Canad. Math. Bull. Vol. **42** (2), 1999 pp. 155–161

Non-Cohen-Macaulay Vector Invariants and a Noether Bound for a Gorenstein Ring of Invariants

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Abstract. This paper contains two essentially independent results in the invariant theory of finite groups. First we prove that, for any faithful representation of a non-trivial p-group over a field of characteristic p, the ring of vector invariants of m copies of that representation is not Cohen-Macaulay for $m \geq 3$. In the second section of the paper we use Poincaré series methods to produce upper bounds for the degrees of the generators for the ring of invariants as long as that ring is Gorenstein. We prove that, for a finite non-trivial group G and a faithful representation of dimension n with $n > 1$, if the ring of invariants is Gorenstein then the ring is generated in degrees less than or equal to $n(|G| - 1)$. If the ring of invariants is a hypersurface, the upper bound can be improved to $|G|$.

Introduction

Let V be a vector space of dimension n over a field $\mathbf{k}$, let V^* denote the dual of V and let $\mathbf{k}[V]$ denote the symmetric algebra of V^*. By choosing a basis, $\{x_1, \ldots, x_n\}$, for V^*, we can identify $\mathbf{k}[V]$ with the polynomial algebra $\mathbf{k}[x_1, \ldots, x_n]$. Let G be a finite subgroup of $\mathrm{GL}(V)$. The elements of G act as degree preserving algebra automorphisms on $\mathbf{k}[V]$. We denote the subring of G-invariant polynomials by $\mathbf{k}[V]^G$. A *homogeneous system of parameters* for $\mathbf{k}[V]^G$ is a collection of homogeneous elements, $\{a_1, \ldots, a_n\}$, of $\mathbf{k}[V]^G$ such that $\mathbf{k}[V]^G$ is a finitely generated $\mathbf{k}[a_1, \ldots, a_n]$-module. By the Noether normalization theorem $\mathbf{k}[V]^G$ contains a homogeneous system of parameters (see, for example, [5, Theorem 5.3.3] or [1, Theorem 2.2.7]). $\mathbf{k}[V]^G$ is *Cohen-Macaulay* if for every homogeneous system of parameters $\{a_1, \ldots, a_n\}$, $\mathbf{k}[V]^G$ is a free $\mathbf{k}[a_1, \ldots, a_n]$-module. If the characteristic of $\mathbf{k}$ does not divide the order of G, then $\mathbf{k}[V]^G$ is always Cohen-Macaulay ([3], see [2, Section 6.4]). However, when the order of G divides the characteristic of $\mathbf{k}$, $\mathbf{k}[V]^G$ often fails to be Cohen-Macaulay. Let mV denote the faithful representation of G formed by taking the direct sum of m copies of V. $\mathbf{k}[mV]^G$ is known as the ring of vector invariants (see [9]). In the first section of the paper we prove that if the characteristic of $\mathbf{k}$ is p and V is a faithful representation of a non-trivial p-group P, then $\mathbf{k}[mV]^P$ is not Cohen-Macaulay when $m \geq 3$.

When G is finite, $\mathbf{k}[V]^G$ is finitely generated. In [4], Noether proved that if $\mathbf{k}$ has characteristic zero then $\mathbf{k}[V]^G$ is generated by elements in degrees less than or equal to $|G|$. This is not true when the characteristic of $\mathbf{k}$ divides $|G|$. Any two minimal homogeneous generating sets for $\mathbf{k}[V]^G$ have the same number of elements of each degree. The maximum such degree is called the *Noether number* of the representation. We will refer to an upper

Received by the editors July 23, 1997; revised February 4, 1998.
This research is supported in part by the NSERC of Canada.
AMS subject classification: 13A50.

https://doi.org/10.4153/CMB-1999-018-4 Published online by Cambridge University Press

bound on the Noether number as a *Noether bound*. If the characteristic of $\mathbf{k}$ does not divide $|G|$ then, as long as $n > 1$ and G is non-trivial, $n(|G| - 1)$ is a Noether bound for the representation (see [6, Corollary 2.5]).

