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LOCAL REPRESENTATIONS OF THE LOOP BRAID GROUP

ZOLTÁN KÁDÁR[1], PAUL MARTIN

Department of Pure Mathematics
University of Leeds
Leeds, LS2 9JT
United Kingdom
e-mail: zokadar@gmail.com, P.P.Martin@leeds.ac.uk

ERIC ROWELL

Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
USA
e-mail: rowell@math.tamu.edu

and ZHENGHAN WANG

Microsoft Research, Station Q and Department of Mathematics
University of California
Santa Barbara, CA 93106
USA
e-mail: zhenghwa@micros.oft.com, zhenghwa@math.ucsb.edu

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Abstract. We study representations of the loop braid group LB_n from the perspective of extending representations of the braid group $\mathcal{B}_n$. We also pursue a generalization of the braid/Hecke/Temperlely–Lieb paradigm – uniform finite dimensional quotient algebras of the loop braid group algebras.

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1. Introduction. Non-abelian statistics of anyons in two spatial dimensions has attracted considerable attention largely due to topological quantum computation [**27**, **32**]. Recently, non-abelian statistics is extended to statistics of point-like topological defects projectively [**6**]. But an extension of non-abelian statistics of point-like excitations to three spatial dimensions is not possible. However, loop or closed string excitations occur naturally in condensed matter physics and string theory. Therefore, it is important to study statistics of extended objects in three spatial dimensions.

A systematical way to produce interesting and powerful representations of the braid group is via $(2+1)$-topological quantum field theories (TQFTs) [**32**]. Since the loop braid group is a motion group of sub-manifolds[†], we expect that interesting representations of the loop braid group could result from extended $(3+1)$-TQFTs.

[†]Roughly speaking, a motion of a submanifold N of a smooth manifold M is a path f_t in the diffeomorphism group Diff(M) such that the start and end points are in the subgroup Diff(M, N) of elements that restrict to elements of Diff(N) [**12**, **16**, **22**]. When N is n points in $\mathbb{R}^2$ the group of motions up to suitable equivalence is a braid group; and when N is the trivial link with n components in $\mathbb{R}^3$ we get a loop braid group. See also later.

But $(3+1)$-TQFTs are much harder to construct, and the largest known explicit class is the Crane-Yetter TQFTs based on pre-modular categories [**11**, **34**]. The difficulty of constructing interesting representations of the loop braid group reflects the difficulty of constructing non-trivial $(3+1)$-TQFTs. Potentially, given a pre-modular category $\mathcal{C}$, there are representations of all motion groups of sub-manifolds including the loop braid group associated to $\mathcal{C}$, but no explicit computation has been carried out for any non-trivial theory. Hence, we will take a closely related, but different first step in the study of representations of the loop braid group.

The tower of group algebras of Artin's braid group $\mathcal{B}_n$, for $n \geq 1$ have topologically interesting quotients, such as the Temperley–Lieb algebras [**17**], Hecke algebras [**18**] and BMW-algebras [**9**, **26**]. Each of these algebras support a Markov trace which then produces polynomial knot and link invariants. Moreover, at roots of unity many such quotient algebras can be realized as endomorphism algebras in unitary modular categories–the algebraic structure underlying certain $(2+1)$-TQFTs [**30**]. These, in turn, describe the quantum symmetries of topological phases of matter in two spatial dimensions [**33**]. The braid group representations associated with unitary modular categories would be physically realized as the motion of point-like particles in the disk D^2. Our goal is to generalize this picture to topological systems in three spatial dimensions with loop-like excitations.

The loop braid group LB_n is the motion group of the n-component oriented unlink inside the 3-dimensional ball D^3 [**12**, **16**, **22**]. It has appeared in other contexts as well: it is isomorphic to the braid-permutation group (see [**3**]), the welded-braid group (see [**14**]) and the group of conjugating automorphisms of a certain free group (see [**23**]), the group of ribbon tubes [**2**], the group of flying rings [**7**] and the fundamental group of the configuration space of Euclidian circles [**10**]. For an exploration of the structure as a semidirect product, see [**4**]. Very little is known about the linear representations of LB_n. We investigate when a given representation of $\mathcal{B}_n$ may be extended to LB_n. Some results in this direction are found in [**5**] and [**31**]. For example, it is known that the faithful Lawrence–Krammer–Bigelow (LKB) representation of $\mathcal{B}_n$ does not extend to LB_n for $n \geq 4$ except at degenerate values of the parameters ([**5**]), but the Burau representation of $\mathcal{B}_n$ does extend.

It seems to be a rather hard problem to discover interesting finite-dimensional quotients of the tower of loop braid group algebras of LB_n. Considering that the LKB representation appears in the BMW-algebra, we should not expect to simply extend known $\mathcal{B}_n$ quotients. Our approach is to consider extensions of $\mathcal{B}_n$ representations associated with solutions to the parameter-free Yang–Baxter equation. This ensures that the quotient algebras are finite dimensional. The main problem we study is when such representations extend. One particular family of extendible representations are studied in some detail: the so-called affine group-type solutions.

The contents of the paper are as follows. In Section 2, we recall a presentation of the loop braid group. In Section 3, we study representations of the loop braid group from braided vector spaces, and hence make the connection to Drinfeld doubles. In Section 4, we initiate a general program to generalize the braid/Hecke/Temperley–Lieb paradigm – uniform finite dimensional quotient algebras of the loop braid quotient algebras, and report some preliminary analysis. In particular, we answer a question that has been open for some time, raised in [**24**, Section 12.1], about the structure of certain 'cubic' braid group representations that lift to loop braid representations.

2. The loop braid group and its relatives. Let us start with a group presentation.

THEOREM. [**14**] *The **loop braid group** LB_n is isomorphic to the abstract group generated by $2(n-1)$ generators σ_i and s_i for $1 \leq i \leq (n-1)$, satisfying the following three sets of relations:*

*The **braid relations**:*

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| > 1$,

*the **symmetric group relations**:*

(S1) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$

(S2) $s_is_j = s_js_i$ *for* $|i-j| > 1$,

(S3) $s_i^2 = 1$,

*and the **mixed relations**:*

(L0) $\sigma_is_j = s_j\sigma_i$ *for* $|i-j| > 1$

(L1) $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$

(L2) $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}.$

□

The images of the generators σ_i, s_i in the motion group *per se* are given for example in [**10**, **7**, **3**]. The subgroup generated by the $\{\sigma_i\}$ is Artin's braid group $\mathcal{B}_n$. (There is an isomorphism of LB_n to the automorphism group of the free group with n generators [**8**] which takes this subgroup to $\mathcal{B}_n$ [**19**].) The second set $\{s_i\}$ generate the symmetric group $\mathfrak{S}_n$. The loop braid group is a quotient of the **virtual braid group** VB_n [**31**] which satisfies all relations above except $(L2)$.

The relations $(L1)$ also hold if read backwards, *i.e.* $s_{i+1}s_i\sigma_{i+1} = \sigma_is_{i+1}s_i$, but $(L2)$ is not equivalent to its reverse:

(L3) $s_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_is_{i+1}$.

However, in the transposed group OLB_n (*i.e.* the group that coincides with LB_n as a set, but with the opposite multiplication $a * b = ba$) one has all relations as in LB_n except $(L2)$ is replaced by $(L3)$. Every group is isomorphic to its transposed group (via inversion) so we may freely work with either LB_n or OLB_n.

We define the **symmetric loop braid group** SLB_n to be LB_n modulo the relations $(L3)$. In particular, we have surjections $VB_n \twoheadrightarrow LB_n \twoheadrightarrow SLB_n$. Note, that this group was called unrestricted virtual braid group in [**21**].

3. LB_n representations from braided vector spaces. Several authors (see e.g. [**31**]) have considered the question of extending representations of $\mathcal{B}_n$ to LB_n. In this section, we consider extending certain local representations of $\mathcal{B}_n$ (see [**29**]).

A **braided vector space** (BVS) (V, c) is a solution $c \in GL(V^{\otimes 2})$ to the Yang–Baxter equation:

$$(c \otimes Id_V)(Id_V \otimes c)(c \otimes Id_V) = (Id_V \otimes c)(c \otimes Id_V)(Id_V \otimes c).$$

Any BVS gives rise to a **local** representation ρ^c of $\mathcal{B}_n$ via $\sigma_i \to Id_V^{\otimes i-1} \otimes c \otimes Id_V^{\otimes n-i-1}$. If an extension of ρ^c to LB_n or OLB_n is given via $s_i \to Id_V^{\otimes i-1} \otimes S \otimes Id_V^{\otimes n-i-1}$ where $S \in \mathrm{GL}(V^{\otimes 2})$, then it will be also called **local**. The corresponding triple (V, c, S) will be called a **loop braided vector space**.

