PHOTOCONVECTION

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Convection under the influence of dynamically significant radiation fields occurs routinely in hot stars (Underhill 1949 ab) and probably also in a variety of other objects near the Eddington limit (Joss, Salpeter, and Ostriker 1973). Yet this topic, which is here called photoconvection, has not been actively investigated prior to the present decade. Except for limiting cases, the stability condition does not seem to have been worked out and only some preliminary notions exist about the highly unstable case. This is somewhat surprising since it has long been suspected that some of the vigorous dynamical activity observed in hot stars (Huang and Struve 1960, Reimers 1976) is caused by radiative forces (Underhill 1949 ab). In the hope that this neglect may be compensated for by the application of some of the techniques described at this meeting, I shall sketch some of the main features of this topic. Three aspects are considered. First, I list a set of approximate equations for plane-parallel photoconvection. Then I give a schematic treatment of the onset of instability. And finally, I shall outline some of the arguments for believing that photon bubbles occur in the nonlinear regime.

I. EQUATIONS OF PHOTOHYDRODYNAMICS

The interaction of electromagnetic radiation with a plasma is a complicated subject with a long and controversial history. However, many of the difficulties are avoided if we consider densities and radiation frequencies that keep the index of refraction of the medium quite close to unity. In that case, we can describe the radiation field by transfer theory if we take due notice of the motion of the material medium. The simplest description arises if we simply take the first two moments of the transfer equation and supply a constitutive relation for the radiative pressure tensor. For the matter, we shall adopt the model of a perfect gray gas. Then the matter field is described by the velocity $\vec{u}$, the density ρ, and the pressure p, while the radiation field is characterized by the flux $\vec{F}$, the energy density E, and the pressure tensor $\overleftrightarrow{P}$.

These variables are expressed in the inertial frame of the system (star), in which we will generally be working. It will be useful, however, to make use of the expressions for radiative flux and energy density in the local rest frame of the matter. These are

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(1.1a) $$\breve{E} = E - 2\vec{u}\cdot\vec{F}/c^2$$

(1.1b) $$\breve{\vec{F}} = \vec{F} - E\vec{u} - \overleftrightarrow{P}\cdot\vec{u},$$

where c is the speed of light. These expressions are valid only to order $|\vec{u}|/c$, which is the level of accuracy (at best) aimed for here. Nevertheless, in the equations used below, we shall see some factors of c^{-2}, because the radiation field is relativistic. In particular, the quantity $\vec{F}/E$ qualitatively plays the role of a velocity for the radiation field and in the surface layers of stars the magnitude of this velocity may be comparable with c.

In addition to the field variables, we have to specify certain quantities that measure the effective interactions between the two fields. These interactions we shall take to be Thomson scattering, absorption, and emission. We shall assume that the Compton effect can be modeled by a suitable choice of absorption coefficient. We shall call κ the absorption coefficient and σ the scattering coefficient (both per unit mass); σ will be constant and κ may depend on density and temperature. The source function (divided by c) is denoted by S and depends only on the matter's temperature, as indicated below.

The equations describing the conservation of matter and the force balance of the medium are

(1.2) $$\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{u}$$

and

(1.3) $$\rho\frac{d\vec{u}}{dt} = -\nabla p - g\rho\hat{z} + \frac{\rho(\kappa+\sigma)}{c}\breve{\vec{F}}$$

where $g\hat{z}$ is the acceleration of gravity, $\hat{z}$ is a unit vector, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla \ .$$

The last term on the right of (1.3) is the usual expression for the radiative force.

Analagous equations exist for the radiative fluid:

(1.4) $$\frac{\partial E}{\partial t} + \nabla\cdot\vec{F} = \rho\kappa c(S-\breve{E}) - \frac{\rho(\kappa+\sigma)}{c}\vec{u}\cdot\breve{\vec{F}}$$

and

(1.5) $$\frac{1}{c^2}\frac{\partial\vec{F}}{\partial t} + \nabla\cdot\overleftrightarrow{P} = -\frac{\rho(\kappa+\sigma)}{c}\breve{\vec{F}} + \frac{\rho\kappa}{c}(S-\breve{E})\vec{u}.$$

For the pressure tensor of the radiation field a standard form is

(1.6) $$\overleftrightarrow{P} = \frac{1}{3}E\overleftrightarrow{I} + (\vec{u}\vec{F} + \vec{F}\vec{u})/c^2 - \overleftrightarrow{\tau}$$

where $\overleftrightarrow{I}$ is the idemtensor and $\overleftrightarrow{\tau}$ is a viscous tensor. In component form,

(1.6a) $$\tau_{ij} = \eta\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}(\nabla\cdot\vec{u})\delta_{ij}\right]$$

where δ_{ij} is the Kronecker symbol and the viscosity is approximated by

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(1.6b) $$\eta = \frac{8E}{3\rho(10\kappa + 9\sigma)c} .$$

Expression (1.6) arises when the radiation pressure tensor is approximated in the matter frame by the usual Eddington approximation plus a viscosity tensor.