A Cohen-Macaulay ring which is isomorphic to its own canonical module is called a *Gorenstein* ring. If $\mathbf{k}[V]^G$ is a Cohen-Macaulay domain then $\mathbf{k}[V]^G$ is Gorenstein if and only if the Poincaré series satisfies the duality condition

$$P\big(\mathbf{k}[V]^G, 1/t\big) = (-1)^n t^m P\big(\mathbf{k}[V]^G, t\big)$$

for some integer m (see [8, Theorem 8.1]). In the second section of the paper we use Poincaré series methods to find a Noether bound when $\mathbf{k}[V]^G$ is Gorenstein. We show that if G is non-trivial and $n > 1$ then $\mathbf{k}[V]^G$ is generated in degrees less than or equal to $n(|G| - 1)$. If p is the characteristic of $\mathbf{k}$, G is a p-group and $\mathbf{k}[V]^G$ is Cohen-Macaulay, then $\mathbf{k}[V]^G$ is Gorenstein so this bound applies. If $\mathbf{k}[V]^G$ is a hypersurface then the Noether bound can be improved to $|G|$.

1 Vector Invariants

In this section we assume that the characteristic of $\mathbf{k}$ is p and that P is a non-trivial p-subgroup of $\mathrm{GL}(V)$. We use V^* to denote the dual to V and $(V^*)^P$ to denote the subspace of V^* fixed by P. We remind the reader that every linear action of a p-group over a field of characteristic p has a non-zero fixed point. Choose $z \in V^*$ so that z represents a non-zero element in $\big(V^*/(V^*)^P\big)^P$. There is an $\mathbf{F}_p$-subspace of $(V^*)^P$, say W, such that the P-orbit of z is the set $z + W$. Choose y to be a non-zero element in W and let U be an $\mathbf{F}_p$-subspace of W which is complementary to $\mathbf{F}_p y$. For any $g \in P$, $g(z) = z + \beta_g y + u_g$ for some $\beta_g \in \mathbf{F}_p$ and $u_g \in U$. For any $t \in V^*$ define

$$N(t; U) = \prod_{u \in U} t - u.$$

In particular, let $a = N(y; U)$ and $b = N(z; U)$.

Lemma 1.1 *For any $g \in P$, $g(b) = b + \beta_g a$.*

Proof Let $x_1, \dots, x_r$ be a basis for U. Then

$$b = N(z; U) = \sum_{i=0}^{r} d_{i,r} z^{p^{r-i}}$$

where $d_{i,r}$ is the i-th Dickson invariant over $\mathbf{F}_p$ in the variables $x_1, \dots, x_r$ (see [5, Section 8.1]). Since g fixes U,

$$\begin{aligned} g(b) &= \sum_{i=0}^{r} d_{i,r}\big(g(z)\big)^{p^{r-i}} \\ &= \sum_{i=0}^{r} d_{i,r}(z + \beta_g y + u_g)^{p^{r-i}}. \end{aligned}$$

The p-th power map is additive and therefore

$$g(b) = N(z;U) + N(\beta_g y;U) + N(u_g;U).$$

However, $u_g \in U$ and therefore $N(u_g;U) = 0$. Furthermore, since $\beta_g \in \mathbf{F}_p$, we see that $(\beta_g)^{p^{r-i}} = \beta_g$ and $N(\beta_g y;U) = \beta_g N(y;U)$. Hence $g(b) = b + \beta_g a$. ∎

Since there is a $g \in P$ for which $\beta_g = 1$, it follows from the preceding lemma that b is not in $\mathbf{k}[V]^P$.

Let mV denote the direct sum of m copies of V, with $m \geq 3$, and consider the diagonal action of P on mV. Choose $z \in V^*$ and use z_i, for $i \in \{1,2,3\}$, to denote the element in the i-th copy of V^* corresponding to z. Define y_i, a_i, b_i and U_i similarly. Further define $u_{ij} = a_i b_j - a_j b_i$. Using Lemma 1.1 and the fact that $a \in \mathbf{k}[V]^P$, observe that, for any $g \in P$, $g(u_{ij}) = a_i(b_j + \beta_g a_j) - a_j(b_i + \beta_g a_i) = u_{ij}$. Also observe that, by direct computation, $a_1u_{23} + a_2u_{31} + a_3u_{12} = 0$.

