is no longer conserved.

Noting the factor 1/2 in the Lagrangian density, the Hamiltonian operator H and momentum operator **P** for Majorana neutrinos are (see Section 6.5)

$$\begin{aligned}
\mathrm{H} &= \frac{1}{2}\sum_{\mathbf{p},\varepsilon}\left(b^*_{\mathbf{p}\varepsilon}b_{\mathbf{p}\varepsilon} - b_{\mathbf{p}\varepsilon}b^*_{\mathbf{p}\varepsilon}\right)E_{\mathbf{p}} = \sum_{\mathbf{p},\varepsilon}\left(b^*_{\mathbf{p}\varepsilon}b_{\mathbf{p}\varepsilon}\right)E_{\mathbf{p}},\\
\mathbf{P} &= \frac{1}{2}\sum_{\mathbf{p},\varepsilon}\left(b^*_{\mathbf{p}\varepsilon}b_{\mathbf{p}\varepsilon} - b_{\mathbf{p}\varepsilon}b^*_{\mathbf{p}\varepsilon}\right)\mathbf{p} = \sum_{\mathbf{p},\varepsilon}\left(b^*_{\mathbf{p}\varepsilon}b_{\mathbf{p}\varepsilon}\right)\mathbf{p},
\end{aligned} \tag{21.6}$$

where $\varepsilon = \pm 1$ is the helicity index.

21.3 Majorana field equations

A variation $\delta\nu^*$ in the Majorana action yields the field equation

$$\mathrm{i}\tilde{\sigma}^\mu\partial_\mu\nu = m\left(\mathrm{i}\sigma^2\right)\nu^*.$$

(Note that there are two contributions from the mass term in the Lagrangian density.)

In a frame K' in which the Majorana neutrino is at rest, $p'_i\nu' = -\mathrm{i}\partial'_i\nu' = 0\,(i = 1, 2, 3)$, and the field equation reduces to

$$\mathrm{i}\frac{\partial\nu'}{\partial t'} = m\left(\mathrm{i}\sigma^2\right)\nu'^* \tag{21.7}$$

It is easy to verify that this equation has two solutions of the form

$$\nu'_1 = b\mathrm{e}^{-\mathrm{i}Et'}\begin{pmatrix}1\\0\end{pmatrix} + b^*\mathrm{e}^{\mathrm{i}Et'}\begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad \nu'_2 = b\mathrm{e}^{-\mathrm{i}Et'}\begin{pmatrix}0\\1\end{pmatrix} - b^*\mathrm{e}^{\mathrm{i}Et'}\begin{pmatrix}1\\0\end{pmatrix},$$

with $E = m$. (21.8)

We may then, as in Section 6.3, transform to a frame K in which the Majorana neutrino is moving with velocity $v > 0$ in the Oz direction:

$$\begin{aligned}
\nu_1 = \mathrm{M}^{-1}\nu'_1 &= \begin{pmatrix}e^{-\theta/2} & 0\\ 0 & \mathrm{e}^{\theta/2}\end{pmatrix}\left[b\mathrm{e}^{-\mathrm{i}mt'}\begin{pmatrix}1\\0\end{pmatrix} + b^*\mathrm{e}^{\mathrm{i}mt'}\begin{pmatrix}0\\1\end{pmatrix}\right]\\
&= b\mathrm{e}^{-mt'}\mathrm{e}^{-\theta/2}\begin{pmatrix}1\\0\end{pmatrix} + b^*\mathrm{e}^{\mathrm{i}mt'}\mathrm{e}^{\theta/2}\begin{pmatrix}0\\1\end{pmatrix}.
\end{aligned}$$

Substituting $t' = t\cosh\theta - z\sinh\theta$,

$$\nu_1 = b\mathrm{e}^{-\theta/2}\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(pz-Et)} + b^*\mathrm{e}^{\theta/2}\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}(-pz+Et)}. \tag{21.9}$$

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Similarly there are solutions of the form

$$\nu_2 = b\mathrm{e}^{\theta/2}\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}(pz-Et)} - b^*\mathrm{e}^{-\theta/2}\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{\mathrm{i}(-pz+Et)}. \tag{21.10}$$

All other plane wave solutions may be generated from these by rotations, and we recover the general field (21.1).