A special case of local $\mathcal{B}_n$ representations through *group-type* BVSs were introduced by Andruskiewitsch and Schneider [**1**]. These play an important role in their

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classification program for pointed finite-dimensional Hopf algebras. We extend their definition slightly and say that a BVS (V, c) is of **left group-type** (resp. **right group-type**) if there is an ordered basis $X := [x_1, \ldots, x_n]$ of V and $g_i \in GL(V)$ such that $c(x_i \otimes z) = g_i(z) \otimes x_i$ (resp. $c(z \otimes x_j) = x_j \otimes g_j(z)$) for all i, j and $z \in V$. There is a one-to-one correspondence between left and right group-type BVSs, since the Yang–Baxter equation is invariant under $c \leftrightarrow c^{-1}$. Indeed, the inverse of $c(x_i \otimes x_j) = g_i(x_j) \otimes x_i$ is $c^{-1}(x_i \otimes x_j) = x_j \otimes g_j^{-1}(x_i)$, so that (V, c) is a BVS of left group-type if and only if c^{-1} is a BVS of right group-type.

LEMMA 3.1. *Suppose that (V, c) is a BVS of left group-type with respect to $X := [x_1, \ldots, x_n]$ and corresponding g_i defined on X by $g_i(x_j) := \sum_{k=1}^n g_i^{j,k} x_k$. If $g_i^{j,k} \neq 0$ then $g_i g_j = g_k g_i$.*

Proof. we compute

$$(c \otimes I)(I \otimes c)(c \otimes I)(x_i \otimes x_j \otimes z),$$

and

$$(I \otimes c)(c \otimes I)(I \otimes c)(x_i \otimes x_j \otimes z),$$

and compare the two sides. This yields the equality:

$$\sum_{k=1}^{n} g_i^{j,k} g_i g_j(z) \otimes x_k \otimes x_i = \sum_{k=1}^{n} g_i^{j,k} g_k g_i(z) \otimes x_k \otimes x_i.$$

Thus, we see that if $g_i^{j,k} \neq 0$ then $g_i g_j(z) = g_k g_i(z)$ for all z, and the result follows. □

The proof of Lemma 3.1 shows that the Yang-Baxter equation for (V, c) of left group type is equivalent to the matrix equation:

$$g_i^{j,k} g_i g_j = g_i^{j,k} g_k g_i \quad \text{for all } i, j, k. \tag{3.1}$$

A similar result may be derived for right group type BVSs.

If g_i acts diagonally with respect to the basis X so that $c(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$ for some scalars q_{ij} then we say (V, c) is of **diagonal type**. More generally, we will say that (V, c) is **diagonalizable** if there exists a basis of V with respect to which (V, c) is a BVS of diagonal type. We do not need to specify a handedness for diagonal type BVS, indeed we have:

LEMMA 3.2. *A BVS (V, c) is of both left and right group type if and only if (V, c) is diagonalizable.*

Proof. If c is of left group type with respect to X and $g_i \in GL(V)$ and right group type with respect to $Y := [y_1, \ldots, y_n]$ and $h_j \in GL(V)$ then $x_i \otimes y_j$ is a basis for V, and $c(x_i \otimes y_j) = g_i(y_j) \otimes x_i = y_j \otimes h_j(x_i)$. This implies that the g_i are simultaneously diagonalized in the basis Y so that the g_i pairwise commute. Denote by G the (abelian) group generated by the g_i and let $g_i^{(j,k)}$ be the coefficient of x_k in $g_i(x_j)$. Since the g_i pairwise commute, Lemma 3.1 shows that $g_i^{(j,k)} \neq 0$ implies $g_j = g_k$. Now note that the spaces $W_k := \mathbb{C}\{x_j : g_j = g_k\}$ are G-invariant, and denote by $I_k := \{j : x_j \in W_k\}$, so

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that the distinct I_k partition $[n]$. So choose a basis for each W_k with respect to which each g_i is diagonal, and denote the union of these bases by Z. It is clear that g_i are diagonal with respect to the basis Z, but we must check that (V, c) is of group type with respect to this basis. Let $z_k = \sum_{j\in I_k} z_j^k x_j \in W_k \cap Z$. Then,

$$c(z_k \otimes z_s) = \sum_{j\in I_k} z_j^k c(x_j \otimes z_s) = \sum_{j\in I_k} g_j(z_s) \otimes z_j^k x_j = q_{k,s} z_s \otimes z_k,$$

since all the g_j with $j \in I_k$ are identical and so act by a common scalar $q_{k,s}$ on z_s.

The other direction is clear, diagonal type BVSs are of both left and right group type. □

BVSs of group type *always* extend to loop BVSs, with left group-type BVSs giving representations of OLB_n and right group-type BVSs giving representations of LB_n:

PROPOSITION 3.3. *Define* $S(x_i \otimes x_j) := x_j \otimes x_i$. *If* (V, c) *is a BVS of left (resp. right) group-type then* (V, c, S) *is a loop braided vector space.*

Proof. Define $\rho^c(s_i) = Id_V^{\otimes i-1} \otimes S \otimes Id_V^{\otimes n-i-1}$. Relations $(B1), (B2), (S1), (S2), (S3)$ and $(L0)$ are immediate. Since inversion gives an isomorphism from LB_n to OLB_n and produces a left group-type BVS from a right group-type BVS it suffices to check the relations $(L1)$ and $(L3)$ for $i = 1$. First,

$$\rho^c(s_1 s_2 \sigma_1)(x_i \otimes x_j \otimes x_k) = (x_k \otimes g_i(x_j) \otimes x_i) = \rho^c(\sigma_2 s_1 s_2)(x_i \otimes x_j \otimes x_k),$$

verifying $(L1)$. Similarly,

$$\rho^c(\sigma_2 \sigma_1 s_2)(x_i \otimes x_j \otimes x_k) = g_i(x_k) \otimes g_i(x_j) \otimes x_i = \rho^c(s_1 \sigma_2 \sigma_1)(x_i \otimes x_j \otimes x_k).$$

so we have $(L3)$. □

Suppose that (V, c) is of left group-type, and we define $\rho^c(s_i)$ via S as in the proof of Proposition 3.3. Then, $(L2)$ is satisfied if and only if the g_i pairwise commute:

$$\rho^c(\sigma_1 \sigma_2 s_1)(x_i \otimes x_j \otimes x_k) = g_j g_i(x_k) \otimes j \otimes i = g_i g_j(x_k) \otimes x_j \otimes x_i = \rho^c(s_2 \sigma_1 \sigma_2).$$

In particular, if (V, c) is both of left and right group-type then ρ^c extends to a representation of SLB_n. More generally, we have:

PROPOSITION 3.4. *Suppose that* (V, c) *and* (V, S) *are of diagonal type with respect to the (same) basis* X *and* $S^2 = Id_{V^{\otimes 2}}$. *Then,* ρ^c *extends to a representation of* SLB_n *via* $\rho^c(s_i) = Id_V^{\otimes i-1} \otimes S \otimes Id_V^{\otimes n-i-1}$.

Proof. It suffices to check $(L1), (L2)$ and $(L3)$, which are straightforward calculations. □

Note, that in case (V, c) is of group type (either right or left), c takes a canonical form in terms of the basis $X = [x_1, \ldots, x_n]$ and in terms of that basis $S(x_i \otimes x_j) = \pm x_j \otimes x_i$ then (V, c) is of diagonal type if $cS = Sc$. In this case, the index of the subgroup $\rho^c(B_n)$ in $\rho^c(LB_n)$ is finite. The representations τ_N in section 4.4 belong to this class.

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3.1. Affine group-type BVSs. We are interested in local representations of LB_n that detect symmetry, *i.e.* that do not factor over SLB_n. Fix $m \in \mathbb{N}$ and let V be an m-dimensional vector space with basis $[x_1, \dots, x_m]$. For each $1 \leq j \leq m$ define $h_j(x_i) = x_{\alpha i + \beta j}$ for some $\alpha, \beta \in \mathbb{N}$, where indices are taken modulo m. We will determine sufficient conditions on α and β so that $c(x_i \otimes x_j) := x_j \otimes h_j(x_i)$ gives (V, c) the structure of a right BVS. We will call these **affine** group-type BVSs. For notational convenience we will identify x_i with $i \pmod m$ and define $h_j(i) = \alpha i + \beta j$ where now α, β are integers modulo m, and denote $x_i \otimes x_j$ by (i, j).