For the constitutive relations for the matter we adopt

(1.7) $$p = R\rho T$$

and

(1.8) $$S = aT^4$$

where T is the temperature and R and a are constants. (We shall not specify κ here.) The introduction of the temperature calls for another equation, as in normal convection.

If s is the specific entropy of the matter, we may write

(1.9a) $$\rho T \frac{ds}{dt} = -\rho\kappa c(S-E),$$

or, if we use the expression for the entropy of an ideal gas,

(1.9b) $$\rho C_p \frac{dT}{dt} - \frac{dp}{dt} = -\rho\kappa c(S-\tilde{E}),$$

where C_p is the specific heat at constant pressure.

These governing equations are consistent and moderately accurate sets of governing equations. I have said little about the basis of them (but see Simon 1963 or Hsieh and Spiegel 1976) since their physical content is reasonably clear. If anything, these equations are, for present purposes, too complete. It appears that there are a number of generally small terms which will hinder calculations and obscure meanings. But many of these terms are unfamiliar, and the challenge is to discover when we can discard which terms. In what follows, I shall make a number of guesses about this; I hope that these are not too misleading. In fact, much of the discussion is just aimed at seeing what some of these terms do and in such a schematic treatment you would not expect to see boundary conditions. I shall hardly disappoint you. But before I commit mayhem on the equations, let us modify the appearance of the last one by combining it with (1.4). We obtain, with the help of (1.2),

(1.10) $$\rho C_p \frac{dT}{dt} - \frac{dp}{dt} + \frac{dE}{dt} - \frac{4}{3}\frac{E}{\rho}\frac{d\rho}{dt} = -\nabla\cdot(\vec{F} - \frac{4}{3}E\vec{u}) - \frac{1}{3}\vec{u}\cdot\nabla E - \frac{\rho(\kappa+\sigma)}{c}\vec{u}\cdot\vec{F}.$$

We may note that the left hand side of this equation is $\rho T ds_{tot}/dt$ where s_{tot} is the total (matter plus radiation) specific entropy.

II. THE HYDROSTATIC STATE

As background to the problem of photoconvection it is useful to know the solution of the basic equations which describe the state in which the matter is static. But note that this solution is not photostatic; the radiation is flowing through matter like a fluid through a porous medium.

We consider stationary solutions whose properties are independent of horizontal coordinate. If $\kappa \neq 0$, equations (1.9), (1.1), and (1.8) indicate that

(2.1) $$E = aT^4;$$

if $\kappa = 0$ this relation is not forced and T is an arbitrary function of the vertical coordinate, z. In either case $\vec{F}$ is constant and is in the $\hat{z}$-direction.

Now (1.2) is identically satisfied and (1.3) gives the hydrostatic equation

(2.2) $$\frac{dp}{dz} = g_* \rho ,$$

where

(2.3) $$g_* = g - \frac{\kappa+\sigma}{c} F$$

is the effective gravity. (In the Eddington limit, $g_* = 0$.) The radiative flow equation (1.5) becomes

(2.4) $$\frac{dE}{dz} = -3\rho \frac{(\kappa+\sigma)}{c} F,$$

and (1.7) is unmodified. Thus all the governing equations are accounted for and we have a simple system to solve once κ is known. In general the problem is handled numerically, but some analytically tractable cases exist. Let us look briefly at the simplest: $\kappa+\sigma$ = constant.

We may introduce the total pressure

(2.5) $$P = p + \frac{1}{3} E,$$

and combine (2.2) and (2.4). We find that

(2.6) $$\frac{dP}{dz} = -g\rho,$$

and, on dividing by (2.4), that

(2.7) $$3 \frac{dP}{dE} = \frac{gc}{(\kappa+\sigma)F} .$$

The integral of this equation, after some rearrangement, may be written

(2.8) $$p = \frac{g_* c}{3(\kappa+\sigma)F} (E-E_1) ,$$

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where E_1 is an arbitrary constant. It is often convenient to choose E_1 as the value of E at the top of the "atmosphere ".

We may now write a simple differential equation for E, or T, and find the solution

$$(2.9) \qquad -z = \frac{4R}{g_*}\left[T - \frac{1}{2}T_1 \tan^{-1}\left(\frac{T}{T_1}\right) - \frac{1}{2}T_1 \tanh^{-1}\left(\frac{T_1}{T}\right)\right] ,$$

where $E_1 = aT_1^4$. If $T_1 = 0$, this represents a complete polytropic atmosphere. In any case, the medium is polytropic for $z << 0$ and T is proportional to $-z$ down there. For $z >> 0$, $T - T_1$ decays exponentially as we move upward and the atmosphere extends to infinity for $T_1 \neq 0$.