21.4 Majorana neutrinos: mixing and oscillations

The most general Lorentz invariant Majorana mass term that can be introduced into a Lagrangian density is

$$\mathcal{L}_{\mathrm{mass}}(x) = -\frac{1}{2}\sum_{\alpha,\beta}\nu_\alpha^{\mathrm{T}}\left(-\mathrm{i}\sigma^2\right)\nu_\beta m_{\alpha\beta} + \text{Hermitian conjugate}. \tag{21.11}$$

α and β run over the three neutrino types, e, μ and τ; $\nu_\alpha,\ \nu_\beta$ are left-handed Majorana fields; $m_{\alpha\beta}$ is an arbitrary complex matrix. In contrast to the case of Dirac neutrinos, $m_{\alpha\beta}$ can be taken to be symmetric. This is because fermion fields anticommute, so that $\nu_\alpha^{\mathrm{T}}\left(-\mathrm{i}\sigma^2\right)\nu_\beta$ is symmetric on the interchange of α and β (see Problem 21.2).

A general symmetric complex matrix can be transformed into a real diagonal matrix with positive diagonal elements by means of a single unitary matrix **U** (see, for example, Horn and Johnson (1985)). If $m_{\alpha\beta} = m_{\beta\alpha}$, we can write

$$m_{\alpha\beta} = \sum_{i=1}^{3} U_{\alpha i}\, m_i\, U_{\beta i}, \tag{21.12}$$

where the m_i are three positive masses. Note that **U** has no phase ambiguities, whereas Dirac neutrinos have phase ambiguities (see (19.2)).

If we now define the fields

$$\nu_i(x) = \sum_\alpha U_{\alpha i}\nu_\alpha(x), \tag{21.13}$$

the mass term takes the standard Majorana form:

$$\mathcal{L}_{\mathrm{mass}} = -\frac{1}{2}\sum_i m_i\nu_i^{\mathrm{T}}\left(-\mathrm{i}\sigma^2\right)\nu_i + \text{Hermitian conjugate}.$$

The dynamical terms in the Lagrangian density keep the same form under the transformation:

$$\mathcal{L}_{\mathrm{dyn}} = \sum_\alpha \mathrm{i}\nu_\alpha^\dagger\tilde{\sigma}^\mu\partial_\mu\nu_\alpha = \sum_{\mathrm{i}}\mathrm{i}\nu_{\mathrm{i}}^\dagger\tilde{\sigma}^\mu\partial_\mu\nu_i.$$

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$(\mathcal{L}_{\text{dyn}} + \mathcal{L}_{\text{mass}})$ is the Lagrangian density of free Majorana neutrinos of masses m_1, m_2, m_3. Inverting equation (21.13), the neutrino fields $\nu_\alpha(x)$ appear as mixtures of the neutrino fields of definite mass:

$$\nu_\alpha(x) = \sum_i U^*_{\alpha i} \nu_i(x). \tag{21.14}$$

This is of the same form as equation (19.6) for Dirac neutrinos. The consequences for the weak currents and neutrino oscillations are the same as in Section 19.2 and Section 19.3 for Dirac neutrinos but antineutrinos are interpreted as the neutrinos that accompany a negative charge lepton in weak interaction decays.

21.5 Parameterisation of U

A 3×3 unitary matrix $\mathbf{U}$ is specified by nine real parameters, but by absorbing phase factors into the definition of the lepton fields, as in Section 19.6, $U_{\alpha \mathrm{i}}$ can be redefined as

$$U'_{\alpha i} = \mathrm{e}^{\mathrm{i}\theta\alpha} U_{\alpha i},$$

without changing the physical content of the theory. Thus U can be characterised by $9 - 3 = 6$ parameters. The Dirac neutrino mixing matrix (Section 19.6) is determined by four parameters, and requires extension, to include two more parameters. One may take

$$U_{\text{Majorana}} = U_{\text{Dirac}} \times \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta 1} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Delta 2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{21.15}$$

Potentially we have two more CP violating parameters. However Δ_1 and Δ_2 make no contribution to the CP violation of the oscillation phenomena of Chapters 19 and 20 (see (19.19) and Problem 21.3)

21.6 Majorana neutrinos in the Standard Model

To bring Majorana neutrinos carrying mass into the Standard Model, we must maintain the $SU(2)$ symmetry of the weak interaction. As in the case of Dirac neutrinos, a suitable $SU(2)$ invariant expressions that we can construct from the Higgs doublet field Φ and a lepton doublet L_α is $(\Phi^{\mathrm{T}}\, \varepsilon\, L_\alpha)$ (See Section 19.5). On symmetry breaking, this becomes $(\Phi^{\mathrm{T}}\, \varepsilon\, L_\alpha) = -(\phi_0 + h/\sqrt{2})\nu_\alpha$.