The operator h_j is invertible if and only if $\gcd(\alpha, m) = 1$. Since we are interested in finding BVSs that do not factor over SLB_n, we should look for non-diagonalizable affine BVSs. By the proof of Lemma 3.2 we see that a BVS corresponding to $\{h_j : 1 \leq j \leq m\}$ is diagonalizable if and only the h_j pairwise commute. Computing $h_i h_j(k) = h_j h_i(k)$ we see that this happens precisely when $(\alpha - 1)\beta \equiv 0 \pmod m$. In particular, we must assume that $\alpha \not\equiv 1 \pmod m$ and $\beta \not\equiv 0 \pmod m$.

By Proposition 3.3 as soon as we have determined values α, β so that (V, c) is a (right) BVS we may extend ρ^c to LB_n by taking $S(x_i \otimes x_j) = x_j \otimes x_i$. Computing, we have:

$$\sigma_1\sigma_2\sigma_1(i, j, k) = (k, h_k(j), (h_k \circ h_j)(i)) = \sigma_2\sigma_1\sigma_2(i, j, k) = (k, h_k(j), (h_{h_k(j)} \circ h_k)(i)).$$

Therefore, we must have

$$(h_k \circ h_j)(i) = \alpha^2 i + \alpha\beta j + \beta k = (h_{h_k(j)} \circ h_k)(i) = \alpha^2 i + \alpha\beta(k + j) + \beta^2 k,$$

that is, $\beta(\alpha + \beta) = \beta$. One family of solutions corresponds to $\alpha + \beta = 1$ so we set $t = \alpha$ and $\beta = (1 - t)$. In this case $(\alpha - 1)\beta = -(t - 1)^2$, so we have proved:

THEOREM 3.5. *Let $m, t \in \mathbb{N}$ with $\gcd(m, t) = 1$ and $(t - 1)^2 \not\equiv 0 \pmod m$. Then defining $h_j(x_i) = x_{ti + (1-t)j}$ and $S(x_i \otimes x_j) = x_j \otimes x_i$ (indices modulo m) on the basis $X := [x_1, \dots, x_m]$ gives rise to a loop braided vector space (V, c, S) of LB_n such that the corresponding LB_n representation, φ, does not factor over SLB_n.*

REMARK 3.6. For m prime, the family of loop braided vector spaces in Theorem 3.5 are all possible non-diagonalizable affine BVSs, but for m composite there are other solutions. We will only focus on these solutions in the present work.

It is clear from the construction that the representations φ act by permutation on the standard basis vectors of $V^{\otimes n}$. By passing to the action on indices, we may identify the $\mathbb{C}$-representation φ in Theorem 3.5 with the following homomorphism $\rho_{m,t} : LB_n \to \mathrm{GL}_n(\mathbb{Z}_m)$ via

$$\rho_{m,t}(\sigma_i) = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix}, \rho_{m,t}(s_i) = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix},$$

where $M = \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with entries in $\mathbb{Z}_m$. For later use, we point out that evaluating $\rho_{m,t}(\sigma_i)$ at $t = 1$ gives $\rho_{m,t}(s_i)$.

We now investigate the images of these representations.

The restriction of $\rho_{m,t}$ to $\mathcal{B}_n$ may look familiar: it is nothing more than the (inverse of) the (unreduced) Burau representation, specialized at an integer t with entries modulo m. In light of [**31**] it is not surprising that the Burau representation should admit an extension to LB_n (although we caution the reader that [**31**] may have a different composition convention than ours). Note, that the extended Burau representation at integer t, reduced mod m is also found in [**5**]. The form of the representation here differs from that of loc. cit. because there the (isomorphic) group OLB_n is considered. The precise relationship is that the image of σ_i is replaced by its inverse, followed by a parameter change $t- > t^{-1}$.

Observe that the row-sums of $\rho_{m,t}(\sigma_i)$ and $\rho_{m,t}(s_i)$ are 1; therefore, they are $n \times n$ (row)-stochastic matrices (modulo m). In particular, since the affine linear group $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$ is isomorphic to the group of $n \times n$ stochastic matrices modulo m (see [**28**], where m prime is considered, but the proof is valid for any m), we see that the image of $\rho_{m,t}$ is a subgroup of $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$. The question we wish to address is: When is $\rho_{m,t} : LB_n \to \mathrm{AGL}_{n-1}(\mathbb{Z}_m)$ surjective?

The group $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$ is the semidirect product of $(\mathbb{Z}_m)^{n-1}$ with $\mathrm{GL}_{n-1}(\mathbb{Z}_m)$ (with the obvious action). The standard way to view $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$ is as the subgroup of $\mathrm{GL}_n(\mathbb{Z}_m)$ consisting of matrices of the form $\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$ where $A \in \mathrm{GL}_{n-1}(\mathbb{Z}_m)$ and $v \in \mathbb{Z}_m^{n-1}$ (a column vector). For economy of notation, we will denote these elements by $g(A, v)$. In this notation the multiplication rule is

$$(A_1, v_1)(A_2, v_2) = (A_1A_2, A_1v_2 + v_1).$$

To determine the conditions on m, t so that $\rho_{m,t}$ is surjective, we need some additional notation and technical results.

- For $i \neq j$, define $\Delta_{i,j} \in Mat(n)$ to be the matrix with (i, j)-entry equal to 1 and all other entries zero.
- For $i \neq j$, define $E_{i,j}(\alpha) = I + \alpha\Delta_{i,j}$, *i.e.* the elementary matrix corresponding to the row operation which adds α times the jth row to the ith row.
- Let $D(\alpha, i) := I + (\alpha - 1)\Delta_{i,i}$ be the diagonal matrix with the (i, i)-entry equal to α and the remaining (diagonal) entries equal to 1.

LEMMA 3.7. *Let* $B = g(I, e_i) \in \mathrm{AGL}_{n-1}(\mathbb{Z}_m)$*, with* $e_i \in (\mathbb{Z}_m)^{n-1}$ *a standard basis vector. Then* $\mathrm{AGL}_{n-1}(\mathbb{Z}_m) \subset \mathrm{GL}_n(\mathbb{Z}_m)$ *is generated by* B *and the following matrices:*
(a) $E_{i,j}(1)$*, all* $1 \leq i \neq j \leq n-1$ *and*
(b) $D(\alpha, 1)$ *all* $\alpha \in \mathbb{Z}_m^\times$.

Proof. Let $e_j \in (\mathbb{Z}_m)^{n-1}$ be an arbitrary standard basis vector and choose A so that $Ae_i = e_j$. Then

$$g(A, 0)g(I, e_i)g(A^{-1}, 0) = g(I, e_j).$$

Since the matrices $g(I, e_j)$ generate all elements of the form $g(I, b)$, $b \in (\mathbb{Z}_m)^{n-1}$, it is enough to show that matrices in (a) and (b) generate all matrices of the form $g(A, 0)$ with $A \in \mathrm{GL}_{n-1}(\mathbb{Z}_m)$.

Since $[E_{i,j}(1)]^k = I + k\Delta_{i,j} = E_{i,j}(k)$ we see that we can obtain all elementary matrices corresponding to replacing row/column i with a multiple of row/column j plus row i. Moreover, we may obtain all matrices that permute rows and all matrices of the form $D(\alpha, j)$ inductively from $D(-1, 1)$ via:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus we obtain all elementary matrices in $\mathrm{GL}_{n-1}(\mathbb{Z}_m)$ as products of matrices as in (a) and (b).

