J. Aust. Math. Soc. **95** (2013), 315–328
doi:10.1017/S1446788713000335

PARABOLIC HIGGS BUNDLES AND Γ-HIGGS BUNDLES

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(Received 31 August 2012; accepted 5 June 2013; first published online 19 August 2013)

Communicated by D. Chan

Abstract

We investigate parabolic Higgs bundles and Γ-Higgs bundles on a smooth complex projective variety.

2010 *Mathematics subject classification*: primary 14D23; secondary 14H60.
Keywords and phrases: parabolic Higgs bundle, root stack, Γ Higgs bundle.

1. Introduction

Let X be a compact connected Riemann surface and E a holomorphic vector bundle on X. The infinitesimal deformations of E are parametrized by $H^1(X, \text{End}\, E)$, where $\text{End}\ E = E \otimes E^*$ is the sheaf of endomorphisms of the vector bundle E. By Serre duality, we have $H^1(X, \text{End}\, E)^* = H^0(X, (\text{End}\, E) \otimes \Omega^1_X)$, where Ω^1_X is the holomorphic cotangent bundle of X. A Higgs field on E is defined to be a holomorphic section of $(\text{End}\, E) \otimes \Omega^1_X$; they were introduced by Hitchin [Hi87a, Hi87b]. A Higgs bundle is a holomorphic vector bundle equipped with a Higgs field. Hitchin proved that stable Higgs bundles of rank r and degree zero on X are in bijective correspondence with the irreducible flat connections on X of rank r [Hi87a]. He also proved that the moduli space of Higgs bundles on X of rank r is a holomorphic symplectic manifold, and the space of holomorphic functions on this holomorphic symplectic manifold gives it the structure of an algebraically completely integrable system [Hi87b]. Simpson arrived at Higgs bundles via his investigations of variations of Hodge structures [Si88]. He extended the results of Hitchin to Higgs bundles over higher dimensional complex projective manifolds.

A parabolic structure on a holomorphic vector bundle E on X is roughly a system of weighted filtrations of the fibers of E over some finitely many given points. A parabolic vector bundle is a holomorphic vector bundle equipped with a parabolic

MLW would like to express his gratitude to the Fonds québécois de la recherche sur la nature et les technologies for the support of a Bourse de recherche postdoctorale (B3).

structure; parabolic vector bundles were introduced by Mehta and Seshadri [MS80]. Parabolic vector bundles with Higgs structure were introduced by Yokogawa [Yo95].

Our aim here is to investigate the parabolic vector bundles equipped with a Higgs structure. More precisely, we study the relationship between the parabolic Higgs bundles and the Higgs vector bundles on a root stack.

Root stacks are important examples of smooth Deligne–Mumford stacks; see [Cad07, Bo07] for root stacks.

Let Y be a smooth complex projective variety on which a finite group Γ acts as a group of automorphisms satisfying the condition that the quotient $X = \Gamma \setminus Y$ is also a smooth variety. There is a bijective correspondence between the *parabolic Higgs bundles* on X and the Γ-*Higgs bundles* on Y. We prove that the parabolic Higgs bundles on X are identified with the Higgs bundles on the associated root stack.

The organization of the paper is as follows. In Section 3 we review the notions of parabolic bundle and Γ-vector bundle and define the parabolic Higgs bundle and Γ-Higgs bundles. As was done for parabolic vector bundles in [Bi97] we describe an equivalence between the category of Γ-Higgs bundles on Y and parabolic Higgs bundle on X.

Section 4.1 describes the construction of a root stack as done in [Cad07]. In Section 4.2, we investigate vector bundles on root stacks. Section 4.3 generalizes Theorem 3.5 to the case of the root stack over the base space.

2. Preliminaries

Let Y be a smooth complex projective variety of dimension m endowed with the action $\lambda : \Gamma \times Y \longrightarrow Y$ of a finite group Γ such that:

(1) $X := \Gamma \setminus Y$ is also a smooth variety; and
(2) the projection map $\pi : Y \longrightarrow X$ is a Galois covering with Galois group Γ.

The closed subset of Y consisting of points with nontrivial isotropy subgroups for the action of Γ is a divisor $\widehat{D} \subset Y$ [Bi97, Lemma 2.8]. Let $D \subset X$ be its (reduced) image under π and $D = D_1 + \cdots + D_h$ its decomposition into irreducible components. We will always be working with the assumption that the D_μ, $1 \le \mu \le h$, are smooth, and D is a normal crossing divisor; this means that all the intersections of the irreducible components of D are transversal. For an effective divisor D' on Y, by D'_{red} we will denote the corresponding reduced divisor. So D'_{red} is obtained from D' by setting all the multiplicities to be one. We set

$$\widetilde{D}_\mu := (\pi^* D_\mu)_{\mathrm{red}}, \quad 1 \le \mu \le h, \quad \widetilde{D} := \sum_{\mu=1}^{h} \widetilde{D}_\mu.$$

There exist $k_\mu, r \in \mathbb{N}$ for $1 \le \mu \le h$ such that $\pi^* D_\mu = k_\mu r \widetilde{D}_\mu$; with this,

$$\pi^* D = r \sum_{\mu=1}^{h} k_\mu \widetilde{D}_\mu.$$

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It should be clarified that there are many choices of k_μ and r. We will write $\overline{D} := \sum_{\mu=1}^{h} k_\mu \widetilde{D}_\mu$, so that $\pi^* D = r\overline{D}$.