Theorem 1.2 *$\mathbf{k}[mV]^P$ is not Cohen-Macaulay for $m \geq 3$.*

Proof By way of contradiction, assume that $\mathbf{k}[mV]^P$ is Cohen-Macaulay. Choose a homogeneous system of parameters for $\mathbf{k}[mV]^P$ containing $\{a_1, a_2, a_3\}$. Since $\mathbf{k}[mV]^P$ is Cohen-Macaulay, the homogeneous system of parameters is a regular sequence and therefore (a_1, a_2, a_3) is a regular sequence. Since $a_1u_{23} + a_2u_{31} + a_3u_{12} = 0$, u_{12} is in the $\mathbf{k}[mV]^P$-ideal generated by a_1 and a_2. Therefore there exist homogeneous r and s in $\mathbf{k}[mV]^P$ such that $u_{12} = ra_1 + sa_2$. Referring to the definition of u_{12} gives $ra_1 + sa_2 = a_1b_2 - a_2b_1$. Thus $(s + b_1)a_2 = (b_2 - r)a_1$. Since a_1 and a_2 have no common factors over $\mathbf{k}[V]$, a_1 divides $s + b_1$. Furthermore, a_1 and $s + b_1$ have the same degree. Therefore, for some non-zero $\alpha \in \mathbf{k}$, $a_1 = \alpha(s + b_1)$. Thus $b_1 = (\alpha)^{-1}a_1 - s \in \mathbf{k}[mV]^P$. However, from Lemma 1.1, we see that b_1 is not invariant, giving a contradiction. ∎

2 Noether Bounds for Gorenstein Invariants

We begin with a brief discussion of Poincaré series. We will consider finitely generated graded algebras, A, of Krull dimension n, whose Poincaré series have the form

$$P(A,t) = \frac{a(t)}{\prod_{i=1}^{n} 1 - t^{d_i}}$$

for some polynomial $a(t)$. $P(A,t)$ has a pole of order n at $t = 1$ so $a(1)$ is not zero. The series may be expanded about $t = 1$. The first coefficient in this expansion is, by definition, the degree of A and the second coefficient is denoted by $\psi(A)$. In other words

$$P(A,t) = \frac{\deg(A)}{(1-t)^n} + \frac{\psi(A)}{(1-t)^{n-1}} + O\left(\frac{1}{(1-t)^{n-2}}\right)$$

where $O\left(\frac{1}{(1-t)^{n-2}}\right)$ is some rational function whose Laurent series about $t = 1$ starts with $c \cdot \frac{1}{(1-t)^{n-2}}$ for some constant c. By substituting $t = 1$ into $(1-t)^n P(A,t)$ we see that

$$\deg(A) = \frac{a(1)}{\prod_{i=1}^{n} d_i}.$$

https://doi.org/10.4153/CMB-1999-018-4 Published online by Cambridge University Press

We will use $a'(t)$ to denote the first derivative of $a(t)$.

Lemma 2.1

$$\psi(A) = \frac{\deg(A)}{2}\left(\left(\sum_{i=1}^{n} d_i\right) - n - \frac{2a'(1)}{a(1)}\right).$$

Proof See [7, Lemma 2.5.9]. ∎

Now suppose that $A = \mathbf{k}[V]^G$ for a finite group G and that n is the dimension of V. When $n = 1$, G is a non-modular reflection group and hence $\mathbf{k}[V]^G$ is a polynomial algebra. Since this case is well understood we will assume, for the rest of this section, that $n > 1$. Recall that $\deg(\mathbf{k}[V]^G) = 1/|G|$ and, if $\mathbf{k}$ has characteristic zero, then $2|G|\psi(\mathbf{k}[V]^G)$ is the number of (pseudo) reflections in G (see, for example, [1, Chapter 2]). Regardless of the characteristic of $\mathbf{k}$ we have the following.

Proposition 2.2 $\psi(\mathbf{k}[V]^G) \geq 0$.

Proof See [1, Section 3.12]. ∎

Suppose that $\mathbf{k}[V]^G$ is Gorenstein and that $d_1, \ldots d_n$ are the degrees of a homogeneous system of parameters. Then

$$P(\mathbf{k}[V]^G, t) = \frac{a(t)}{\prod_{i=1}^{n} 1 - t^{d_i}}$$

with $a(t) = a_0 + a_1 t + \cdots + a_s t^s$ and $a_i = a_{s-i}$. Thus $a'(1) = s \cdot a(1)/2$. Using Lemma 2.1 and Proposition 2.2 we get the following.

Corollary 2.3 *If $\mathbf{k}[V]^G$ is Gorenstein then*

$$\psi\big(\mathbf{k}[V]^G\big) = \frac{1}{2|G|}\left(\left(\sum_{i=1}^{n} d_i\right) - n - s\right)$$

and

$$\left(\sum_{i=1}^{n} d_i\right) - n \geq s.$$

We wish to use Dade's construction (see [8, p. 483]) to produce a system of parameters for $\mathbf{k}[V]^G$ with degrees less than or equal to the order of G. However in order to use Dade's construction it may be necessary to extend the field.