Finally, observe that the gcd of the entries in any row/column of $A \in \mathrm{GL}_{n-1}(\mathbb{Z}_m)$ must be a unit in $\mathbb{Z}_m$. Using elementary row/column operations (left/right multiplication by elementary matrices) we may transform A into a matrix with the $(1, 1)$ entry equal to 1 and the remaining entries equal to zero. It then follows by induction that every $A \in \mathrm{GL}_{n-1}(\mathbb{Z}_m)$ is a product of matrices as in (a) and (b), as required. □

PROPOSITION 3.8. *Suppose that $t \in \mathbb{Z}$ is chosen so that t and $(1-t)$ are units in $\mathbb{Z}_m$ and $\mathbb{Z}_m^\times = \langle t, -1\rangle$. Then, $\rho_{m,t}(LB_n) \cong \mathrm{AGL}_{n-1}(\mathbb{Z}_m)$.*

Proof. We proceed by induction on n. For the case, $n = 2$ we must show that M and P as above generate $\mathrm{AGL}_1(\mathbb{Z}_m)$. By taking the transpose of M and P followed by a change of basis we can transform these into our standard $\mathrm{AGL}_1(\mathbb{Z}_m)$ form as: $\sigma = g(-t, t) = \begin{pmatrix} -t & t \\ 0 & 1 \end{pmatrix}$, $s = g(-1, 1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

Now $g(-t, t)g(-1, 1) = g(t, 0)$, and $g(-1, 1)g(t, 0)g(-1, 1)g(1/t, 0) = g(1, 1-t)$. Since $(1-t)$ is invertible and $g(1, a)g(1, b) = g(1, a+b)$, we obtain all $g(1, a)$. Since one of t or $1-t$ is even, 2 is a unit in $\mathbb{Z}_m$, with multiplicative inverse, say i_2. Now, we compute $g(1, -i_2)g(-1, 1)g(1, i_2) = g(-1, 0)$. Since $\mathbb{Z}_m^\times = \langle t, -1\rangle$ we obtain all $g(x, 0)$ where $x \in \mathbb{Z}_m^\times$. Therefore, we have all $g(1, a)g(x, 0) = g(x, a) \in \mathrm{AGL}_1(\mathbb{Z}_m)$.

Now we again take the transpose of $\rho_{m,t}(\sigma_i)$ and $\rho_{m,t}(s_i)$ for $1 \le i \le n-1$ and then change to the ordered basis: $[(1, \ldots, 1), (0, 1, \ldots, 1), \ldots, (0, \ldots, 0, 1)]$, so that the generators have the form $g(A, a)$ with $A \in \mathrm{GL}_{n-1}(\mathbb{Z}_m)$ and $a \in (\mathbb{Z}_m)^{n-1}$. By the induction hypothesis, the images of σ_i, s_i for $1 \le i \le n-2$ generate all matrices of the form $g(B, 0)$ where $B \in \mathrm{AGL}_{n-2}(\mathbb{Z}_m)$. That is, we have all $g(g(C, c), 0)$ with $C \in \mathrm{GL}_{n-2}(\mathbb{Z}_m)$ and $c \in (\mathbb{Z}_m)^{n-2}$. With respect to this basis the image of the generator σ_{n-1} has the form $\Sigma_{n-1}(t) := g(J, te_{n-1})$ where $J = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 1 & -t \end{pmatrix}$, and the image of the generator s_i is obtained by evaluating $\Sigma_{n-1}(t)$ at $t = 1$.

We have now reduced to showing that $g(g(C, c), 0)$ together with $\Sigma_{n-1}(t)$ and $\Sigma_{n-1}(1)$ generate all of $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$. By Lemma 3.7 it suffices to obtain $g(I, e_{n-1})$ as well as all $g(E_{i,j}(1), 0)$ for $1 \le i, j \le n-1$ and $g(D(\alpha, 1), 0)$ for all $\alpha \in \mathbb{Z}_m^\times$. Since $C \in \mathrm{GL}_{n-2}(\mathbb{Z}_m)$ and $c \in (\mathbb{Z}_m)^{n-2}$ can be chosen arbitrarily, we immediately obtain all $g(D(\alpha, 1), 0)$ as well as the $g(E_{i,j}(1), 0)$ for $i \le n-2$ and $1 \le j \le n-1$.

Let e_{n-1} denote the standard basis vector in $(\mathbb{Z}_m)^{n-1}$ and set

$$T(t) := \Sigma_{n-1}(t)\Sigma_{n-1}(1)\Sigma_{n-1}(t)^{-1}\Sigma_{n-1}(1) = I + (1-t)(\Delta_{n-1,n-2} - \Delta_{n-1,n}).$$

We compute $T(t)^k = I + k(1-t)(\Delta_{n-1,n-2} - \Delta_{n-1,n})$ and since $(1-t)$ is invertible modulo m we may choose $k = (1-t)^{-1}$ to obtain $T(0) = I + (\Delta_{n-1,n-2} - \Delta_{n-1,n})$. Now we compute:

$$g(D(-1, n-2), 0)T(0)g(D(-1, n-2), 0)T(0) = g(I, -2e_{n-1}).$$

Since -2 is invertible modulo m, we may appeal to Lemma 3.7 to produce all elements of the form $g(I, b)$, once we obtain the remaining generators of $\mathrm{GL}_{n-1}(\mathbb{Z}_m)$.

Thus, it remains to produce $g(E_{n-1,j}(1), 0)$ for all $1 \le j \le n-2$. For this, we set $X = g((I + \sum_{j=1}^{n-2} a_i\Delta_{n-3,i})D(a_{n-2}, n-2), 0)$, that is, the $n \times n$ matrix with the

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$(n-2)$th row equal to $(a_1, \ldots, a_{n-2}, 0, 0)$ and $X_{i,j} = \delta_{i,j}$ for $i \neq (n-2)$. Notice that X is of the form $g(g(C, 0), 0)$ with $C \in \mathrm{GL}_{n-2}(\mathbb{Z}_m)$, assuming that a_{n-2} is invertible. Setting $Z = X^{-1}\Sigma_{n-1}(1)X\Sigma_{n-1}(1)$ we find that the $(n-1)$th row of Z has entries $(a_1, \ldots, a_{n-3}, a_{n-2} - 1, 1, 0)$ and $Z_{i,j} = \delta_{i,j}$ for $i \neq (n-1)$. Specializing at appropriate values of a_i (e.g. $a_{n-2} \in \{1, 2\}$, $a_i \in \{0, 1\}$ for $i < n-2$) we obtain all $g(E_{n-1,j}(1), 0)$ for $1 \leq j \leq n-2$. Thus, by Lemma 3.7 we have completed the induction and the result follows. □

REMARK 3.9. We conjecture that Prop. 3.8 is sharp.

Clearly $\{\det(T) : T \in \mathrm{AGL}_{n-1}(\mathbb{Z}_m)\} = \mathbb{Z}_m^\times$. Since $\det(M) = -t$ and $\det(S) = -1$, the image of $\rho_{m,t}(LB_n)$ consists of matrices with determinant $\pm t^k$. This shows if $\rho_{t,m}(LB_n) \cong \mathrm{AGL}_{n-1}(\mathbb{Z}_m)$ then $\langle t, -1\rangle = \mathbb{Z}_m^\times$. In particular, if $\mathbb{Z}_m^\times$ is not a cyclic group or the direct product of $\mathbb{Z}_2$ with a cyclic group $\mathbb{Z}_d$ then $\rho_{t,m}(LB_n)$ is a proper subgroup of $\mathrm{AGL}_{n-1}(\mathbb{Z}_m)$. Clearly, t and $(1-t)$ can both be units only if m is odd. In this case, the group $\mathbb{Z}_m^\times \cong \mathbb{Z}_d \times \mathbb{Z}_2$ if only if $m = p^a q^b$ is a product of at most 2 odd primes and $\gcd(p^a - p^{a-1}, q^b - q^{b-1}) = 2$.

3.2. Relationship with Drinfeld doubles. In [15] it is observed that a BVS (V, c) with corresponding operators $g_1, \ldots, g_n$ may be realized as a Yetter–Drinfeld module over the group $G = \langle g_1, \ldots, g_n\rangle$. When G is finite, these can be identified with objects in $\mathrm{Rep}(DG)$ (regarded as a braided fusion category) where DG is the Drinfeld double of the group G.

As a vector space $DG = G^{\mathbb{C}} \otimes \mathbb{C}[G]$ where $G^{\mathbb{C}}$ is the Hopf algebra of functions on G with basis $\delta_g(h) = \delta_{g,h}$ and $\mathbb{C}[G]$ is the (Hopf) group algebra. The Hopf algebra structure on DG is well-known. For an account of the associated braid group representations (and further details) see [13].

The irreducible representations of DG are labeled by pairs $(\overline{g}, \chi)$ where $\overline{g}$ is a conjugacy class in G and χ is the character of an irreducible representation of the centralizer of g in G: $C_G(g)$. The representation $\rho_{m,t}$ of Theorem 3.5 can be obtained in this way. We now describe this explicitly.