Lemma 2.1. *With the assumption that D is a normal crossing divisor with smooth components, given a point $y \in Y$ with $\pi(y) \in \bigcap_1^l D_\mu$, one can choose coordinates $w_1, \ldots, w_m$ in an analytic neighborhood of y and $z_1, \ldots, z_m$ in an analytic neighborhood of $\pi(y)$ such that D_μ is defined by z_μ for $1 \le \mu \le l$, $\widetilde{D}_\mu$ is defined by w_μ for $1 \le \mu \le l$ and π is given in the local coordinates by*

$$z_1 = w_1^{k_1 r}, \ldots, z_l = w_l^{k_l r}, \quad z_{l+1} = w_{l+1}, \ldots, z_m = w_m.$$

Proof. This is proved in [Na, page 11, Theorem 1.1.14]. □

Recall [Del70, Section II.3] that the sheaf $\Omega_X^1(\log D)$ of logarithmic differentials with poles at D is the locally free sheaf on X, a basis for which in the neighborhood of a point in $\bigcap_1^l D_\mu$ with coordinates chosen as in the lemma is given by

$$\frac{dz_\mu}{z_\mu}, 1 \le \mu \le l, \quad dz_\mu, l+1 \le \mu \le m.$$

Therefore, the dual $\Omega_X^1(\log D)^*$ is the subsheaf of the holomorphic tangent bundle TX given by the sheaf of vector fields that preserve $O_X(-D) \subset O_X$.

We have the following analogue of Hurwitz's theorem.

Lemma 2.2. *With $\pi : Y \longrightarrow X$ as above, one has*

$$\Omega_Y^1(\log \widetilde{D}) \cong \pi^* \Omega_X^1(\log D).$$

Proof. This follows immediately from [EV92, page 33, Lemma 3.21]. □

Observe that we have inclusions of sheaves

$$\Omega_X^1 \subseteq \Omega_X^1(\log D) \subseteq \Omega_X^1(D) := \Omega_X^1 \otimes O_X(D),$$
$$\Omega_Y^1 \subseteq \Omega_Y^1(\log \widetilde{D}) \subseteq \Omega_Y^1(\widetilde{D}) := \Omega_Y^1 \otimes O_Y(\widetilde{D}).$$

Fixing an irreducible component D_μ of D, there is a residue map (see [Del70, Section II.3.7])

$$\mathrm{Res}_{D_\mu} : \Omega_X^1(\log D_\mu) \longrightarrow \mathscr{O}_{D_\mu}.$$

In local coordinates $z_1, \ldots, z_m$ on X where D_μ is defined by $z_\mu = 0$, if ω is a section of $\Omega_X^1(\log D_\mu)$ with local expression

$$\omega = f_1\, dz_1 + \cdots + f_\mu \frac{dz_\mu}{z_\mu} + \cdots + f_m\, dz_m,$$

where the f_i, $1 \le i \le m$, are holomorphic functions, then the residue has the local expression

$$\mathrm{Res}_{D_\mu} \omega = f_\mu|_{z_\mu = 0}.$$

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3. Parabolic Higgs bundles and Γ-Higgs bundles

3.1. Parabolic Higgs bundles. Let E be a torsion-free coherent sheaf on X. We recall that a *parabolic structure* on E with respect to the divisor D is the data of a filtration

$$E = E_{\alpha_1} \supset E_{\alpha_2} \supset \cdots \supset E_{\alpha_l} \supset E_{\alpha_{l+1}} = E(-D),$$

where $0 \leq \alpha_1 < \cdots < \alpha_l < 1$ are real numbers called *weights* (see [MY92, Definition 1.2]). The α_j will be chosen without redundancy in the sense that if $\epsilon > 0$, then $E_{\alpha_j+\epsilon} \neq E_{\alpha_j}$. We will often shorten E_{α_j} to E_j. The sheaf E together with a parabolic structure is called a *parabolic sheaf* and is often denoted by E_*. If E is a locally free sheaf, then we will call E_* a *parabolic vector bundle*. See [MY92] for more on parabolic sheaves.

We will always assume that the parabolic weights are rational numbers whose denominators all divide $r \in \mathbb{N}$, that is, $\alpha_j \in (1/r)\mathbb{Z}$, for $1 \leq j \leq l$; this way, we may write $\alpha_j = m_j/r$ for some integers $0 \leq m_j \leq r-1$. It should be clarified that there are many choices for r. Further, we will make the same assumptions as in [Bi97, Assumptions 3.2].

A *parabolic Higgs field*, respectively *strongly parabolic Higgs field*, will be defined as a section $\phi \in H^0(X, (\operatorname{End} E) \otimes \Omega^1_X(\log D))$ satisfying

$$\phi \wedge \phi = 0$$

and

$$(\operatorname{Res}_{D_\mu}\phi)(E_j|_{D_\mu}) \subseteq E_j|_{D_\mu}, \text{ respectively } (\operatorname{Res}_{D_\mu}\phi)(E_j|_{D_\mu}) \subseteq E_{j+1}|_{D_\mu}, \tag{3.1}$$

for $1 \leq j \leq l$, $1 \leq \mu \leq h$. By a *parabolic Higgs bundle* we will mean a pair (E_*, ϕ) consisting of a parabolic vector bundle E_* and a strongly parabolic Higgs field ϕ.