$\phi_0 \approx$ 180 GeV is the Higgs field vacuum expectation value and $h(x)$ is the Higgs boson field.

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From these SU(2) invariant expressions we can construct an SU(2) invariant Lagrangian density that on symmetry breaking becomes

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}(\phi_0 + h/\sqrt{2})^2\, \nu_\alpha^{\mathrm{T}}(-\mathrm{i}\sigma^2)\nu_\beta K_{\alpha\beta} + \text{Hermitian conjugate.} \tag{21.16}$$

The matrix $K_{\alpha\beta}$ couples the neutrino fields to the Higgs field, and we can identify the mass term

$$m_{\alpha\beta} = \phi_0^2 K_{\alpha\beta}. \tag{21.17}$$

Hence the coupling matrix K has dimension $(\text{mass})^{-1}$, which implies (see Section 8.4) that it is an 'effective' Lagrange density. Coupling terms such as this render the theory unrenormalisable.

21.7 The seesaw mechanism

To address the question of renormalisability consider the Lagrangian density

$$\mathcal{L} = \mathrm{i}\nu_{\mathrm{L}}^\dagger\tilde{\sigma}^\mu\partial_\mu\nu_{\mathrm{L}} + \mathrm{i}R^\dagger\sigma^\mu\partial_\mu R - \frac{M}{2}\left(\mathrm{i}R^{\mathrm{T}}\sigma^2 R - \mathrm{i}R^\dagger\sigma^2 R^*\right) - \mu\nu_L^\dagger R - \mu R^\dagger\nu_{\mathrm{L}}. \tag{21.18}$$

M and μ are mass parameters; ν_{L} and R are two component left-handed and right-handed spinor fields respectively. Discarding the terms coupling ν_{L} and R, the Lagrangian density is that of a massless left-handed neutrino field ν_{L}, and a right-handed Majorana neutrino field carrying mass M.

We now suppose that M is so large that the dynamical term $\mathrm{i}R^\dagger\sigma^\mu\partial_\mu R$ may be neglected, to leave

$$\mathcal{L} = \mathrm{i}\nu_{\mathrm{L}}^\dagger\tilde{\sigma}^\mu\partial_\mu\nu_{\mathrm{L}} - \frac{M}{2}(R^{\mathrm{T}}(\mathrm{i}\sigma^2)R - R^\dagger(\mathrm{i}\sigma^2)R^*) - \mu\nu_{\mathrm{L}}^\dagger \mathrm{R} - \mu R^\dagger\nu_{\mathrm{L}}. \tag{21.19}$$

A variation δR^* in the action gives the field equation for R:

$$M\mathrm{i}\sigma^2 R^* - \mu\nu_{\mathrm{L}} = 0.$$

And multiplying by $\mathrm{i}\sigma^2/M$ we obtain

$$R = -(\mu/M)\mathrm{i}\sigma^2\nu_{\mathrm{L}}^*. \tag{21.20}$$

Substituting back into (21.19) gives the effective Lagrangian density

$$\mathcal{L} = \mathrm{i}\nu_{\mathrm{L}}^\dagger\tilde{\sigma}^\mu\partial_\mu\nu_{\mathrm{L}} + (\mu^2/2M)\left(\nu_{\mathrm{L}}^\dagger\mathrm{i}\sigma^2\nu_{\mathrm{L}}^* + \nu_{\mathrm{L}}^{\mathrm{T}}(-\mathrm{i}\sigma^2)\nu_L\right). \tag{21.21}$$

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The sign of the mass term can be changed by making the phase change $\nu_L \to \nu'_L = i\nu_L$. The effective $\mathcal{L}$ is then a free neutrino field of mass $m = \mu^2/M$. Taking for μ a typical lepton mass, say the mass of the muon (10^2 MeV), we can make m the magnitude of a neutrino mass by taking M sufficiently large, $>10^7$ GeV. The generalisation of the seesaw mechanism to include three neutrino types is straightforward.

Taking R to be an $SU(2)$ singlet, the Lagrangian density (21.19) can be made compatible with the Standard Model by replacing $\mu\nu_L^\dagger R$ with the $SU(2)$ invariant $C(L_L^\dagger\phi)R$, and similarly replacing $\mu R^\dagger \nu$, where C is a dimensionless coupling constant. After symmetry breaking, $\mu\nu_L^\dagger R$ becomes $C\left(\phi_O + h(x)/\sqrt{2}\right)\nu_L^\dagger R$ and setting aside the coupling to the Higgs boson, the mass $\mu = C\phi_0$. It should be noted though that although there are no dimensioned coupling constants the mass M is not generated by the Higgs mechanism.