Let m, t be positive integers with $\gcd(m, t) = 1$ and $t \neq 1 \pmod{m}$. Let ℓ be the order of t modulo m, and $\mathbb{Z}_m = \langle r\rangle$ be the cyclic group modulo m with generator r. The map $\tau(r) = r^t$ defines an automorphism of $\mathbb{Z}_m$, which generates a cyclic subgroup $\mathbb{Z}_\ell$ of $\mathrm{Aut}(\mathbb{Z}_m)$. Therefore, we may form the semidirect product $G = \mathbb{Z}_m \rtimes \mathbb{Z}_\ell$ via

$$srs^{-1} = r^t,$$

where $\langle s\rangle = \mathbb{Z}_\ell$. Let us further assume that $\gcd(m, t-1) = 1$. It follows from the relations above that $r^i s r^{-i} = r^{i(1-t)}s$ for all i, and the conjugacy class of s is $\{r^{i(1-t)}s : 0 \leq i \leq m-1\}$. For notational convenience, let $q = r^{1-t}$ so that q has order m and the conjugacy class of s is $\{q^i s : 0 \leq i \leq m-1\}$. Then $V = V_{(s,1)}$ has basis $\{q^i \mid 0 \leq i \leq m-1\}$, a set of coset representatives of $G/C_G(s)$. The action of the R-matrix of DG $\check{R}$

on $V \otimes V$ is (where P denotes the usual transposition):

$$\begin{aligned}
\check{R}(q^i \otimes q^j) &= PR(q^i \otimes q^j) \\
&= P\left(\sum_{g \in G} \delta_g \otimes g\right)(q^i \otimes q^j) \\
&= P(q^i \otimes q^{i(1-t)} s q^j) \\
&= P(q^i \otimes q^{i(1-t)+jt}) \\
&= q^{i(1-t)+jt} \otimes q^i.
\end{aligned}$$

Clearly, we may identify $\check{R}$ with the $\mathbb{Z}$-linear operator on $\mathbb{Z}_m \times \mathbb{Z}_m$ given by

$$(i, j) \mapsto ((1-t)i + tj, i).$$

This is the transpose of the braided vector space described in Theorem 3.5.

4. Finite dimensional quotient algebras. In order to study certain local and finite-dimensional representations ρ of LB_n, such as the BVS representations ρ^c described in Section 3 above, we are interested in certain finite-dimensional quotient algebras of the group algebra $\mathbb{C}[LB_n]$, namely the algebras

$$L_n^\rho := \mathbb{C}[LB_n]/\ker\rho.$$

In passing to the group algebra, we linearize. Thus, $\ker\rho = \{x \in \mathbb{C}[LB_n] \mid \rho(x) = 0\}$. This should be contrasted with the group representation version: $\ker_G \rho = \{g \in LB_n \mid \rho(g) = 1\}$. It can easily happen that $\ker_G \rho = \{1\}$ but $\ker\rho \neq \{0\}$. * This raises some questions. (1): What is a good presentation of L_n^ρ for each n? Can the kernel be described in closed form for all n? (2): What are the irreducible representations of L_n^ρ?

In this section, we first use an analogy to show why the answers to these questions will be useful. This analogy shows that the study of the quotient algebras L_n^ρ is of *intrinsic* interest. Then, we analyse these representations, and answer (2) in certain cases. All will be reasonably self-contained, but further background and references for relevant concepts from the representation theory of Artin's braid group can be found e.g. in [**20**, **18**, **24**, **25**, **30**].

4.1. A braid group quotient analogy. Consider the ordinary braid group $\mathcal{B}_n$. For each N, $V = \mathbb{C}^N$ and $q \in \mathbb{C}^*$ there is a well-known BVS with $c = c_N$, where in the case $N = 2$:

$$c_2 = q\begin{pmatrix} 1 & & & \\ & 1-q^{-2} & -q^{-1} & \\ & -q^{-1} & 0 & \\ & & & 1 \end{pmatrix} . = \begin{pmatrix} q & & & \\ & (q-q^{-1}) & -1 & \\ & -1 & 0 & \\ & & & q \end{pmatrix}.$$

We write ρ_N for the representation ρ^{c_N}.

*Note also that while group representations and group algebra representations are closed under tensor products, the linear kernel is not preserved in general.

For $\underline{q} = (q_1, q_2, \ldots)$ a tuple in $\mathbb{C}^*$ define $\chi_{\underline{q}} = \prod_i(\sigma_1 - q_i) \in \mathbb{C}[\mathcal{B}_n]$. The Hecke algebra is $H_n = \mathbb{C}[B_n]/I_q$, where I_q being the ideal generated by $\chi_{(q,-q^{-1})} = (\sigma_1 - q)(\sigma_1 + q^{-1}) \in \mathbb{C}[B_n]$ for some $q \in \mathbb{C}^*$. In the following, we work with a fixed q and often omit the q-dependence from the notation (as done for $H_n \equiv H_n(q)$). Evidently, ρ_N factors through H_n, but it is not linearly faithful for all n. The quotients H_n^N are defined by $H_n^N = \mathbb{C}[\mathcal{B}_n]/\ker \rho_N$.

So what is a good presentation for H_n^N for given N? There is an element $\mathbf{f}$ of H_m for $m = N + 1$ such that

$$\ker \rho_N \;=\; H_n \mathbf{f} H_n, \tag{4.1}$$

for all n (with the kernel understood to be 0 for $n < m$). To construct $\mathbf{f}$ for a given N, recall that for each m there is a nonzero element $\mathbf{f}_m$ of H_m unique up to scalars such that $\sigma_i \mathbf{f}_m = \mathbf{f}_m \sigma_i = (-q^{-1})\mathbf{f}_m$ for all i. For example, we can take $\mathbf{f}_2 = U_1$ where $U_i := \sigma_i - q$ and $\mathbf{f}_3 = U_1 U_2 U_1 - U_1$. We may take $\mathbf{f} = \mathbf{f}_{N+1}$ or any nonzero scalar multiple thereof. That is, there is a single additional relation that characterises H_n^N as a quotient of H_n for all n and q, namely $\mathbf{f}_{N+1} = 0$ [**25**]. Thus, H_n^2 is the Temperley–Lieb algebra and so on.

4.2. On localization. Given an algebra A, let $\Lambda(A)$ be the set of irreducible representations up to isomorphism. Another feature of the braid/Hecke/Temperley–Lieb paradigm is *localization*.

Given an algebra A and idempotent $e \in A$, then eAe is also an algebra (not a subalgebra) and the functors G_e, F_e:

$$A\text{ –mod}, \;\underset{F_e}{\overset{G_e}{\rightleftarrows}}\; eAe\text{ –mod}, \tag{4.2}$$

('globalization' and 'localization', respectively) given on modules by

$$G_e N = Ae \otimes_{eAe} N$$

$$F_e M = eM,$$

are an adjoint pair. Useful corollaries to this include the following:

(LI) Let L_i, L_j be distinct simple A-modules, with eL_i and eL_j nonzero. Then, eL_i, eL_j are distinct simple eAe-modules.

(LII) If L_i has composition multiplicity m_i in A-module M, and $eL_i \neq 0$, then eL_i has multiplicity m_i in eAe-module eM:

$$[M : L_i] \;=\; [eM : eL_i], \tag{4.3}$$

(LIII) The set $\Lambda(A)$ of irreducible representations of A (up to isomorphism) is in bijection with the disjoint union of those of eAe and those of A/AeA:

$$\Lambda(A) \;\cong\; \Lambda(eAe) \sqcup \Lambda(A/AeA), \tag{4.4}$$

The idea here is very general. Given an algebra A to study, we find an idempotent in it, then study A by studying eAe and A/AeA. In general eAe and A/AeA are also

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unknown and this subdivision does not help much. But for H_n^N we have an e such that

$$eH_n^N e \cong H_{n-N}^N, \tag{4.5}$$

so we can consider eAe to be known by an induction on n. The analysis goes as follows.

For H_n^N, in addition to the property (4.1), there is also an element $\mathbf{e}$ of H_n^N for some n (in fact $n = N$ and $\mathbf{e} = \mathbf{f}_N$) such that the matrix $\rho_N(\mathbf{e})$ is rank=1. It follows that

$$\rho_N(\mathbf{e})\ \rho(H_N^N)\ \rho_N(\mathbf{e}) \quad \subseteq \quad \mathbf{k}\rho_N(\mathbf{e}), \tag{4.6}$$

Indeed, we have the following ('localization property'): For all n,

$$\rho_N(\mathbf{e})\,\rho_N(H_n^N)\,\rho_N(\mathbf{e}) \quad = \quad \underbrace{\rho_N(\mathbf{e})}_{on\ V^N} \otimes \underbrace{\rho_N(H_{n-N}^N)}_{on\ V^{n-N}}, \tag{4.7}$$

(cf. (4.5)).