REMARK 3.1. Observe that this definition of a parabolic Higgs field differs from that given in [Yo95, Definition 2.2], where one takes $\phi \in H^0(X, (\operatorname{End} E) \otimes \Omega^1_X(D))$.

3.2. Γ-Higgs bundles. Let W be a vector bundle on Y admitting an action $\Lambda : \Gamma \times W \longrightarrow W$ compatible with the action λ on Y. If we think of W as a space with projection $r : W \longrightarrow Y$, then this means that

$$\begin{array}{ccc} \Gamma \times W & \xrightarrow{\Lambda} & W \\ {\scriptstyle \mathbb{1}_\Gamma \times r}\downarrow & & \downarrow{\scriptstyle r} \\ \Gamma \times Y & \xrightarrow[\lambda]{} & Y \end{array}$$

commutes. Alternatively, if we think of W as a locally free sheaf then this means that there is an isomorphism

$$L : \lambda^* W \xrightarrow{\sim} p_Y^* W$$

of sheaves on $\Gamma \times Y$ satisfying a suitable cocycle condition. When such an action exists, we will call W a Γ-*vector bundle*. In this realization, if W' is another Γ-vector

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bundle with $L' : \lambda^* W' \longrightarrow p_Y^* W'$ giving the action on W', then compatible actions on $W \oplus W'$ and $W \otimes W'$ are readily defined since direct sums and tensor products commute with pullbacks.

For each $\gamma \in \Gamma$, the restrictions $L_{\{\gamma\}\times Y} : \lambda_\gamma^* W \xrightarrow{\sim} W$ yield isomorphisms $L_\gamma : W \longrightarrow \lambda_{\gamma *} W$ (by adjunction) satisfying

$$L_e = \mathbb{1}_W \quad \text{and} \quad \lambda_{\gamma *} L_\delta \circ L_\gamma = L_{\gamma\delta}$$

for all $\gamma, \delta \in \Gamma$. In our case, since Γ is discrete, knowledge of the L_γ is enough to reconstruct L.

Example 3.2. There are three examples of Γ-bundles that will be of particular interest to us.

(a) The action λ on Y induces a natural action on the sheaf of differentials Ω_Y^1 which will be compatible with λ.

(b) Since $X = \Gamma \setminus Y$, we have $\pi \circ \lambda = \pi \circ p_Y$ as maps $\Gamma \times Y \longrightarrow X$. Thus, if E is any vector bundle on X, there is a canonical isomorphism $\lambda^* \pi^* E \xrightarrow{\sim} p_Y^* \pi^* E$. Hence the pullback $\pi^* E$ carries a Γ-action for which the action on the fibers is induced by the action on Y.

(c) By the previous example, $\mathscr{O}_Y(\pi^* D) = \pi^* \mathscr{O}_X(D)$ carries a compatible Γ-action. Since $\widetilde{D} \subseteq \pi^* D$ is a Γ-invariant subset we have an induced action on the line bundle $\mathscr{O}_Y(\widetilde{D})$ making it into a Γ-line bundle.

Let W, W' be as above. A homomorphism $\Phi : W \longrightarrow W'$ is said to *commute with the* Γ-*actions* or is a Γ-*homomorphism* if the diagram

$$\begin{array}{ccc} \lambda^* W & \xrightarrow{\lambda^*\Phi} & \lambda^* W' \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L'} \\ p_Y^* W & \xrightarrow[p_Y^*\Phi]{} & p_Y^* W' \end{array}$$

commutes.

If $\Phi \in H^0(Y, (\operatorname{End} W) \otimes \Omega_Y^1)$ is a Higgs field on W, that is, $\Phi \wedge \Phi = 0$, then we will call it a Γ-*Higgs field* if as a map $W \longrightarrow W \otimes \Omega_Y^1$ it commutes with the Γ-actions, where $W \otimes \Omega_Y^1$ has the tensor product action. Thus, for every $\gamma \in \Gamma$, there is a commutative diagram.

$$\begin{array}{ccc} W & \xrightarrow{\Phi} & W \otimes \Omega_Y^1 \\ {\scriptstyle L_\gamma}\downarrow & & \downarrow{\scriptstyle \widetilde{L}_\gamma} \\ \lambda_{\gamma *} W & \xrightarrow[\lambda_{\gamma *}\Phi]{} & \lambda_{\gamma *}(W \otimes \Omega_Y^1) \end{array} \tag{3.2}$$

If Φ is a Γ-Higgs field, the pair (W, Φ) will be referred to as a Γ-*Higgs bundle*.