In principle, all the other details could be worked out from this, but numerical work is generally needed. However, some things are still simply expressible in terms of the optical depth

$$(2.10) \qquad \tau = \int_z^\infty (\kappa+\sigma)\rho dz .$$

In particular,

$$(2.11) \qquad E = \frac{F}{c}(\tau+\tau_1),$$

where τ_1 is a constant of order unity.

Another quantity of interest in the static atmosphere is the temperature gradient. In the present instance this is most simply expressed in the familiar nondimensional form

$$(2.12) \qquad \nabla \equiv \frac{d\ln T}{d\ln P} = -\frac{R}{g\beta}\frac{dT}{dz} \equiv \frac{\gamma-1}{\gamma}\frac{C_p}{g\beta}\frac{dT}{dz} .$$

where

$$(2.13) \qquad \beta = \frac{p}{P} .$$

For the atmosphere with $\kappa+\sigma$ constant we find

$$(2.14) \qquad \nabla = \frac{1}{4}\frac{1-\alpha}{1-\beta} ,$$

where

$$(2.15) \qquad \alpha = \frac{g_*}{g} .$$

III. THE ONSET OF CONVECTION

The action of radiative forces under suitable conditions may promote wave amplification (Hearn 1972, 1973; Berthomieu, Provost, and Rocca 1976) and possibly overstability (e.g., Spiegel 1976). The nature of this overstability seems to place it more in the domain of stellar pulsation theory than convection theory, though the two may become enmeshed in the nonlinear regime. On the other hand, monotonic instability, that is exponential growth without oscillation, is more clearly linked to the development of convection when the time scales are dynamic, and I shall confine myself here to discussing that topic.

The procedure for deciding whether convective instability arises is straightforward in principle, especially when we are not trying to study overstability or finite-amplitude instability. We decompose each dependent variable into a hydrostatic part and small perturbation. Here we shall indicate the latter type of quantity by a prime, except in the case of velocity. We restrict ourselves to the situation where $\partial/\partial t = 0$. Then, on linearizing in the usual way, we find from (1.10) that

$$\rho C_p b_1 w = \nabla\cdot\overset{\frown}{\vec{F}}{}' , \tag{3.1}$$

where

$$\rho C_p b_1 = -(\rho C_p \frac{dT}{dz} - \frac{dp}{dz} + \frac{dS}{dz} - \frac{4}{3}\frac{S}{\rho}\frac{d\rho}{dz}) . \tag{3.2}$$

But we may also proceed in this way on the basis of (1.9b) and in that case we obtain the equation

$$\rho C_p b_2 w = \rho\kappa c(S' - \breve{E}') , \tag{3.3}$$

where

$$\rho C_p b_2 = -(\rho C_p \frac{dT}{dz} - \frac{dp}{dz}) . \tag{3.4}$$

Now in a full treatment of the problem it would not matter which of these two routes is taken since the final answer would be the same. But the stability criteria that are normally used are obtained with approximations and the two approaches may differ in that case since they have suggested different approximations to different people. In particular, people have simply written down criteria for instability with respect to adiabatic disturbances with differing notions of what they mean by adiabatic. Thus, the commonly encountered criterion results from equating $\nabla\cdot\overset{\frown}{\vec{F}}{}'$ to zero. If we do this we find that $b_1 w$ must vanish at marginal stability. Since w in that case has small but arbitrary amplitude, we obtain the critical condition $b_1 = 0$, which is the conventional one (Chandrasekhar 1939). On the other hand, if we set the right hand side of (3.4) equal to zero (e.g., Wentzel 1970, Spiegel 1976) we obtain $b_2 = 0$ as the condition for marginal stability. This criterion holds strictly when absorption and Compton scattering are omitted and its use otherwise is dangerous.

The two criteria represent valid approximations under certain circumstances and P. Vitello (private communication) has recently investigated what these are.

The discussion of this question shows that the conditions under which one or other neutral stability criterion holds depends on the perturbation being made. This is a common situation and we expect that the correct instability criterion is to be found by choosing the most unstable mode.

To see how the problem goes let us begin to do the stability calculation. From (1.5) we find for marginal linear perturbations that

$$\overset{\leftrightarrow}{F}{}' = -\frac{c}{\rho(\kappa+\sigma)}\nabla\cdot\overset{\leftrightarrow}{P}{}' - \left[\frac{c}{\rho(\kappa+\sigma)}\right]'\nabla\cdot\overset{\leftrightarrow}{P} \tag{3.5}$$

with

$$\overset{\leftrightarrow}{P}{}' = \frac{1}{3}E'\overset{\leftrightarrow}{I} + (\vec{u}\,\hat{z} + \hat{z}\,\vec{u})\cdot F/c^2 - \overset{\leftrightarrow}{\tau}. \tag{3.6}$$