21.8 Are neutrinos Dirac or Majorana?

The principal feature that distinguishes massive Majorana neutrinos from massive Dirac neutrinos is that Majorana neutrinos do not conserve lepton number. As pointed out in Section 21.2, in the Majorana case the $U(1)$ symmetry that gives lepton number conservation in the Dirac case is lost. The experimental observation of a lepton number violating process would therefore be of great interest. ‘Double β decay’ is the most promising phenomenon for investigation.

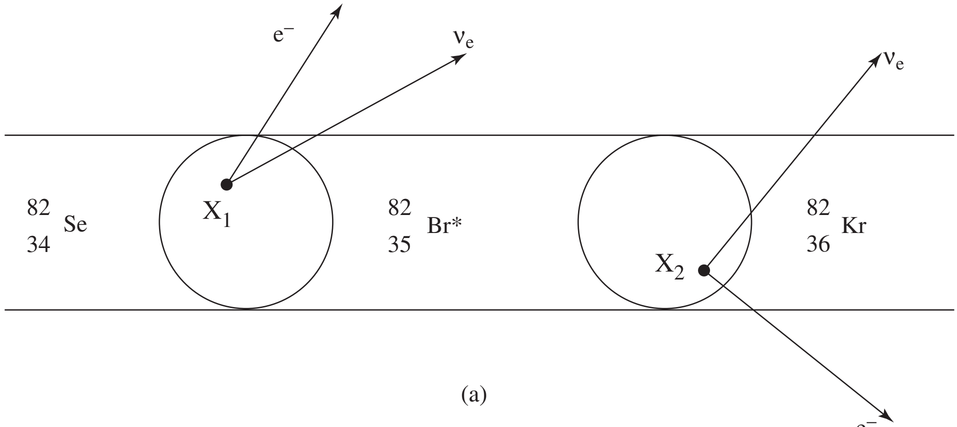
The first direct laboratory observation of double β decay was made in 1987, with the decay

$$ {}^{82}_{34}\mathrm{Se} \to {}^{82}_{36}\mathrm{Kr} + e^- + e^- + \bar{\nu}_e + \bar{\nu}_e + 3.03\,\mathrm{MeV}. $$

The mean lifetime for this decay has been measured to be $(9.2 \pm 1)\ 10^{19}$ yrs.

If neutrinos are Dirac particles, $\bar{\nu}_e$ is the appropriate symbol in this decay. If neutrinos are Majorana particles, ν and $\bar{\nu}$ are identical. The observed decay does not distinguish between the two interpretations. The process is illustrated in Fig. 21.1a. An electron and a $\bar{\nu}$ in the Dirac case, or a ν in the Majorana case, are created at each interaction point at which a d quark is transformed into a u quark. The nucleus becomes ${}^{82}_{35}\mathrm{Br}$, possibly in an excited state, between the interaction points.

If neutrinos are Majorana, the decay might be a neutrinoless double β decay, as envisaged in Fig. 21.1b. The neutrino created at X_1 is annihilated at X_2, giving a change of 2 in lepton number. This process is not available if neutrinos are Dirac particles. In the absence of neutrinos to share the energy, the sum of the energies of