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3.3. From Γ-Higgs bundles to parabolic Higgs bundles. We now begin with a Γ-Higgs bundle (W, Φ) and from it construct a parabolic Higgs bundle (E_*, ϕ). The underlying vector bundle E is defined as $E := \pi_* W^\Gamma$, the sheaf of Γ-invariant sections of $\pi_* W$, and as in [Bi97, Section 2c], the parabolic structure on E is defined by

$$E_j := \pi_* W\Big(\sum_{\mu=1}^{h} \lfloor -k_\mu r \alpha_j \rfloor \widetilde{D}_\mu \Big)^\Gamma .$$

Suppose $\Phi \in H^0(Y, (\operatorname{End} W) \otimes \Omega^1_Y)$ is a Γ-Higgs field on W. We will think of Φ as a homomorphism $\Phi : W \longrightarrow W \otimes \Omega^1_Y$. Since $\Omega^1_Y \subseteq \Omega^1_Y(\log \widetilde{D})$,

$$\pi_*(W \otimes \Omega^1_Y) \subseteq \pi_*(W \otimes \Omega^1_Y(\log \widetilde{D})) = \pi_*(W \otimes \pi^* \Omega^1_X(\log D)) = \pi_* W \otimes \Omega^1_X(\log D),$$

where the first equality is due to Lemma 2.2 and the last step by the projection formula. Therefore, $\phi := \pi_* \Phi$ may be considered as a map $\pi_* W \longrightarrow \pi_* W \otimes \Omega^1_X(\log D)$, and we have a candidate for a parabolic Higgs field.

Let $U \subseteq X$ be open and let s be an invariant section of $\pi_* W$ over U, so that we may think of s as a section $\widehat{s}$ of W over $\pi^{-1}(U)$ with $L_\gamma \widehat{s} = \widehat{s}$ for all $\gamma \in \Gamma$. Then by definition $\phi s := \Phi \widehat{s}$, and for $\gamma \in \Gamma$, using (3.2),

$$\widetilde{L}_\gamma(\phi s) = \widetilde{L}_\gamma(\Phi \widehat{s}) = \Phi(L_\gamma \widehat{s}) = \Phi \widehat{s} = \phi s,$$

so ϕs is a Γ-invariant section, and hence $\phi : E \longrightarrow E \otimes \Omega^1_X(\log D)$.

Proposition 3.3. *To a* Γ*-Higgs bundle* (W, Φ) *there is a naturally associated parabolic Higgs bundle* (E_*, ϕ)*.*

Proof. We have constructed (E_*, ϕ). We must prove that ϕ is strongly parabolic. This is a condition on the residues of ϕ along the components of the divisor D, so we may concentrate on those points of D_μ that do not belong to any other component of D. Therefore, we may assume that we are in the neighborhood of a point y of $\widetilde{D}_1$ that lies on no other $\widetilde{D}_\mu$. In this neighborhood, for $1 \le j \le l$,

$$E_j = \pi_* W(-m_j k_1 \widetilde{D}_1)^\Gamma .$$

We now choose coordinates on Y and X as in Lemma 2.1, so that the divisor $\widetilde{D}_1$ is defined by w_1 and the divisor D_1 is defined by z_1; we will write $p := k_1 r$ so that π is given in these coordinates by

$$z_1 = w_1^p, \quad z_2 = w_2, \ldots, z_m = w_m.$$

In these coordinates, near y, we may write

$$\Phi = A_1 dw_1 + \cdots + A_m dw_m$$

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for some holomorphic sections A_i of End W. We may then consider $A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$ as a locally defined map $W \longrightarrow W \otimes \mathscr{O}_Y(-\widetilde{D}_1) \otimes \pi^*\Omega^1_X(\log D_1)$, or more generally, as a map

$$W(-m_j k_1 \widetilde{D}_1) \longrightarrow W(-(m_j k_1 + 1)\widetilde{D}_1) \otimes \pi^*\Omega^1_X(\log D_1)$$

for $1 \le j \le l$. It is easily verified that $W(-(m_j k_1 + 1)\widetilde{D}_1) \subseteq W(-rk_1(\alpha_j + \epsilon)\widetilde{D}_1)$, where $0 \le \epsilon \le 1/rk_1$. So taking invariants we see that $\pi_* A_1 dw_1 = (1/p)A_1 w_1 dz_1/z_1$ gives a locally defined map

$$E_j \longrightarrow E_{\alpha_j + (1/rk_1)} \otimes \Omega^1_X(\log D_1) = E_{j+1} \otimes \Omega^1_X(\log D_1).$$

Since, by definition,

$$\mathrm{Res}_{D_1}\phi = \tfrac{1}{p} w_1 A_1|_{z_1=0},$$

and noting that $(\mathrm{Res}_{D_\mu}\phi)(E_j|_{D_1}) \subseteq E_{j+1}|_{D_1}$, it follows that the strong parabolicity condition (3.1) is satisfied.

That $\phi \wedge \phi = 0$ is easily seen, since if s is any section of E, then

$$(\phi \wedge \phi)s = (\Phi \wedge \Phi)\widehat{s} = 0$$

since $\Phi \wedge \Phi = 0$. □

3.4. From parabolic Higgs bundles to Γ-Higgs bundles. Recall that we are working under assumptions as in [Bi97, Assumptions 3.2], hence we can use the construction from [Bi97, Section 3b] in the following proposition.