Also from (1.9b) we obtain

$$\tilde{E}' = S' - \frac{C_p b_2}{\kappa c}w. \tag{3.7}$$

If we combine these results with (3.1), making use of other equations as needed, we find an equation of the form

$$\rho C_p[\mathcal{B}_2\nabla^2 w + \rho\kappa A\frac{\partial w}{\partial z} - 3\rho^2\kappa(\kappa+\sigma)\mathcal{B}_1 w] = 3\rho^2\kappa(\kappa+\sigma)\nabla\cdot(K\nabla T)', \tag{3.8}$$

where

$$K = \frac{4acT^3}{3\rho(\kappa+\sigma)} \tag{3.9}$$

and $\mathcal{B}_1$, $\mathcal{B}_2$, and A are quantities whose dimensions are temperature over length. If $\eta = 0$ and C_p is constant,

$$\mathcal{B}_1 = b_1 - \frac{1}{3\rho^2(\kappa+\sigma)}\frac{d^2}{dz^2}\left(\frac{b_2}{\kappa}\right) - \frac{1}{3\rho}\frac{d}{dz}[\rho(\kappa+\sigma)]^{-1}\frac{d}{dz}\left(\frac{b_2}{\kappa}\right) - \frac{F}{c}\frac{d}{dz}\left[(\kappa+\sigma)\frac{d\rho}{dz}\right]^{-1} - \frac{F}{\rho(\kappa+\sigma)c}\frac{d^2\ln\rho}{dz^2}, \tag{3.10a}$$

$$\mathcal{B}_2 = b_2 \tag{3.10b}$$

and

$$A = \frac{2}{\rho}\frac{d}{dz}\left(\frac{b_2}{\kappa}\right) - \frac{b_2}{\rho\kappa}\frac{d}{dz}[\ln\rho(\kappa+\sigma)] + \frac{3F}{\rho C_p c}\frac{d}{dz}[\ln\rho^2(\kappa+\sigma)]. \tag{3.10c}$$

Now consider the case in which the right hand side of (3.8) is set equal to zero. That is, instead of trying to speak of an adiabatic disturbance, let us simply ask what happens to a perturbation when radiative conductivity is suppressed. If geometrically small horizontal scales are the most unstable, as they are in ordinary inviscid, non-conducting convection, we may replace ∇^2 by $-k^2$ where k is the horizontal wave number. For qualitative purposes, we may also omit the term

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with coefficient A since this is not important in the limiting cases we wish to consider. Also, for the present argument, I shall set $B_1 = b_1$, since this discussion is merely schematic. The approximate condition then becomes

$$k^2 b_2 + 3\rho^2 \kappa(\kappa+\sigma) b_1 = 0, \tag{3.11}$$

and in terms of the quantities defined in (2.12) and (2.15) we find the instability criterion

$$\nabla \geq \frac{(\gamma-1)[\beta\xi^2(4-3\beta)]}{\beta[\gamma\beta+4(\gamma-1)(1-\beta)] + \xi^2[\gamma\beta^2+4(\gamma-1)(1-\beta)(4+\beta)]}, \tag{3.12}$$

where

$$\xi^2 = \frac{3\rho^2\kappa(\kappa+\sigma)}{k^2}. \tag{3.13}$$

This criterion holds approximately in the limit of zero viscosity and with the omission of radiative conduction terms as indicated. The dimensionless quantity ξ^2, which arises in radiative cooling problems (Unno and Spiegel 1966), should be chosen so as to minimize the right hand side of (3.12). The resulting value of ξ is then inserted to give the local stability criterion. Of course, if we are led to extreme values of ξ we should worry about the possible violation of physical constraints that have been removed in this simplified analysis. (_In extremis_, we could just solve the problem properly.)

To make the appropriate choice we observe that for $\gamma > 4/3$ the right side of (3.12) increases as ξ decreases. In that case, the instability criterion is obtained with the largest possible values of ξ, hence with modes of large horizontal scale in the length unit $[3\rho^2\kappa(\kappa+\sigma)]^{-1/2}$. In stellar interiors most scales of interest satisfy this condition and the conventional criterion would apply. In transparent regions, however, it may be that geometrical constraints intervene and large ξ cannot be achieved. In that case, the maximum values allowed for ξ should be taken, and here we should note that once ξ exceeds unity there is not a large difference from the results at very large ξ.

In cases where $\gamma < \frac{4}{3}$, the situation is reversed and the right hand side of (3.12) decreases as ξ decreases. The preferred value of ξ is now the smallest one possible; that is we want the largest allowed value of k. If the particle mean free path is much less than the photon mean free path we can choose small ξ without worrying about the breakdown of fluid dynamics. But we do have to make sure that we don't choose a k which is so large that diffusive effects wipe out the instability. In fact this amounts to finding the preferred mode in the usual way, but here the choice determines not just diffusive corrections to the critical gradient, but also the effective adiabatic gradient itself. Unfortunately, there is a complication that arises in this situation.