We have from (4.7) that $\mathbf{e}H_n^N\mathbf{e} \cong H_{n-N}^N$ and, since $\mathbf{e} = \mathbf{f}_N$, that $H_n^N/H_n^N\mathbf{e}H_n^N \cong H_n^{N-1}$. So, by (4.4) the irreducible representations of H_n^N can be determined by an iteration on n (and N).

It is sometimes possible to lift this to the loop-braid case. How might the braid group paradigm generalize? Of course every finite-dimensional quotient of the group algebra of the braid group $\mathcal{B}_n$ has a local relation – a polynomial relation $\chi_{\underline{q}} = 0$ obeyed by each braid generator. Thus we can start, organisationally, by fixing such a relation. If this relation is quadratic then the quotient algebra is finite dimensional for all n, in particular it is the Hecke algebra. If the local relation is cubic or higher order then this quotient alone is not enough to make the quotient algebra finite-dimensional for all n [**9**] (and also not enough to realise the localisation property, as in Section 4.2).

Below, we study group-type representations of LB_n in this context.

4.3. Some more preparations: the BMW algebra. We define the BMW algebra over $\mathbb{C}$ as follows. For $n \in \mathbb{N}$ and $r, q \in \mathbb{C}^*$ with $q^2 \neq 1$, $\mathbb{C}$-algebra $\mathcal{C}_n(r, q)$ is generated by $b_1, b_2, \ldots, b_{n-1}$ and inverses obeying the braid relations

$$b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}, \qquad\qquad b_ib_j = b_jb_i \qquad (|i-j| > 1), \tag{4.8}$$

and, defining

$$u_i = 1 - \frac{b_i - b_i^{-1}}{q - q^{-1}} \quad = \quad \frac{b_i^{-1}}{q^{-1} - q}(b_i - q)(b_i + q^{-1}), \tag{4.9}$$

obeying the additional relations

$$u_ib_i = r^{-1}u_i \tag{4.10}$$

$$u_ib_{i-1}^{\pm 1}u_i = r^{\pm 1}u_i. \tag{4.11}$$

Relation (4.10) is equivalent to a 'cubic local relation'

$$(b_i - r^{-1})(b_i - q)(b_i + q^{-1}) = 0. \tag{4.12}$$

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Relation (4.10) also implies

$$u_i^2 = \left(1 + \frac{r - r^{-1}}{q - q^{-1}}\right) u_i.$$

Relation (4.11) implies

$$u_i u_{i\pm 1} u_i = u_i.$$

Of course we also have from (4.8):

$$u_i u_j = u_j u_i \qquad (|i - j| > 1).$$

Indeed the u_i's generate a Temperley–Lieb subalgebra of $\mathcal{C}_n(r, q)$. This subalgebra realizes a different quotient of the braid group algebra: the images of the braid generators are $a_i = 1 - q'(q, r)\, u_i$, where q' is defined by $q' + q'^{-1} = 1 + \frac{r - r^{-1}}{q - q^{-1}}$ with a quadratic local relation, and with the two eigenvalues depending on q and r.

For us the interesting case of $\mathcal{C}(r, q)$ is $r = q$, where the braid generators of the TL subalgebra obey the symmetric group relations. In this case, then, we have images of both the braid group and the symmetric group in $\mathcal{C}_n(q, q)$, as for LB_n. Indeed, we have the following.

LEMMA 4.1. *There is a map $\psi : LB_n \to \mathcal{C}_n(q, q)$ given by $\sigma_i \mapsto b_i$, $s_i \mapsto a_i = 1 - u_i$.*

Proof. With $r = q$ we have $u_i^2 = 2u_i$ and $q' = 1$ so $a_i = 1 - u_i$ and $a_i^2 = 1$ as already noted. Relations (L1,L2) can be directly checked.

4.4. The representations τ_N of LB_n. For each N, and $x \in \mathbb{C}$, there is a well-known local representation τ_N^x of $\mathbb{C}[\mathcal{B}_n]/I_{x,1,-1}$, with $I_{x,1,-1}$ as defined in section 4.1 (trivially rescalable, setting $x = q^2$, to a representation τ_N of $\mathbb{C}[\mathcal{B}_n]/I_{q,q^{-1},-q^{-1}}$; and that in case $N = 2$ is also a representation of $\mathcal{C}_n(q, q)$). One takes the diagonal BVS with

$$g_1 = \begin{pmatrix} x & 0 & & \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} 1 & 0 & & \\ 0 & x & & \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ldots, \; g_N = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & x \end{pmatrix}.$$

We abbreviate the basis element $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}$ of V^n as $|i_1 i_2 \ldots i_n\rangle$, so that $e_1 \otimes e_1 \otimes e_2$ becomes $|112\rangle$ and so on. Then,

$$\sigma_j |i_1 i_2 \ldots i_n\rangle = \begin{cases} x|i_1 i_2 \ldots i_n\rangle & i_j = i_{j+1} \\ \quad |i_1 i_2 \ldots i_{j+1} i_j \ldots i_n\rangle & \text{otherwise.} \end{cases} \tag{4.13}$$

Specifically for $N = 2$ (with basis elements of V^2 ordered $|11\rangle, |12\rangle, |21\rangle, |22\rangle$):

$$\sigma_i \overset{\tau_2^x}{\mapsto} Id_2 \otimes Id_2 \otimes \ldots \otimes \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} . \otimes Id_2 \otimes \ldots \otimes Id_2$$

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Strictly speaking we need to rescale: $g_1 = \begin{pmatrix} q & 0 \\ 0 & 1/q \end{pmatrix}$.; $g_2 = \begin{pmatrix} 1/q & 0 \\ 0 & q \end{pmatrix}$.. So

$$\sigma_i \overset{\tau_2}{\mapsto} Id_2 \otimes Id_2 \otimes \ldots \otimes \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ 0 & 1/q & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} . \otimes Id_2 \otimes \ldots \otimes Id_2$$

This gives, for example,

$$\frac{\sigma_i - \sigma_i^{-1}}{q - q^{-1}} \overset{\tau_2}{\mapsto} Id_2 \otimes Id_2 \otimes \ldots \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \otimes Id_2 \otimes \ldots \otimes Id_2 \qquad (4.14)$$

Let us define quotient $\mathbb{C}$-algebra

$$B_n^{\tau_N} = \mathbb{C}[\mathcal{B}_n] / \ker \tau_N .$$

PROPOSITION 4.2. The algebra $B_n^{\tau_N}$ is semisimple.

Proof. In case x is real the algebra is evidently generated by hermitian (indeed real symmetric) matrices. In other cases, one can show that the same is true for a different generating set. □

PROPOSITION 4.3. The map $s_i \mapsto \tau_N(\frac{\sigma_i - \sigma_i^{-1}}{q - q^{-1}})$ extends τ_N to a representation of LB_n. That is to say, $B_n^{\tau_N}$ is a quotient of $\mathbb{C}[LB_n]$.

PROPOSITION 4.4. The case τ_2 factors through $\mathcal{C}_n(q, q)$. That is $b_i \mapsto \tau_2(\sigma_i)$ gives a representation of $\mathcal{C}_n(q, q)$.

Given any realization of $\mathcal{B}_n$, and $q \in \mathbb{C}$, we define u_i as in (4.9). (The image $\tau_2(u_i)$ obeys the BMW relation (4.11), but $\tau_N(u_i)$ for $N > 2$ does not.) As noted in (4.3), $s_i \mapsto \tau_N(a_i = 1 - u_i)$ gives a representation of $\mathfrak{S}_n$ for each N. Indeed, the τ_N representation of $\mathfrak{S}_n$ coincides with the classical case, $q = 1$, of the ρ_N Hecke algebra representation:

$$\rho_N^{q=1}(U_i) = \tau_N(u_i). \qquad (4.15)$$

Thus from Section 4.1, we have $\mathbf{f}_N^1 := \rho_N^{q=1}(\mathbf{f}_N) \in \tau_N$.

Given a loop BVS one obvious question is: Do we have an analogue of (4.7) here together with corresponding strong representation theoretic consequences? We are particularly interested in cases that do not factor over SLB_n. But the question is hard in general and it is instructive to start with a 'toy' such as the class of loop BVSs above.

4.5. Fixed-charge submodules of τ_N. One aim is to decompose the representations τ_N into irreducible representations. To this end, note that the subspaces of τ_N of fixed N-colour-charge (the colour-charge is the composition of n giving for each i in $\{1, 2, \ldots, N\}$ the number of i's in a basis element $|i_1 i_2 \ldots i_n\rangle$) are invariant under the $\mathbb{C}[LB_n]$ action.