Proposition 3.4. *Given a parabolic Higgs bundle (E_*, ϕ) on X, we can associate a Γ-Higgs bundle (W, Φ) on Y.*

Proof. We will begin by constructing a parabolic vector bundle on X of rank m. The holomorphic vector bundle underlying the parabolic vector bundle is Ω^1_X. To define the parabolic structure, take any irreducible component D_i of D. Let $\iota : D_i \hookrightarrow X$ be the inclusion map. We have a short exact sequence of vector bundles on D_i

$$0 \longrightarrow N^*_{D_i} \longrightarrow \iota^*\Omega^1_X \longrightarrow \Omega^1_{D_i} \longrightarrow 0,$$

where N_{D_i} is the normal bundle of D_i. Note that the Poincaré adjunction formula says that $N_{D_i} = \iota^* O_X(D_i)$. The quasiparabolic filtration over D_i is the above filtration

$$N^*_{D_i} \subset \iota^*\Omega^1_X.$$

The parabolic weights are 0 and $(rk_i - 1)/rk_i$. More precisely, $N^*_{D_i}$ has parabolic weight $(rk_i - 1)/rk_i$ and the parabolic weight of the quotient $\Omega^1_{D_i}$ is zero. Note that the nonzero parabolic weight $(rk_i - 1)/rk_i$ has multiplicity one. This parabolic vector bundle will be denoted by $\widetilde{\Omega}^1_X$.

The action of Γ on Y induces an action of Γ on the vector bundle Ω^1_Y making it a Γ-bundle. From the construction of $\widetilde{\Omega}^1_X$ it can be deduced that the parabolic vector bundle corresponding to the Γ-bundle Ω^1_Y is $\widetilde{\Omega}^1_X$. To prove this, first note that if $U_0 \subset X$ is a Zariski open subset such that the complement U_0^c is of codimension at least two, and V, W are two algebraic vector bundles on X that are isomorphic over U_0, then V and W are isomorphic over X; using Hartog's theorem, any isomorphism $V|_{U_0} \longrightarrow W|_{U_0}$ extends to a homomorphism $V \longrightarrow W$, and similarly, we have a homomorphism $W \longrightarrow V$, and these two homomorphisms are inverses of each other because they are so over U_0. Next note that

$$(\pi_*\Omega^1_Y)^\Gamma = \Omega^1_X$$

because $(\pi_*\Omega^1_Y)^\Gamma = \Omega^1_X$ over the complement of the singular locus of D. Therefore, Ω^1_X is the vector bundle underlying the parabolic bundle corresponding to the Γ-bundle Ω^1_Y. It is now straightforward to check that the parabolic weights are of the above type.

Let W be the Γ-bundle on Y corresponding to the parabolic vector bundle E_* on X (using [Bi97, Section 3b]). Let ϕ be a strongly parabolic Higgs field on E_*. It is straightforward to check that ϕ defines a homomorphism of parabolic vector bundles

$$\phi' : E_* \longrightarrow E_* \otimes \widetilde{\Omega}^1_X,$$

where $E_* \otimes \widetilde{\Omega}^1_X$ is the parabolic tensor product of E_* and $\widetilde{\Omega}^1_X$.

Since the correspondence between parabolic bundles and Γ-vector bundles is compatible with the operation of tensor product, we conclude that the parabolic tensor product $E_* \otimes \widetilde{\Omega}^1_X$ corresponds to the Γ-bundle $W \otimes \Omega^1_Y$. Therefore, the above homomorphism ϕ' pulls back to a Γ-equivariant homomorphism Φ from W to $W \otimes \Omega^1_Y$. □

THEOREM 3.5. *We have an equivalence of categories between Γ-Higgs bundles on Y and parabolic Higgs bundles on X which satisfy the assumptions as in* [Bi97, Assumptions 3.2].

PROOF. Proof is clear from Propositions 3.3 and 3.4 and [Bi97, Sections 2c, 3b]. □

REMARK 3.6. In Borne's formalism, the parabolic bundle E_* may be considered as a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathfrak{Vect}(X)$, with

$$\frac{j}{r} \longmapsto E_j(D),$$

and composing with $\pi^* : \mathfrak{Vect}(X) \longrightarrow \mathfrak{Vect}(Y)$, we get a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \longrightarrow \mathfrak{Vect}(Y)$. We also have a covariant functor $(1/r)\mathbb{Z} \longrightarrow \mathfrak{Vect}(Y)$ given by

$$\frac{j}{r} \longmapsto \mathscr{O}_Y(m_{j-1}\overline{D}).$$

Therefore, we obtain a functor $((1/r)\mathbb{Z})^{\mathrm{op}} \times \frac{1}{r}\mathbb{Z} \longrightarrow \mathfrak{Vect}(Y)$

$$\frac{j}{r} \longmapsto \pi^* E_j(D) \otimes \mathscr{O}_Y(m_{j-1}\overline{D}).$$

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An end for this functor [Ma98, Section IX.5] consists of a vector bundle $V \in \mathrm{Ob}\mathfrak{Vect}(Y)$ and diagrams for $i \leq j$

$$\begin{array}{ccccc} & & \pi^*E_i(D) \otimes \mathcal{O}_Y(m_{i-1}\overline{D}) & & \\ & \nearrow & & \searrow & \\ V & & & & \pi^*E_i(D) \otimes \mathcal{O}_Y(m_{j-1}\overline{D}) \\ & \searrow & & \nearrow & \\ & & \pi^*E_j(D) \otimes \mathcal{O}_Y(m_{j-1}\overline{D}) & & \end{array}$$

such that the diagram is terminal among all such diagrams. It is not difficult to check that W is a universal end for the functor defined above, that is, it is an end, and given an end V as in the diagram, there is a unique morphism $V \longrightarrow W$ which yields the appropriate commuting diagrams.