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The case $\gamma < \frac{4}{3}$ will normally occur in ionization zones and therefore has to be treated with some care. In fact, Underhill (1949b) has evaluated the "adiabatic" temperature gradient with partial ionization and in the presence of an important radiation field. But that calculation was what I have been calling the conventional criterion. That is, she has applied the condition of zero total (matter plus radiation) entropy gradient which corresponds to the marginal stability conditions with $\xi = \infty$. However, the possibility exists that finite values of ξ may be more correct since the zones of partial ionization tend to occur in stellar envelopes. We could then have a somewhat increased convective instability but the modes involved, being radiatively leaky, might not carry heat effectively. It appears therefore that for most purposes the standard convection criterion is good. However, it would be more comfortable to have a detailed treatment of this problem, and I predict that there soon will be one.

IV. PHOTON BUBBLES

In thinking about ordinary stellar convection we may be guided by solar observations, but we have not such direct experience to guide us in photoconvection. Instead, we may appeal to observations of a laboratory flow that is analogous to photoconvection. We have already seen that the radiation in this problem behaves (in the Eddington approximation) like a fluid flowing through a deformable porous medium. This closely resembles the situation in a fluidized bed (Thorne 1973, Prendergast and Spiegel 1973). Though the analogy is not a perfect one (Spiegel 1976), it can be used to suggest the qualitative nature of nonlinear photoconvection. And one of the most striking implications of this analogy is that instead of convective thermals having relatively low densities, we should expect real bubbles in photoconvection. These are filled with radiation and contain virtually no matter. How this modification of the normal convective process may influence the heat flux can only be crudely estimated (Thorne 1973), but there are also other features of convection which are strongly affected. In particular, bubbles feel the full effect of gravity rather than the reduced gravity of ordinary convection, hence large (that is, sonic) convective speeds may be anticipated.

In this section, I shall sketch an approach to the treatment of photon bubbles borrowing heavily from the literature on fluidized beds (Jackson 1970, Rowe 1971). In comparison to fluidization, this theory suffers from the disadvantage that we have not yet seen a photon bubble. However, John Lin at Columbia is looking seriously at the prospects for removing this drawback experimentally.

We wish then to study a photon bubble of radius r_o rising at speed V. We shall assume that $r_o << H_*$, where $H_* = \mathcal{R} T/g_*$, and that the bubble may be taken to be quasi-steady when described in its own reference frame. We may nevertheless introduce the dynamical time scale r_o/V. Let us assume that this time is much shorter

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than the thermal time of a region of size r_o and then presume from this that there is validity in neglecting thermal effects. Then we may tentatively set $\kappa = 0$ in the basic equations.

Now let E and F be representative values for the ambient radiant energy density and flux and let ρ_* be a representative ambient matter density. The following dimensionless parameters are of interest:

$$\varepsilon = \frac{V}{c}, \delta = \frac{EV}{F}, f^2 = \frac{V^2}{gr_o}, \tau_* = \rho_*\sigma r_o . \tag{4.1}$$

We assume that $\varepsilon \ll 1$ and, from the analogy to fluidization, we anticipate that f^2 is of order unity. If E and F may be estimated from their static values (see (1.11)), we have $E/F \sim c/\tau$, for $\tau \gg 1$, where τ is the optical depth. Hence $\delta \sim \varepsilon\tau$. Then, when the bubble is only a few radii below the surface, $\tau_* \sim \tau$ and we have that $\delta \sim \varepsilon\tau_*$, which is the regime we shall study here. A further restriction to be used in the following analysis is $\delta \ll 1$, but I shall mention at the end what may happen at larger depths when δ becomes of order unity.

If we nondimensionalize the basic equations and make use of the foregoing approximations, we obtain a greatly reduced set of equations. In dimensionful form these are

$$\rho(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}) = -\nabla p - g\rho\hat{z} + \frac{\rho\sigma}{c} F \tag{4.2}$$

$$\frac{\partial p}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \tag{4.3}$$

$$\frac{\partial}{\partial t}(p/\rho^\gamma) + \vec{u}\cdot\nabla(p/\rho^\gamma) = 0 \tag{4.4}$$

$$\nabla\cdot\vec{F} = 0 \tag{4.5}$$

$$\frac{1}{3}\nabla E = \frac{\rho\sigma}{c}\vec{F}. \tag{4.6}$$

This description is about as primitive as it can be while still involving the elements of photohydrodynamics. Let us now seek approximate solutions for a bubble rising at constant speed V. We presume that the medium is unstable, which is true if $\alpha \lesssim 0.2$.