Lemma 2.4 *If $\tilde{\mathbf{k}}$ is a field extension of $\mathbf{k}$ then the Noether number of $\tilde{\mathbf{k}} \otimes_{\mathbf{k}} V$ is the same as the Noether number of V.*

Proof Let $\tilde{V}$ denote $\tilde{\mathbf{k}} \otimes_{\mathbf{k}} V$. It is well known that $\tilde{\mathbf{k}}[\tilde{V}]^G = \tilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{k}[V]^G$. For any connected graded algebra, A, let A_+ denote the set of homogeneous elements of A with positive degree. The degrees of a minimal generating set for A correspond to the degrees of a vector space

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basis for the graded algebra $A/(A_+)^2$. Therefore, to prove the lemma, it is sufficient to prove that $\tilde{\mathbf{k}} \otimes_{\mathbf{k}} (\mathbf{k}[V]^G_+)^2 = (\tilde{\mathbf{k}}[\tilde{V}]^G_+)^2$. Clearly $\tilde{\mathbf{k}} \otimes_{\mathbf{k}} (\mathbf{k}[V]^G_+)^2$ is a subset of $(\tilde{\mathbf{k}}[\tilde{V}]^G_+)^2$. To see the reverse inclusion start with $a \cdot b \in (\tilde{\mathbf{k}}[\tilde{V}]^G_+)^2$. Let S be a basis for $\tilde{\mathbf{k}}$ over $\mathbf{k}$. Since $\tilde{\mathbf{k}}[\tilde{V}]^G = \tilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{k}[V]^G$ we may write $a = \sum_{\lambda \in S} \lambda \otimes a_\lambda$ for some choice of $a_\lambda \in \mathbf{k}[V]^G_+$ and $b = \sum_{\gamma \in S} \gamma \otimes b_\gamma$ for $b_\gamma \in \mathbf{k}[V]^G_+$. Therefore

$$a \cdot b = \sum_{\gamma \in S} \sum_{\lambda \in S} (\lambda\gamma) \otimes (a_\lambda b_\gamma) \in \tilde{\mathbf{k}} \otimes_{\mathbf{k}} \left(\mathbf{k}[V]^G_+\right)^2. \qquad \blacksquare$$

Theorem 2.5 *If $\mathbf{k}[V]^G$ is Gorenstein and $n > 1$, then $\mathbf{k}[V]^G$ is generated in degrees less than or equal to $n(|G| - 1)$.*

Proof By Lemma 2.4, extending the field does not change the Noether number, so we may assume that $\mathbf{k}$ is algebraically closed. We use Dade's construction to produce a homogeneous system of parameters for $\mathbf{k}[V]^G$ with degrees less than or equal to $|G|$ (see [8, p. 483]). Let $d_1, \ldots, d_n$ be the degrees of such a homogeneous system of parameters. Since, for all i, $d_i \leq |G|$, applying Corollary 2.3 gives

$$n(|G| - 1) \geq \left(\sum_{i=1}^{n} d_i\right) - n \geq s.$$

Furthermore, any additional generators for $\mathbf{k}[V]^G$ occur in degrees less than or equal to s and, as long as $n > 1$ and $|G| > 1$, $n(|G| - 1) > |G|$. $\blacksquare$

Corollary 2.6 *Suppose that the characteristic of $\mathbf{k}$ is p, G is a p-group and $\mathbf{k}[V]^G$ is Cohen-Macaulay. Then $\mathbf{k}[V]^G$ is generated in degrees less than or equal to $n(|G| - 1)$.*

Proof Since G is a p-group, $\mathbf{k}[V]^G$ is a unique factorization domain (see, for example, [5, Proposition 1.5.7]). A Cohen-Macaulay unique factorization domain is Gorenstein (see [2, Corollary 3.3.19]). Now apply Theorem 2.5. $\blacksquare$

Now suppose that $\mathbf{k}[V]^G$ is a hypersurface. In this case $\mathbf{k}[V]^G$ has $n+1$ generators and a single relation. Suppose that the degrees of the generators are $d_1, \ldots, d_{n+1}$ and the degree of the relation is e. Thus

$$P\left(\mathbf{k}[V]^G, t\right) = \frac{1 - t^e}{\prod_{i=1}^{n+1} 1 - t^{d_i}}.$$