4. Root stacks

The notion of a root stack is something of a generalization of the notion of an orbifold with cyclic isotropy groups over a divisor. Of course, our main interest in this construction is in the case when X is a smooth complex projective variety, but giving the definition for an arbitrary $\mathbb{C}$-scheme imposes no further conceptual or technical difficulties, so we will give the definition and describe some of the basic properties in this generality. We largely follow the presentations of [Bo07] and [Cad07] here (as well as [The], [Vis08] for generalities), so we direct the reader requiring further illumination on issues raised below to these references.

4.1. Definition and construction. We fix a $\mathbb{C}$-scheme X, an invertible sheaf L on X and $s \in H^0(X, L)$, so that if s is nonzero, it defines an effective divisor D on X. We will also fix $r \in \mathbb{N}$. Let $\mathfrak{X} = \mathfrak{X}_{(L,r,s)}$ denote the category whose objects are quadruples

$$(f : U \longrightarrow X, N, \phi, t), \tag{4.1}$$

where U is a $\mathbb{C}$-scheme, f is a morphism of $\mathbb{C}$-schemes, N is an invertible sheaf on U, $t \in H^0(U, N)$ and $\phi : N^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism of invertible sheaves with $\phi(t^{\otimes r}) = f^*s$. A morphism

$$(f : U \longrightarrow X, N, \phi, t) \longrightarrow (g : V \longrightarrow X, M, \psi, u)$$

consists of a pair (k, σ), where $k : U \longrightarrow V$ is a $\mathbb{C}$-morphism making

$$\begin{array}{ccc} U & \xrightarrow{k} & V \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

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commute and $\sigma : N \xrightarrow{\sim} k^*M$ is an isomorphism such that $\sigma(t) = k^*(u)$. Moreover, the following diagram must commute:

$$\begin{array}{ccc} N^{\otimes r} & \xrightarrow{\phi} & f^*L \\ \sigma^{\otimes r}\downarrow & & \downarrow \text{can} \\ k^*M^{\otimes r} & \xrightarrow[k^*\psi]{} & k^*g^*L \end{array}$$

If

$$(g : V \longrightarrow X, M, \psi, u) \xrightarrow{(l,\tau)} (h : W \longrightarrow X, J, \rho, v)$$

is another morphism, then the composition is defined as

$$(l, \tau) \circ (k, \sigma) := (l \circ k, k^*\tau \circ \sigma), \tag{4.2}$$

using the canonical isomorphism $(l \circ k)^*J \cong k^*l^*J$.

We will often use the symbols $\mathfrak{f}$, $\mathfrak{g}$ to denote objects of $\mathfrak{X}$. If it is understood that $\mathfrak{f} \in \mathfrak{X}_U$, then by $\mathfrak{f}$ we will denote the quadruple $\mathfrak{f} = (f : U \longrightarrow X, N_{\mathfrak{f}}, \phi_{\mathfrak{f}}, t_{\mathfrak{f}})$.

The category $\mathfrak{X}$ comes with a functor $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ which simply takes $\mathfrak{f}$ to U and (k, σ) to h.

Proposition 4.1 [Cad07, Theorem 2.3.3]. *The morphism of categories* $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ *makes* $\mathfrak{X}$ *a Deligne–Mumford stack.*

Remark 4.2. The previous statement implies that $\mathfrak{X} \longrightarrow \mathfrak{Sch}/\mathbb{C}$ is a category fibered in groupoids. Let $\mathfrak{f} \in \mathrm{Ob}\mathfrak{X}_U$ be an object of $\mathfrak{X}$ lying over U as given in (4.1) and let $g : V \longrightarrow U$ be a morphism of schemes. A choice of pullback $g^*\mathfrak{f} \in \mathrm{Ob}\mathfrak{X}_V$ can easily be described by the tuple

$$(g \circ f : V \longrightarrow X, g^*N_{\mathfrak{f}}, g^*\phi_{\mathfrak{f}}, g^*t_{\mathfrak{f}})$$

and the Cartesian arrow $g^*\mathfrak{f} \longrightarrow \mathfrak{f}$ is given by $(g, \mathbb{1}_{g^*N_{\mathfrak{f}}})$.

Example 4.3 [Cad07, Example 2.4.1]. Suppose $X = \mathrm{Spec}\, A$ is an affine scheme, $L = \mathscr{O}_X$ is the trivial bundle and $s \in H^0(X, \mathscr{O}_X) = A$ is a function. Consider $U = \mathrm{Spec}\, B$, where $B = A[t]/(t^r - s)$. Then U admits an action of the group of rth roots of unity (more precisely, of the group scheme of the rth roots of unity) $\boldsymbol{\mu}_r$ of order r, where the induced action of $\zeta \in \boldsymbol{\mu}_r$, a generator, is given by

$$\zeta \cdot a = a, a \in A, \qquad \zeta \cdot t = \zeta^{-1}t.$$

In this case, the root stack $\mathfrak{X}_{(\mathscr{O}_X,s,r)}$ coincides with the quotient stack $[U/\boldsymbol{\mu}_r]$. Thus, as a quotient by a finite group (scheme), the map $U \longrightarrow \mathfrak{X}$ is an étale cover.