Suppose, in first approximation, that the bubble is a spherical hole of radius r_o. If the bubble does not greatly disturb the ambient density, we see that equations (4.5) and (4.6) are simply the transfer equations for a static medium with a hole in it. This results because in the present approximation the radiation field adjusts quickly to the state of the medium ($\varepsilon \ll 1$); also the motion is so slow that the difference between $\vec{F}$ and $\vec{\mathcal{F}}$ may be neglected ($\delta \ll 1$). We have also assumed that the bubble radius is much less than the local scale height, hence ρ in (4.6) is approximately constant outside the hole. Equations (4.5) and (4.6) may then be solved separately with $\rho = 0$ inside the spherical cavity.

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Let us introduce a spherical coordinate system with origin at the center of the hole and with $\theta = 0$ at the top of the hole. Far from the hole, the flux is $F_o\hat{z}$, where F_o is a constant given by the static solution and we have the condition

$$\nabla E \to -\frac{3\rho\sigma F_o}{c}\hat{z} \quad \text{as} \quad r \to \infty. \tag{4.7}$$

Moreover, E and the component of $\vec{F}$ normal to the bubble surface are continuous across the surface. Since (4.6) implies that E is constant inside the hole, we have the boundary condition

$$E = E_o \quad \text{on} \quad r = r_o, \tag{4.8}$$

where E_o is the constant value of E inside the hole. Now (4.5) and (4.6) show that E is a harmonic function and, with conditions (4.7) and (4.8), we find

$$E = E_o - \frac{3\rho\sigma}{c} F_o r_o \left(\frac{r}{r_o} - \frac{r_o^2}{r^2}\right)\cos\theta . \tag{4.9}$$

Since $cE/(3\rho\sigma)$ is a potential for $\vec{F}$, we have

$$\vec{F} = \nabla\cdot\{F_o r_o[\frac{r}{r_o} - (\frac{r_o}{r})^2]\cos\theta\}. \tag{4.10}$$

Alternatively, we can express F in terms of a Stokes stream function:

$$F_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad F_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}, \tag{4.11}$$

where

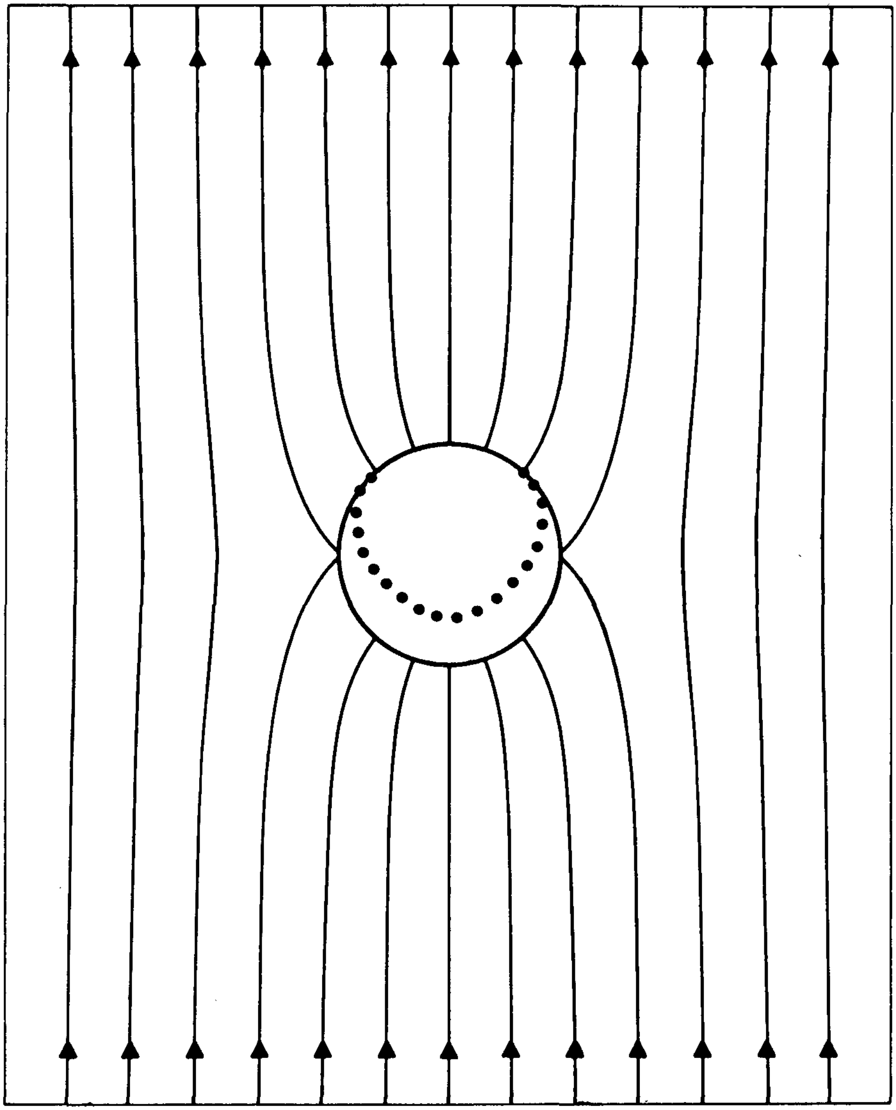
$$\Psi = -\frac{1}{2}F_o r^2[1 + 2(\frac{r_o}{r})^3]\sin^2\theta . \tag{4.12}$$

The flux consists of the original uniform part plus a dipole generated by the hole, a result familiar from analogous problems in, for example, electrostatics. The radiative flow is shown in the neighborhood of the bubble in the figure on the following page. Inside the bubble, the flux is $3F_o$, if equation (4.6) may be used.

