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The Bloch-Nevanlinna conjecture revisited

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In 1929 Rolf Nevanlinna posed a problem attributed to Bloch which has since been known as the Bloch-Nevanlinna conjecture. It can be stated as follows: Is the derivative of a function of bounded characteristic of bounded characteristic? A variety of different counterexamples have provided negative answers to this question. The purpose of the paper is to survey these counterexamples and then give a truly elementary proof of the following theorem.

If $\{b_k\}$ *is any sequence of complex numbers with* $\limsup |b_k| > 0$ *, then there is a bounded analytic function* $f(z)$ *in* $|z| < 1$ *such that* $f'(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$ *has finite radial limits nowhere.*

In 1929, Nevanlinna, [9, p. 138] posed a problem attributed to Bloch which has since been known as the "Bloch-Nevanlinna conjecture". It can be stated as follows: Is the derivative of a function of bounded characteristic always of bounded characteristic? A variety of different counterexamples have provided negative answers to this question, and the purpose of this paper is to survey these counterexamples and then give an elementary way in which any arbitrary sequence of complex numbers with $\limsup |b_k| > 0$ can be used to construct a counterexample.

Let

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$$T(r,\ f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta + \int_0^r \frac{n(t,\infty)-n(0,\infty)}{r}\, dt + n(0,\ \infty) \log r\ ,$$

where $n(t,\ \infty)$ denotes the number of poles of $f(z)$ in $|z| < t$, each being counted according to its multiplicity, and

$$\log^+ |f(re^{i\theta})| = \max(\log|f(re^{i\theta})|,\ 0)\ .$$

A function meromorphic in $|z| < 1$ is said to be of bounded characteristic if its characteristic $T(r,\ f)$ is bounded for $r < 1$; this is equivalent to $f(z)$ being the quotient of two bounded analytic functions [*10*, p. 188]. In particular, any bounded analytic function is of bounded characteristic. Since non-trivial bounded analytic functions have non-zero radial limits almost everywhere [*2*, p. 21], it is obvious that functions of bounded characteristic have finite radial limits at almost all points of $|z| = 1$. Most counterexamples have, therefore, consisted of a bounded analytic function whose derivative does not have finite radial limits almost everywhere.

The first known counterexample is due to Frostman [5] in 1942. He constructed a Blaschke product whose derivative fails to have finite radial limits anywhere. In 1946, Fried [4] proved that if $f(z) = \sum_{k=1}^{\infty} c_k z^{n_k}$ is a function of bounded characteristic with Hadamard gaps, $n_{k+1}/n_k \geq q > 1$, $k = 1,\ 2,\ \ldots$, then necessarily

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty\ . \tag{1}$$

This theorem makes it easy to construct counterexamples. In particular, $f(z) = \sum_{k=1}^{\infty} z^{n_k}/n_k$, $n_{k+1}/n_k \geq q > 1$, is an example of a bounded analytic function whose derivative f' cannot be of bounded characteristic $\left(zf'(z) = \sum_{k=1}^{\infty} z^{n_k}\right.$ violates $\left.(1)\right)$.

In 1955 Rudin [*11*] created a bounded analytic function $f(z) = \sum_{k=1}^{\infty} k^{-2} z^{n_k}$ such that $f'(z)$ had finite radial limits on at most a set of measure zero. Later that year Lohwater, Piranian, and Rudin [8] showed that one could even construct a bounded *univalent* function $f(z)$ for which $f'(z)$ has no radial limit almost everywhere.

In 1964 Hayman [7] proved directly from the definition of bounded characteristic that the derivative of the bounded function

$f(z) = \sum_{k=1}^{\infty} M^{k(\alpha-1)} z^{M^k}$, $0 < \alpha < 1$, M a large positive integer depending on α , is not of bounded characteristic. The result can also be obtained immediately from Fried's Theorem above.

In 1969 Duren [3] showed that the first derivative was not privileged with respect to the Bloch-Nevanlinna conjecture. In particular, he showed that there is a bounded analytic function of the form $f(z) = \sum_{n=1}^{\infty} a_n z^n$ where

$$|a_n| = \begin{cases} k^{-2} & \text{if } n = 2^k , \quad k = 1, 2, \ldots , \\ 0 & \text{otherwise,} \end{cases}$$

such that for every $\alpha > 0$, the fractional derivative of order α ,

$$f^{(\alpha)}(z) = \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+1-\alpha)} a_n z^n ,$$

fails to have finite radial limits almost everywhere.

Duren's result can be put in a multiplier context. A sequence of complex numbers $\{b_n\}$ is a (Hadamard) *multiplier* of functions of bounded characteristic if for every $f(z) = \sum_{n=1}^{\infty} a_n z^n$ of bounded characteristic the function $g(z) = \sum_{n=1}^{\infty} a_n b_n z^n$ is of bounded characteristic. The Bloch-Nevanlinna conjecture essentially asked if $\{n\}_{n=1}^{\infty}$ is such a multiplier. Duren's result shows that for every $\alpha > 0$, $\left\{\frac{n!}{\Gamma(n+1-\alpha)}\right\}_{n=1}^{\infty}$ does not even multiply bounded functions into functions of bounded characteristic.

In 1971 Anderson [1] dramatically extended the set of counterexamples.