REMARK 4.4. If X is any $\mathbb{C}$-scheme, and L, s are as before, we may take an open affine cover $\{X_i = \operatorname{Spec} A_i\}$ such that $L|_{X_i} \cong \mathscr{O}_{X_i}$ and $s|_{X_i}$ corresponds to $s_i \in A_i$. Then by the example above,

$$\coprod_i U_i \longrightarrow \mathfrak{X}$$

is an étale cover, where $U_i = \operatorname{Spec} A[t_i]/(t_i^r - s_i)$.

There is also a functor $\pi : \mathfrak{X} \longrightarrow \mathfrak{Sch}/X$, whose action on objects and morphisms is given by

$$\mathfrak{f} \longmapsto f : U \longrightarrow X, \qquad (k, \sigma) \longmapsto k;$$

this yields a 1-morphism over $\mathfrak{Sch}/\mathbb{C}$, which we will often simply write as $\pi : \mathfrak{X} \longrightarrow X$.

4.2. Vector bundles and differentials on a root stack. Recall (for example [G01, Definition 2.50], [LM00, Lemme 12.2.1], [Vis89, Definition 7.18]) that a quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$ consists of the data of a quasi-coherent sheaf $\mathscr{F}_\mathfrak{f}$ for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ along with isomorphisms $\alpha_k = \alpha_k^{\mathscr{F}} : \mathscr{F}_\mathfrak{f} \longrightarrow k^*\mathscr{F}_\mathfrak{g}$ for any commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\;k\;} & V \\ & {\scriptstyle \mathfrak{f}}\searrow \quad \swarrow{\scriptstyle \mathfrak{g}} & \\ & \mathfrak{X} & \end{array} \tag{4.3}$$

such that for a composition $U \xrightarrow{k} V \xrightarrow{h} W \longrightarrow \mathfrak{X}$ one has

$$\alpha_{h\circ k} = k^*\alpha_h \circ \alpha_k. \tag{4.4}$$

A (global) section $\mathfrak{s} \in H^0(\mathfrak{X}, \mathscr{F})$ of $\mathscr{F}$ over $\mathfrak{X}$ is the data of a global section $\mathfrak{s}_\mathfrak{f} \in H^0(U, \mathscr{F}_\mathfrak{f})$ for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ such that for a diagram (4.3) as above, one has

$$\alpha_k(\mathfrak{s}_\mathfrak{f}) = k^*\mathfrak{s}_\mathfrak{g}.$$

A quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$ is a subsheaf of a quasi-coherent sheaf $\mathscr{G}$ if $\mathscr{F}_\mathfrak{f} \subseteq \mathscr{G}_\mathfrak{f}$ for all étale $\mathfrak{f} : U \longrightarrow \mathfrak{X}$.

LEMMA 4.5. *In the situation of Example 4.3, where $X = \operatorname{Spec} A$ is affine, $U := \operatorname{Spec} A[t]/(t^r - s)$ and $\mathfrak{X} = [U/\boldsymbol{\mu}]$, then for a quasi-coherent sheaf $\mathscr{F}$ on $\mathfrak{X}$, $\mathscr{F}_U$ admits a $\boldsymbol{\mu}_r$-action compatible with that on U.*

PROOF. We have $U \times_\mathfrak{X} U \cong U \times \boldsymbol{\mu}$ and under this isomorphism, the two projection maps from $U \times_\mathfrak{X} U$ correspond to the maps $p_U, \lambda : U \times \boldsymbol{\mu} \longrightarrow U$, where p_U is the projection onto U and λ is the action on U. Then the required action is defined by the composition

$$p_U^*\mathscr{F}_U \xrightarrow{\alpha_{p_U}^{-1}} \mathscr{F}_{U\times_\mathfrak{X} U} \xrightarrow{\alpha_\lambda} \lambda^*\mathscr{F}_U.$$

This concludes the proof. □

https://doi.org/10.1017/S1446788713000335 Published online by Cambridge University Press

4.2.1. The sheaf of differentials on $\mathfrak{X}$. The sheaf of differentials $\Omega^1_{\mathfrak{X}} = \Omega^1_{\mathfrak{X}/\mathbb{C}}$ can be defined as follows. If $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ is an étale map, then we simply set

$$\Omega^1_{\mathfrak{X},\mathfrak{f}} := \Omega^1_{U/\mathbb{C}}.$$

If we are given a diagram (4.3), then from the composition $U \xrightarrow{k} V \longrightarrow \operatorname{Spec}\mathbb{C}$, one obtains a sequence

$$0 \longrightarrow k^*\Omega_{V/\mathbb{C}} \longrightarrow \Omega_{U/\mathbb{C}} \longrightarrow \Omega_{U/V} \longrightarrow 0,$$

which is left exact [The, More on Morphisms, Ch. 33, Lemma 9.9] and whose last term is zero since k is necessarily étale. This defines isomorphisms α_k. The requirement (4.4) will be met because of the universal properties these morphisms possess.