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LEMMA 4.5. *The $\mathfrak{S}_N$ action permuting the standard ordered basis $\{e_1, e_2, \ldots, e_N\}$ of $V = \mathbb{C}^N$ commutes with the LB_n action on V^n.* □

We write the action of $\mathfrak{S}_N$ on the right. So, if M is an LB_n-submodule of V^n then Mw is an isomorphic submodule for any $w \in \mathfrak{S}_N$. This $\mathfrak{S}_N$ action acts on the set of charges. Thus, we can index charge-submodules (up to isomorphism) by the set $\Lambda_{N,n}$ of integer partitions of n of maximum depth N. This is the same as the charge decomposition of the Hecke algebra representation ρ_N (where the submodules are called *Young modules*). But the further decomposition into irreducibles is not the same as in the Hecke case.

For an explicit example, the basis B_λ for the λ subspace in case $\lambda = (2, 1)$ is $B_{21} = \{112, 121, 211\}$. We write Y_λ for the charge λ submodule. Thus we have

$$\tau_{N,n} \cong \bigoplus_{\lambda \in \Lambda_{N,n}} m_\lambda Y_\lambda, \tag{4.16}$$

where m_λ is the multiplicity.

If $\mathfrak{S}_N$ or a subgroup G fixes a submodule Y_λ then this module is itself a right G-module and an idempotent decomposition of 1 in $\mathbb{C}[G]$ induces a decomposition of Y_λ.

For each λ there is a G fixing Y_λ, call it G_λ, a Young subgroup of $\mathfrak{S}_N$ (possibly trivial). As usual an idempotent decomposition of 1 in $\mathbb{C}[G]$ may be characterised by tuples of Young diagrams/integer partitions. For each such label there is also a secondary index running over the dimension of the corresponding irreducible representation of G; but idempotents with the same primary label are isomorphic. If Y_λ has a non-trivial such decomposition we will write $Y_\lambda^{\underline{\mu}}$ for the submodule with primary label $\underline{\mu}$. We call these modules $Y_\lambda^{\underline{\mu}}$ *harmonic modules*. For given λ write Λ_λ for the set of primary labels (the index set for irreducible representations $\Delta_{\underline{\mu}}$ of G_λ). Thus,

$$Y_\lambda = \bigoplus_{\underline{\mu} \in \Lambda_\lambda} \dim \Delta_{\underline{\mu}} Y_\lambda^{\underline{\mu}}. \tag{4.17}$$

Note that the decomposition of Y_λ into irreducible modules for the restriction to the 'classical' subalgebra H^N generated by the u_is (the symmetric group action) is well-known. This gives a lower bound on the size of summands of Y_λ as a module for the full algebra.

LEMMA 4.6. *Actions of subgroup $\mathfrak{S}_n$ and $\mathcal{B}_n$ on $Y_{(1^n)}$ are identical up to sign.* □

Comparing the 'classical' decomposition of $Y_{(1^n)}$ above with the idempotent decomposition with $G = \mathfrak{S}_N = \mathfrak{S}_n$ in this case we see that they are the same.

To apply localization later we will be interested, for each given N, in detecting submodules M of Y_λ on which $\mathbf{e} = \mathbf{f}_N^1$ acts like 0. We call these e-null, or $\mathbf{f}_N^1$-null, submodules. Any such submodule M decomposes also as an $\mathfrak{S}_n$-submodule, and so $\mathbf{f}_N^1$ would have to act like 0 on each of the submodules in this decomposition. For example, in case $N = 2$ only the irreducible $\mathfrak{S}_n$-module $\Delta_{(n)}$ has this property at rank-n. So here there can only be such a submodule M if $\Delta_{(n)}$ is also an LB_n-submodule of Y_λ. A basis element for $\mathfrak{S}_n$-submodule $\Delta_{(n)}$ in Y_λ is known. We take

$$b = \sum_{w \in \mathfrak{S}_n} w \; 111 \ldots 222,$$

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where $111\ldots222$ is the initial basis element of Y_λ in the lex order. Then, for example $b \overset{\lambda=(2,1)}{=} 2(112 - 121 + 211)$. Note here that $q\sigma_1(112 - 121 + 211) = x112 - 211 + 121$, so $\Delta_{(3)}$ is not an LB_3-submodule unless $x = -1$.

LEMMA 4.7. *(I) In case $N = 2$, $x \neq -1$, no Y_λ has e-null proper submodule except in case $\lambda = (1, 1)$, where $Y^{(1^2)}_{(1^2)}$ is $\mathbf{f}^1_2$-null.*
(II) In case $N = 3$ the module $Y^{(1^2)}_{(n-2,1^2)}$ is $\mathbf{f}^1_3$-null for $n > 3$.

Proof. (I) The example above is indicative, except in case $(1, 1)$ where there is no x term.
(II) A basis of $Y^{(1^2)}_{(2,1^2)}$ is $\{1123 - 1132, 1213 - 1312, 1231 - 1321, 2113 - 3112, 2131 - 3121, 2311 - 3211\}$. One readily checks the $\mathbf{f}^1_3$ action on this. The other cases are similar. □

There is an injective algebra map

$$B^{\tau_N}_{n-N} \xrightarrow{\sim} \mathbf{f}^1_N \otimes B^{\tau_N}_{n-N} \hookrightarrow \mathbf{f}^1_N B^{\tau_N}_n \mathbf{f}^1_N.$$

Thus, any $B^{\tau_N}_n$-module gives rise to a $B^{\tau_N}_{n-N}$ module by first localizing (we will write simply F for the localisation functor $F_{\mathbf{f}^1_N}$ here), then restricting.

LEMMA 4.8. *There is an isomorphism of $B^{\tau_N}_{n-N}$-modules*

$$\mathbf{f}^1_N\, Y_{(\lambda_1,\lambda_2,\ldots,\lambda_N)} \cong \begin{cases} Y_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_N-1)} & \lambda_N > 0 \\ 0 & \lambda_N = 0. \end{cases}$$

Proof. For any given N we can write $w \in B_\lambda$ as

$$w = \underbrace{w_1 w_2 \ldots w_N}_{w_-}\, \underbrace{w_{N+1} w_{N+2} \ldots w_n}_{w_+} = w_- w_+.$$

Then

$$\mathbf{f}^1_N w = \mathbf{f}^1_N w_- w_+ = \begin{cases} 0 & \text{unless } w_- \text{ is a permutation of } 12\ldots \text{N.} \\ \overline{12\ldots N} w_+ & w_- \text{ is a perm. of } 12\ldots\text{N.} \end{cases} \tag{4.18}$$

where $\overline{123} = 123 + 213 + 132 + 231 + 312 + 321$ and so on. That is, $\mathbf{f}^1_N V^n \cong V^{n-N}$ and $\mathbf{f}^1_N Y_\lambda \cong Y_{\lambda-(1^N)}$ as vector spaces, and hence modules. □

LEMMA 4.9. *Let $\lambda \in \Lambda$ and $l = l_\lambda$ the number of distinct row-lengths in λ, so that $\underline{\mu}$ in $Y^{\underline{\mu}}_\lambda$ has l_λ distinct components (each μ_i a partition). Let $\underline{\mu}'$ denote $\underline{\mu}$ with the l-th component omitted. There is an isomorphism of $B^{\tau_N}_{n-N}$-modules*

$$\mathbf{f}^1_N\, Y^{\underline{\mu}}_{(\lambda_1,\lambda_2,\ldots,\lambda_N)} \cong \begin{cases} Y^{\underline{\mu}}_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_N-1)} & \lambda_N > 1 \\ Y^{\underline{\mu}'}_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_N-1)} & \lambda_N = 1,\ (\mu_l)_2 = 0 \\ 0 & \lambda_N = 1,\ (\mu_l)_2 > 0 \\ 0 & \lambda_N = 0 \end{cases}.$$

Proof. The decomposition of Y_λ by the right-action of the charge group, commutes with the left-action of $\mathbf{f}^1_N$. So, noting Lemma 4.8, it only remains to verify the $\lambda_N = 1$ cases. In these cases, the first column of λ is uniquely the longest, of length N. Thus, the colours involved in the last component of $\underline{\mu}$ are symmetrised by $\mathbf{f}^1_N$. Any colour symmetry idempotent acting from the right corresponding to μ_l with $(\mu_l)_2 > 0$ involves an antisymmetriser in its construction, and hence annihilates $\mathbf{f}_N^1 Y_\lambda$. □

LEMMA 4.10. *For $x \neq -1$ the harmonic modules of LB_n, i.e. the modules $\{Y_\lambda^{\underline{\mu}} \mid \lambda \in \Lambda_n,\ \underline{\mu} \in \Lambda_\lambda\}$, are pairwise non-isomorphic.*