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He proved that every sequence of the form $\{\phi(n)\}_{n=1}^{\infty}$, $\phi(n)$ monotone increasing to ∞ , fails to multiply the bounded analytic functions satisfying $\sum |a_n| < \infty$ into functions of bounded characteristic; that is, given any function $\phi(n)$ monotonically increasing to ∞ , there is a bounded analytic $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $\sum |a_n| < \infty$, such that $f(z, \phi) = \sum_{n=1}^{\infty} \phi(n) a_n z^n$ does not have bounded characteristic. Similar to Hayman's approach, Anderson does this directly in terms of the characteristic function instead of proving the stronger result that $f(z, \phi)$ does not have a finite radial limit almost everywhere.

In 1972 Hahn [6] gave a soft proof that there is a bounded analytic function which is continuous on $|z| = 1$, $\sum |a_n| < \infty$, whose derivative has no finite radial limit anywhere. In fact, if A denotes the Banach space of all absolutely convergent power series, $\sum |a_n| < \infty$ with norm $\|f\| = \sum |a_n|$, and if E denotes the set of functions in A whose derivative has no finite radial limit anywhere on $|z| = 1$, then E is dense in A . This result disproves the conjecture in a rather emphatic manner. Although Hahn's result guarantees the existence of a very large class of counterexamples, the proof does not display a single concrete example.

Each of these counterexamples have required a good deal of work, usually involving measure theory, powerful results of Fourier series, or techniques involving the characteristic function. We now present a theorem whose proof is truly elementary which will show how a counterexample can be constructed using *any* sequence of complex numbers with $\limsup |b_k| > 0$.

THEOREM. *Let* $\{b_k\}$ *be any sequence of complex numbers with* $\limsup |b_k| > 0$. *Then there is a bounded analytic function* $f(z)$ *such that* $f'(z) = \sum_{k=1}^{\infty} b_k z^{n_k}$ *has finite radial limits nowhere.*

Proof. We will first prove the theorem under the additional assumption that $|b_k| \le k$. Define the sequence of exponents $\{n_k\}$ by $n_k = 4^{k^2}$ and associate with this a sequence of r_k's by letting $r_k = \left(1-2^{-k}\right)^{1/n_k}$. Since

$$(2) \qquad r_{k+1}^{n_{k+1}} = \left(1-2^{-(k+1)}\right) > \left(1-2^{-k}\right) = r_k^{n_k} > r_k^{n_{k+1}},$$

we see that $\{r_k\}$ is a strictly increasing sequence tending to 1.

We now claim that $f(z) = \sum b_k z^{1+n_k}/\left(1+n_k\right)$ is bounded and analytic in $|z| < 1$ and $f'(z)$ has finite radial limits nowhere on $|z| = 1$. The function $f(z)$ is certainly bounded and analytic since $|f(z)| \le \sum_{k=1}^{\infty} k4^{-k^2} < \infty$. Suppose to the contrary that $f'(z)$ has a finite radial limit at $z = e^{i\theta}$; that is,

$$(3) \qquad \lim_{r\to 1^-} \sum_{k=1}^{\infty} b_k e^{i\theta n_k} r^{n_k} = s, \quad s \in \mathbb{C}.$$

Letting $b_k e^{i\theta n_k} = a_k$ we prove that (3) implies $\sum_{k=1}^{\infty} a_k = s$. Let ε be an arbitrary positive number. Since $|a_k| \le k$, we can find a j such that

$$(4) \qquad 2^{-j} \sum_{k=1}^{j} |a_k| < \varepsilon/2 \quad \text{and} \quad \sum_{k=j+1}^{\infty} ke^{-4^k} < \varepsilon/2.$$

Using this fixed j we have for all k, $k \le j$,

$$(5) \qquad 1 - r_j^{n_k} \le 1 - r_j^{n_j} = 2^{-j},$$

while for all k, $k \ge j+1$, we have

$$(6) \qquad r_j^{n_k} \le r_{k-1}^{n_k} = \left(1 - \frac{1}{2^{k-1}}\right)^{4^{2k-1}} < e^{-4^k},$$

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since $(1-x) \le e^{-x}$. Thus by (5), (6), and (4) we have

$$\left|\sum_{k=1}^{j} a_k - \sum_{k=1}^{\infty} a_k r_j^{n_k}\right| \le \sum_{k=1}^{j} |a_k| \left(1-r_j^{n_k}\right) + \sum_{k=j+1}^{\infty} |a_k| r_j^{n_k}$$

$$\le 2^{-j} \sum_{k=1}^{j} |a_k| + \sum_{j=k+1}^{\infty} k e^{-4^k} < \varepsilon .$$

We now have the first equality in (7) while the second follows from (3) and the fact that $r_j \to 1$,

$$\lim_{j\to\infty} \sum_{k=1}^{j} a_k = \lim_{j\to\infty} \sum_{k=1}^{\infty} a_k r_j^{n_k} = \lim_{r\to 1^-} \sum_{k=1}^{\infty} a_k r^{n_k} = s . \tag{7}$$

Thus the existence of a finite radial limit in (3) implies that $\sum_{k=1}^{\infty} a_k$ converges to s . But $\sum_{k=1}^{\infty} a_k$ can not converge since by hypothesis $|a_k| \not\to 0$. Therefore, $f'(z)$ can have no finite radial limits.

Finally, if $|b_k| \nleq k$, then form a new sequence $\{b_k^*\}$ by inserting zeros between the elements of the sequence $\{b_k\}$ so that $|b_k^*| \le k$. The bounded analytic function $f^*(z)$ associated with $\{b_k^*\}$ has a derivative with finite radial limits nowhere and $f^{*\prime}(z) = \sum_{k=1}^{\infty} b_k^* z^{n_k} = \sum_{k=1}^{\infty} b_k z^{n_k^*}$ as desired. This concludes the proof of the theorem.

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