This last point is a delicate one as we have used the Eddington approximation for the transfer theory. However, this approximation holds if the radiation is isotropic and, when $\rho\sigma r_o >> 1$, it probably is. The reason is that for $\tau_* >> 1$ individual photons will scatter off the bubble surface (actually a layer of thickness $(\rho\sigma)^{-1}$) many times before escaping, hence the radiation field inside the bubble should be reasonably isotropic.

The deformation in the radiation field produces an additional force on the matter. The total external force density is

$$-g\rho\hat{z} + \frac{\rho\sigma}{c}F = -g_*\rho\hat{z} - g_R\rho\nabla\left(\frac{r_o^3}{r^2}\cos\theta\right),$$



Streamlines for the radiative flux around a hole, according to (4.12). The solid circle shows the original hole. The dotted curve indicates the estimated deformation of the hole obtained by setting $h = 0$ in (4.22) with $\alpha = 0$ and V chosen as in (4.24).

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where $g_* = g - g_R$ and $g_R = F_o/c$. The additional dipole force produces a fluid circulation which causes the hole to rise and, in general, to deform. Let us study these effects.

If we work in the bubble's frame and assume a stationary situation we have the equations

$$\vec{v}\cdot\nabla\vec{v} = -\nabla h - g_* \hat{z} - \nabla\phi \ , \tag{4.13}$$

$$\vec{v}\cdot\nabla(p/\rho^\gamma) = 0, \tag{4.14}$$

$$\nabla\cdot(\rho\vec{v}) = 0, \tag{4.15}$$

where

$$\phi = \left(\frac{\sigma F_o r_o^3}{c}\right)\frac{z}{r^3} \tag{4.16}$$

and

$$h = \int \frac{dp}{\rho} . \tag{4.17}$$

I have not changed notation to indicate the coordinate transformation except to call $\vec{v} = \vec{u} - \vec{V}\hat{z}$ the new velocity. The correction to $\vec{F}$ due to the motion of the bubble is of order δ and is neglected.

First we shall determine V on the assumption that the bubble remains spherical. This we do with the approximation ρ = const, whence

$$\nabla\cdot\vec{v} = 0. \tag{4.18}$$

We may therefore take $\vec{v}$ to be the incompressible flow around a spherical obstacle. Such a flow has a vanishing normal component on the bubble boundary and it approaches $-V\hat{z}$ as $r \to \infty$. Solutions of this problem are well known and if we also set

$$\nabla \times \vec{v} = 0 \ , \tag{4.19}$$

we find

$$\vec{v} = -V\,\nabla\left[r\cos\theta\left(1 + \frac{r_o^3}{2r^3}\right)\right]. \tag{4.20}$$

Moreover, because of (4.19) we may rewrite (4.13) as

$$\nabla[h+\phi+g_* z+\vec{v}^2/2] = 0; \tag{4.21}$$

hence for $r \geq r_o$,

$$h = gr_o\{1 - [\alpha\left(\frac{r}{r_o}\right) + (1-\alpha)\left(\frac{r_o}{r}\right)^2] \cos\theta\}$$

(4.22)

$$- \frac{1}{2}V^2 \{1 + \left(\frac{r_o}{r}\right)^3 (1-3\cos^2\theta) + \frac{1}{4}\left(\frac{r_o}{r}\right)^6 (1+3\cos^2\theta)\},$$

where $\alpha = g_*/g$ and an arbitrary constant has been chosen so that $h(r_o,0,0) = 0$. On $r = r_o$ we have

(4.23) $$h = gr_o[1 - \cos\theta - \frac{9}{8}f^2(1-\cos^2\theta)].$$

For ρ = const, $h \propto p$. Alternatively, the choice p/ρ^γ = const, which satisfies (4.14), gives $h \propto \rho^{(\gamma-1)/\gamma}$. In either case we would like to have $h = 0$ on $r = r_o$ since $p = 0$ inside the bubble (and E is continuous across the interface). But (4.23) shows this to be impossible with the present approximate treatment. However, we do have the freedom to choose f^2 to match the pressure boundary conditions as well as possible. In fluidization theory the procedure used by Davies and Taylor (1950) for ordinary bubbles is usually adopted. In the present instance this comes down to setting $\partial^2h/\partial\theta^2 = 0$ at $r = r_o$, $\theta = 0$, whence $f^2 = 4/9$ (see also Batchelor 1967). The argument for this is that h and $\partial h/\partial\theta$ are already zero at $r = r_o$, $\theta = 0$, and we would like to extend the region where h is very small as far as possible. Let us adopt this choice. (Any other choice of this type would also give a value of f of order unity. For example we might minimize the integral of h^2 over the surface $r = r_o$.) Thus we have an estimate of the speed of rise of the bubble which can also be used to see the magnitude of the distortion of the spherical hole by the dipole force. For the latter purpose we may simply compute the surface on which $h = 0$ with