Proof. Work by induction on n. Compare $Y = Y_\lambda^{\underline{\mu}}$, $Y' = Y_{\lambda'}^{\underline{\mu}'}$, say, with $\underline{\mu} \neq \underline{\mu}'$. If either $\mathbf{f}_N^1 Y$ or $\mathbf{f}_N^1 Y' \neq 0$ for some $N \geq ||\lambda|| := \lambda_1^t$ then $Y \not\cong Y'$ by Lemma 4.9 and the inductive assumption. The remaining cases are when one or both of Y, Y' are of type-III in Lemma 4.9. These are routine to check. □

How can we understand this proliferation of submodules? Analogous results to the above hold for the Hecke quotients H_n^N. There it is very useful to use a geometrical principle to organise the indexing sets for canonical classes of modules (such as Young modules; or simple modules – except that there it turns out that, roughly speaking, the same index set can be used for these different classes). One way to understand this geometry comes from the theory of weight spaces in algebraic Lie theory. Here, we do not have any such dual picture, but we can naively bring over the same organisational principle. This tells us to consider λ as a vector in $\mathbb{R}^N$, and then to draw the set of λs in $\mathbb{R}^{N-1}$ by projecting down the $(1, 1, \ldots, 1)$ line. One merit of this is that it allows us to draw the entire $N = 3$ 'weight space' of Young module indices in the plane.

4.6. Branching rules for harmonic modules. We consider here the natural restriction from LB_n to LB_{n-1}, and claim Figure 1 gives the branching rules for $N = 3$.

PROPOSITION 4.11. The branching rules for Young modules corresponding to the natural restriction from LB_n to LB_{n-1} are

$$\downarrow Y_\lambda = \bigoplus_i Y_{\lambda - e_i},$$

where the sum is over removable boxes in the Young diagram λ.

Proof. The lB_{n-1} action ignores the last symbol in the colour-word basis for Y_λ. □

PROPOSITION 4.12. *The directed graph in Figure 1 gives the branching rules for harmonic modules for $N = 3$, using the geometric realisation.*

Proof. First note that well in the interior of the picture the Young and harmonic modules coincide and we can use Prop.4.11. Specifically this gives all cases in the forward cone of the point $(4, 2)$.

The remaining cases in the forward cone of $(2, 1)$ may be verified by using Propositions (4.17) and (4.11).

For the remaining 'boundary' cases we split up into cases in the following subsets: (i) the $(1, 0)$-ray of point $(3, 1)$; (ii) the $(1, 0)$-ray of point $(2, 0)$; (iii) the $(1, 1)$-ray of point $(3, 2)$; (iv) the $(1, 1)$-ray of point $(2, 2)$; (v) the point $(2, 1)$; (vi) the point $(1, 0)$; (vii) the point $(1, 1)$; (viii) the point $(0, 0)$.

We indicate the proof with two representative examples. Case (ii): In the fibre over $(2, 0)$ we have

$$\downarrow Y^{(2)}_{(6,4,4)} = Y_{(6,4,3)} \oplus Y^{(2)}_{(5,4,4)},$$

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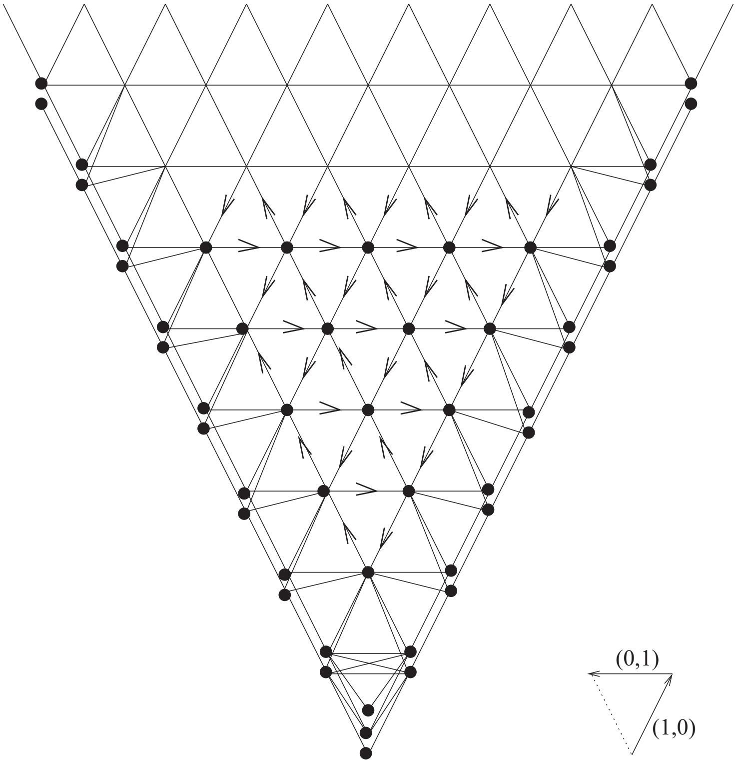

Figure 1. Branching rules for harmonic modules for $N = 3$. All 'parallel' edges are directed in the same direction.

by using 4.11 and commutativity of (left) restriction with the (right) idempotent decomposition.
Case (vi): In the fibre over $(1, 0)$ we have

$$\downarrow Y^{((1),(2))}_{(2,1,1)} = Y_{(2,1)} \oplus Y^{((3))}_{(1^3)} \oplus Y^{((2,1))}_{(1^3)}.$$

Here, the basis is

$$1123 + 1132,\ \ 1213 + 1312,\ \ 1231 + 1321,\ \ 2113 + 3112,\ \ 2131 + 3121,\ \ 2311 + 3211.$$

Since the restriction is defined by disregarding the last symbol in the color-world basis it is clear that on restriction the 1st, 2nd and 4th give a basis of $Y_{(2,1)}$, while the remainder injects into $Y_{(1^3)}$, and indeed into $Y_{(1^3)}(1 + s_2)$. $\square$

THEOREM 4.13. *In cases $N = 2, 3$, $x \neq -1$, the harmonic modules are irreducible.*

Proof. We work by induction on n. Consider a harmonic module Y at level n. By Propositions 4.12, 4.10 and the inductive assumption restriction to $n - 1$ is multiplicity-free. So it is enough to show that there is a basis element b in a good basis with respect to this restriction (a basis that decomposes into bases for the summands of the restriction) such that $Y = B^{\tau_N}_n b$.

In case, Y is also a Young module it is easy to see that $Y = B^{\tau_N}_n b$ for any standard basis element; and that the standard basis is a good basis for the restriction to Young

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modules; and that at least one of these is a summand of the restriction to harmonic modules.

In case Y is not a Young module (i.e. on the boundary) the modification is routine and we content ourselves here with some representative examples:

(1) Recall the restriction Res $Y^{(2)}_{(2,1,1)} = Y^{(3)}_{(1,1,1)} \oplus Y^{(2,1)}_{(1,1,1)} \oplus Y_{(2,1)}$. An element lying in the last summand is $1213 + 1312$. Acting with σ_3 on this we get $1231 + 1321$. It is easy to see that this generates the whole module.

(2) Recall the restriction Res $Y^{(2)}_{(2,2,1)} = Y^{(2)}_{(2,1,1)} \oplus Y^{(1^2)}_{(2,1,1)} \oplus Y^{(2)}_{(2,2)}$. An element lying in the last summand is $11223 + 22113$. Acting with σ_4 on this we get $11232 + 22131$. It is easy to see that this generates the whole module.

(3) We have

$$\text{Res } Y^{(2)}_{(4,2,2)} = Y_{(4,2,1)} \oplus Y^{(2)}_{(3,2,2)}.$$

A good basis is $\{11112233 + 11113322,\ 11112323 + 11113232,\ \ldots,\ 11123213 + 11132312,\ \ldots,\ 32211113 + 23311112,\ \ldots\}$, where all the explicitly written elements lie in the basis for $Y_{(4,2,1)}$ in the restriction (the first word ends in 3). Now, apply σ_7: $\sigma_7(11123213 + 11132312) = 11123231 + 11132321$, which lies in $Y^{(2)}_{(3,2,2)}$. □

Categorical versions of the structure for $N = 2$ and $N = 3$ also can be worked out explicitly. (But in light of Proposition 4.2 these are not as powerful a tool here as in the corresponding Hecke cases.) We will leave them for future publication.

We make the obvious conjecture for the generalisation to higher N: that the harmonic modules are again a complete set of irreducibles.

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