If we further suppose that the first n elements of our generating set form a homogeneous system of parameters then

$$P\left(\mathbf{k}[V]^G, t\right) = \frac{a(t)}{\prod_{i=1}^{n} 1 - t^{d_i}}$$

with $a(t) = 1 + t^{d_{n+1}} + \cdots + t^{\ell d_{n+1}}$. Thus $a(1) = \ell + 1$. However $a(1) = (\prod_{i=1}^{n} d_i)/|G|$ and $s = \ell d_{n+1}$. Hence

$$s = \frac{1}{|G|}\left(\prod_{i=1}^{n+1} d_i\right) - d_{n+1}.$$

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Corollary 2.7 *If $\mathbf{k}[V]^G$ is a hypersurface with a minimal generating set which contains a homogeneous system of parameters then*

$$\psi\big(\mathbf{k}[V]^G\big) = \frac{1}{2|G|}\bigg(\Big(\sum_{i=1}^{n+1} d_i\Big) - n - \frac{1}{|G|}\Big(\prod_{i=1}^{n+1} d_i\Big)\bigg)$$

and

$$|G|\bigg(\Big(\sum_{i=1}^{n+1} d_i\Big) - n\bigg) \geq \prod_{i=1}^{n+1} d_i.$$

Theorem 2.8 *If $\mathbf{k}[V]^G$ is a hypersurface then $\mathbf{k}[V]^G$ is generated in degrees less than or equal to $|G|$.*

Proof Using Lemma 2.4 we see that extending the field does not change the Noether number so we may assume that $\mathbf{k}$ is algebraically closed. Therefore by Dade's construction [8, p. 483] there exists a homogeneous system of parameters for $\mathbf{k}[V]^G$ with degrees less than or equal to $|G|$. The elements of this homogeneous system of parameters are polynomials in the generators. If each of the $n+1$ generators appears in the expression for some element of the homogeneous system of parameters then all of the generators have degree less than or equal to $|G|$ as required. If only n of the generators appear then these n generators form a homogeneous system of parameters. Without loss of generality we may take the degrees of this homogeneous system of parameters to be $d_1, \ldots, d_n$. Furthermore, we may assume that $d_1 \leq \cdots \leq d_n \leq |G|$. Suppose, by way of contradiction, that $d_{n+1} > |G|$. Referring to Corollary 2.7 we have

$$\Big(\sum_{i=1}^{n+1} d_i\Big) - n \geq \frac{1}{|G|}\prod_{i=1}^{n+1} d_i.$$

Since generators of degree one make no contribution to either side of this inequality, we may assume that $d_1 > 1$. Subtracting d_{n+1} from both sides of the inequality gives

$$\Big(\sum_{i=1}^{n} d_i\Big) - n \geq \frac{1}{|G|}\Big(\prod_{i=1}^{n+1} d_i\Big) - d_{n+1} = \frac{d_{n+1}}{|G|}\bigg(\Big(\prod_{i=1}^{n} d_i\Big) - |G|\bigg).$$

Since $d_{n+1} > |G|$,

$$\Big(\sum_{i=1}^{n} d_i\Big) - n > \Big(\prod_{i=1}^{n} d_i\Big) - |G|. \tag{2.1}$$

Note that

$$\deg\big(\mathbf{k}[V]^G\big) = \frac{a(1)}{\prod_{i=1}^{n} d_i} = \frac{1}{|G|}.$$

If $\mathbf{k}[V]^G$ is a polynomial algebra then the conclusion holds. Thus we will assume that $\mathbf{k}[V]^G$ is not a polynomial algebra. Hence $a(1) = \ell + 1 \geq 2$ and therefore

$$\prod_{i=1}^{n} d_i = |G|a(1) \geq 2|G|. \tag{2.2}$$

Combining inequality (2.1) and inequality (2.2) gives

$$\Big(\sum_{i=1}^{n} d_i\Big) - n > \frac{1}{2}\prod_{i=1}^{n} d_i.$$

Subtract d_1 from both sides to get

$$\Big(\sum_{i=2}^{n} d_i\Big) - n > \frac{1}{2}\Big(\prod_{i=1}^{n} d_i\Big) - d_1 = \frac{d_1}{2}\bigg(\Big(\prod_{i=2}^{n} d_i\Big) - 2\bigg).$$

Since $d_1 \geq 2$,

$$\Big(\sum_{i=2}^{n} d_i\Big) - n > \Big(\prod_{i=2}^{n} d_i\Big) - 2.$$

However $n > 1$ and thus

$$\sum_{i=2}^{n} d_i > \prod_{i=2}^{n} d_i.$$

Using the assumption that for $i \leq n$, $d_n \geq d_i \geq 2$, we get

$$(n-1)d_n \geq \sum_{i=2}^{n} d_i > \prod_{i=2}^{n} d_i \geq 2^{n-2} d_n.$$

Thus $n - 1 > 2^{n-2}$, which is a contradiction. ■

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