(4.24) $$V = \frac{2}{3}(gr_o)^{1/2}.$$

For $\alpha = 0$ this surface is the dotted line indicated in the Figure above. The distortion of the hole is caused by the need to balance the fluid-dynamical pressure $\vec{v}^2/2$ and it represents a problem which is also encountered in the theory of ordinary gas bubbles in liquids (Moore 1959). As long as appreciable speeds occur next to the bubble this difficulty arises. In an actual fluidization bubble the problem is resolved by the formation of an indentation at the rear of the bubble. The indentation fills with particles which effectively move with the bubble. This feature has to be built into the theory in a self-consistent way.

With the present estimates a second problem arises, namely that for $r \gg r_o$ and $\theta > 0$ we encounter a region of negative h when $\alpha > 0$. This difficulty does not arise in fluidization theory since that subject is confined to $\alpha = 0$. We therefore have no experimental guide to the meaning of this result. There are some speculations that might be offered here but perhaps the message is simply that bubbles only occur when α is very close to zero.

Now it is evident that the foregoing discussion does not really provide an acceptable theory. It might be different if photon bubbles were an observed phenomenon that we were trying to understand qualitatively. But the real question is whether photon bubbles actually exist and the answer will almost surely have to be given experimentally. In spite of these worries, I would like to close this theoretical discussion of bubbles with one further qualitative remark about what may happen at very large optical depths.

The total radiant energy density includes the usual energy density plus the pressure, hence it is $\sim \frac{4}{3} E$. The energy flux divided by this energy density gives a speed to be associated with the radiant fluid.

$$v_R = \frac{F}{\frac{4}{3}E} .$$

When δ exceeds some critical value $\sim\frac{3}{4}$, V exceeds v_R, and the bubble is moving faster than the radiative fluid. In that case, the radiation does not adjust quickly to the matter. Rather, we may expect the radiation associated with the bubble to be swept along with the bubble, much as in the corresponding case of fluidization where one sees a trapped cloud of fluid circulating in and around the bubble. When this occurs, I expect that photoconvective transport should become very efficient. The optical depth at which this occurs is given approximately by

$$\tau \sim 10^{24/9} \left(\frac{T_{eff}}{10^5}\right)^{4/9} .$$

V. CONCLUSION

The main questions considered here have to do with the nature of photoconvection and the conclusion which is tentatively adopted is that the two-fluid nature of the process may make for some qualitative differences from basic Boussinesq convection. I have tried to sketch how photon bubbles may behave in analogy with fluidization bubbles. The analysis is sufficiently simple that one can easily see what is going on, but there is one point about the results that I want to emphasize. The bubble is not simply held open by an excess of radiation pressure inside it. The radiative force is vital to the process and this is proportional to the flux. The figure in ¶IV is helpful in seeing how this works: flux converges onto the bubble from below and diverges upward from the bubble. This forces the fluid flowing by the bubble to go around it which in turn causes the hole which produced the flux pattern in the first place. This seems to be a dynamically consistent situation, whether or not the equations have been completely solved. Whether the thermodynamics of radiation interacting with matter (and which has not been discussed at all properly here) can spoil the picture, seems difficult to decide, and that is, to me, the biggest question to be faced at present. But if we put doubts aside for now we may imagine some astrophysically interesting aspects of photon bubbles.

The generation of large amplitude, complicated velocity fields in hot stellar atmosphere is one of these. Another is suggested by the rapid separation of particles of differing properties in bubbling, fluidized beds. In this process, called elutriation, particles of relatively large drag are carried up through the bed by the bubbles. Similarly we can imagine that particles with large scattering cross section may be carried swiftly through stellar material by photon bubbles. Moreover, there are some interesting consequences involved when bubbles collapse near a stellar surface. The heating may cause hot bursts of radiation (as J. Pringle has suggested) or radiation of acoustic and shock noise. Also, non-spherical collapse could squirt matter off the stellar surface at high speed, as a preliminary computation by J. Theys confirms.

But these are presently speculative topics and more immediate aims should also command attention in this subject.

We need a more complete stability theory, a study of finite-amplitude stability, and some numerical simulation. In this respect, we should be aware of related work on high density plasmas (Estabrook, Valeo, and Kruer, 1975), though much of what I have said here leaves out plasma kinetic effects and assumes relatively low density, such as is encountered in stars.

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