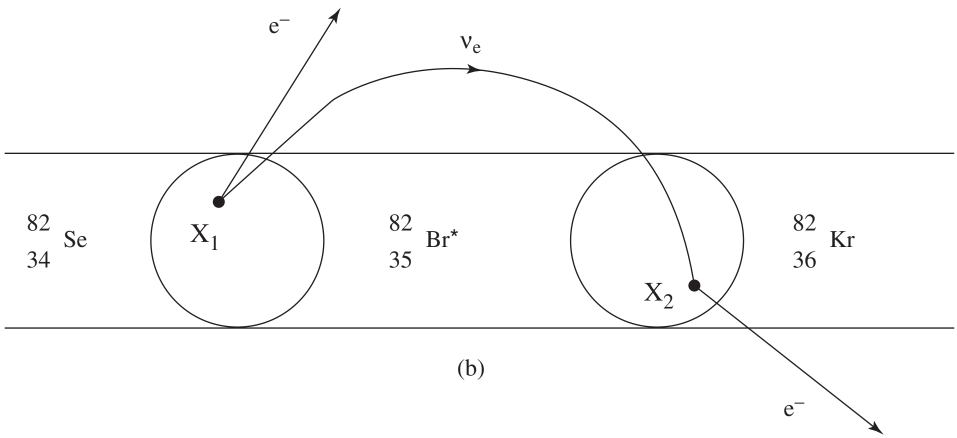


Figure 21.1 (a) Illustrates the two neutrino double β decay of ${}^{82}_{34}$Se. The decay occurs at the second order of perturbation theory in the weak interaction and involves a sum over many states of ${}^{82}_{35}$Br (denoted by ${}^{82}_{35}$Br*).
(b) Illustrates the neutrinoless double β decay, a Majorana neutrino created in the transition ${}^{82}_{34}\mathrm{Se} \rightarrow {}^{82}_{35}\mathrm{Br}^*$ is annihilated in the transition ${}^{82}_{35}\mathrm{Br}^* \rightarrow {}^{82}_{36}\mathrm{Kr}$. In perturbation theory this involves a sum over all momentum states of the neutrino as well as many states of ${}^{82}_{35}$Br.

the two electrons emitted would be sharply peaked at the decay energy. (The recoil energy of the nucleus would be small.)

Double β decay and neutrinoless double β decay occur at the second order of perturbation theory in the effective weak interaction of equation (14.22). For Majorana neutrinos, double β decay and neutrinoless double β decay are competing processes. Neutrinoless decays are heavily suppressed. From the field equation

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Table 21.1. *From Elliot and Vogel hep/ph/0202264 Feb 2002*

Nucleus	$T_{1/2}^{2\nu}$ (years)	Estimate $T_{1/2}^{0\nu}$ (years)	Measured 0υ half life Lower limit (years)
^{48}Ca	$(4.2 \pm 1.2)\,10^{19}$	$(2.2 \pm 1.3)\,10^{25}$	$> 9.5 \times 10^{21}$
^{76}Ge	$(1.3 \pm 0.1)\,10^{21}$	$(3.2 \pm 2.4)\,10^{25}$	$> 1.9 \times 10^{25}$
^{82}Se	$(9.2 \pm 1.0)\,10^{19}$	$(1.3 \pm 1.0)\,10^{25}$	$> 2.7 \times 10^{22}$
^{100}Mo	$(8.0 \pm 0.6)\,10^{18}$	$(8.4 \pm 7.2)\,10^{26}$	$> 5.5 \times 10^{22}$
^{116}Cd	$(3.2 \pm 0.3)\,10^{19}$	$(1.0 \pm 0.9)\,10^{25}$	$> 7.0 \times 10^{22}$

(21.1), the decay amplitude for the neutrinoless mode, with an intermediate neutrino of mass m_i and energy E_ν, is proportional to

$$(m_i/2E_\nu)\left\lfloor \mathrm{e}^{-\theta/2}\,\mathrm{e}^{\theta/2} + \mathrm{e}^{\vartheta/2}\,\mathrm{e}^{-\theta/2}\right\rfloor = (m_\mathrm{i}/E_\nu)\,.$$

The two terms come from the two helicity states. The corresponding factors in two neutrino β decay are dominated by the term $(m_i/2E_\nu)\,\mathrm{e}^\theta$, and $\mathrm{e}^\theta \approx 2\cosh\theta = (2E_\nu/m_i)$, giving unity.

With three neutrino mass eigenstates the decay rate will be proportional to $(1/\bar{E}_\nu^2)|\sum_i m_i U_{\mathrm{e}i}|^2$ where $\bar{E}_\nu$ is some mean neutrino energy that can be expected to be a nuclear excitation energy.

Table 21.1. gives some measured two neutrino β decay half lives, and corresponding estimates of the half lives of the neutrinoless decays. These theoretical estimates are sensitive to the nuclear model used.

Problems

21.1 Show that $(\mathrm{i}\sigma^2\nu^*)^\dagger\sigma^\mu\partial_\mu(\mathrm{i}\sigma^2\nu^*) = \nu^\dagger\tilde{\sigma}^\mu\partial_\mu\nu$.

21.2 Show that, taking account of the anticommuting spinor fields,

$$\nu_\alpha^\mathrm{T}\sigma^2\nu_\beta = \nu_\beta^\mathrm{T}\sigma^2\nu_\alpha.$$

21.3 Denoting the Majorana and Dirac mixing matrices by U^M and U^D, show that $U_{\beta j}^\mathrm{M}U_{\alpha j}^{\mathrm{M}*} = U_{\beta j}^\mathrm{D}U_{\alpha j}^{\mathrm{D}*}$ and hence that the phenomenology of mixing is the same for both Majorana and Dirac neutrinos.