4.2.2. The tautological invertible sheaf on $\mathfrak{X}$. The root stack $\mathfrak{X}$ possesses a tautological invertible sheaf $\mathscr{N}$. For an étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$, we simply take

$$\mathscr{N}_{\mathfrak{f}} := N_{\mathfrak{f}}.$$

Given a diagram (4.3), one has an isomorphism $(\mathbb{1}_U, \sigma) : \mathfrak{f} \longrightarrow k^*\mathfrak{g}$ and one may take

$$\alpha_k^{\mathscr{N}} := \sigma : N_{\mathfrak{f}} \longrightarrow k^* N_{\mathfrak{g}}.$$

The expression in the second component of (4.2) implies that (4.4) is satisfied. This defines the invertible sheaf $\mathscr{N}$ on $\mathfrak{X}$. Furthermore, by definition, we also get a tautological section $\mathfrak{t}$ of $\mathscr{N}$ over $\mathfrak{X}$ by simply taking

$$\mathfrak{t}_{\mathfrak{f}} := t_{\mathfrak{f}}.$$

4.3. Higgs fields on root stacks. Let X be as in Section 2, so that it is a smooth complex projective variety; D will be a normal crossing divisor with smooth components. Let $s \in H^0(X, \mathscr{O}_X(D))$ be a section with $(s) = D$. We also fix $r \in \mathbb{N}$. In all that follows $\mathfrak{X} = \mathfrak{X}_{(\mathscr{O}_X(D),r,s)}$ will be the associated root stack as constructed in Section 4.1.

REMARK 4.6. Consider the fuller situation of Section 2, where $\pi : Y \longrightarrow X$ be a Galois cover of smooth complex projective varieties and there is a divisor $\overline{D}$ on Y such that $\pi^* D = r\overline{D}$. Then there is an isomorphism $\phi : \mathscr{O}_Y(\overline{D})^{\otimes r} \xrightarrow{\sim} \pi^*\mathscr{O}_X(D)$ and a section $t \in H^0(Y, \mathscr{O}_Y(\overline{D}))$ such that $\phi(t^{\otimes r}) = \pi^* s$. Therefore, the quadruple $(\pi : Y \longrightarrow X, \mathscr{O}_Y(\overline{D}), \phi, t)$ defines a morphism

$$\widehat{\pi} : Y \longrightarrow \mathfrak{X}.$$

4.3.1. Higgs fields. Let $\mathcal{V}$ be a vector bundle on $\mathfrak{X}$. A *Higgs field* Φ on $\mathcal{V}$ is a homomorphism $\Phi : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{\mathfrak{X}}$. This means that for each étale morphism $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ we have a homomorphism $\Phi_{\mathfrak{f}} : \mathcal{V}_{\mathfrak{f}} \longrightarrow \mathcal{V}_{\mathfrak{f}} \otimes \Omega^1_U$ such that given a diagram (4.3), we obtain a commutative square.

$$\begin{array}{ccc} \mathcal{V}_{\mathfrak{f}} & \xrightarrow{\Phi_{\mathfrak{f}}} & \mathcal{V}_{\mathfrak{f}} \otimes \Omega^1_U \\ {\scriptstyle \alpha_k^{\mathcal{V}}} \downarrow & & \downarrow {\scriptstyle \alpha_k^{\mathcal{V}} \otimes \alpha_k^{\Omega^1_{\mathfrak{X}}}} \\ k^*\mathcal{V}_{\mathfrak{g}} & \xrightarrow[k^*\Phi_{\mathfrak{g}}]{} & k^*\mathcal{V}_{\mathfrak{f}} \otimes k^*\Omega^1_V \end{array} \tag{4.5}$$

THEOREM 4.7. *There is an equivalence of categories of Higgs bundles on* $\mathfrak{X}$ *and parabolic Higgs bundles on* X.

PROOF. We remark that a parabolic structure is given locally, so we may assume that $X = \mathrm{Spec} A$ is affine and that the parabolic divisor D is defined by $s \in A$. Then as in Example 4.3, we may take $U = \mathrm{Spec} B$ where $B = A[t]/(t^r - s)$, so that $\mathfrak{X} = [U/\boldsymbol{\mu}]$. In this case, the map $\mathfrak{f} : U \longrightarrow \mathfrak{X}$ is étale; we will write $f : U \longrightarrow X$ for the underlying map induced from $A \longrightarrow B$. Given a vector bundle $\mathcal{V}$, by Lemma 4.5, the bundle $\mathcal{V}_{\mathfrak{f}}$ on U carries a compatible $\boldsymbol{\mu}$-action. The fact that $\Phi_{\mathfrak{f}}$ commutes with this action comes from the existence of the diagram (4.5) for the two projection morphisms $U \times_{\mathfrak{X}} U \longrightarrow U$. Thus, we are reduced to the case of Γ-bundles when $\Gamma = \boldsymbol{\mu}$, which comes from Propositions 3.3 and 3.4. □

Acknowledgement

We are very grateful to the referee for helpful comments to improve the exposition.

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