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DELAYED STATE FEEDBACK H_∞ CONTROL FOR PERIODIC-REVIEW INVENTORY SYSTEMS

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Abstract

We investigate delayed state feedback control of a periodic-review inventory management system with perishable goods. The stock under consideration is replenished from multiple supply sources. By using delayed states as well as current states of the inventory system, a delayed feedback H_∞ control strategy is developed to mitigate bullwhip effects of the system. Some conditions on the existence of the delayed feedback H_∞ controller are derived. It is found through simulation results that the proposed delayed H_∞ control scheme is capable of improving the performance of the inventory management significantly. In addition, the delayed controller is better than the traditional delay-free H_∞ controller.

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Keywords and phrases: discrete-time systems, inventory systems, supply chains, time delay, delayed feedback control.

1. Introduction

It is known that the bullwhip effect is one of the key factors to affect performance of inventory management systems [9]. To investigate various causes of bullwhip effects and/or reduce the effects on the performance of inventory systems, in the past decades a large number of schemes have been reported; see [1, 2, 4, 8] and the references therein. Specifically, for one kind of inventory system, a linear–quadratic optimal control strategy was developed to mitigate the bullwhip effect [7]. Such a scheme was then successfully used to improve the dynamical performance of a production-inventory system with perishable goods and multiple supply options [5]. By modelling a production-inventory system with deteriorating stock and an unreliable replenishment process as an uncertain, time-varying discrete-time system with multiple time delays, an efficient stock replenishment policy was addressed to restrain demand variations and the bullwhip effect [6].

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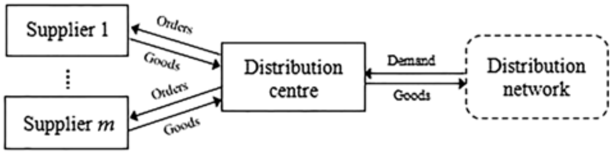

FIGURE 1. An inventory system.

It is worth pointing out that the aforementioned strategies [5–7] are designed by using current state or output signals of the inventory systems. Note that during the implementation process of the inventory management, the history signals, such as history order quantities and history stock levels, of the inventory systems are generally important to make effective order policies to improve performance of the systems. In [5–7], note that only the current stock levels and history order quantities, that is, delayed order quantities, are utilized, while the delayed stock levels of the system are not considered. Therefore, a natural question arises "can the performance of an inventory system be further improved by using both current and proper delayed stock levels of the system?". Properly introducing a time delay is capable of stabilizing some practical systems and improving control performance of them [10]. Inspired by Zhang et al. [10], for a period-review inventory system with perishable goods, we aim to design a supply policy by using both current and delayed states of the system. By choosing a proper time delay in the process of order compensation of the inventory system, a delayed feedback H_∞ control scheme is presented to mitigate the bullwhip effect. A delayed feedback H_∞ controller can be obtained by numerically solving some matrix inequalities; a numerical example is given to illustrate the effectiveness of the proposed method. The performances of the system with the delayed H_∞ control scheme and the traditional delay-free H_∞ control scheme are also investigated.

Throughout this paper, the superscript "T" means the transpose of a matrix; $P > 0$ means that the matrix P is a real symmetric and positive-definite matrix; I denotes the identity matrix of appropriate dimensions. For simplicity, the symbol $*$ is used to denote the entries induced by symmetry.

2. Problem formulation

Consider a perishable inventory system with multiple supply sources shown in Figure 1, where a distribution centre connects suppliers and a distribution network. It is assumed that reviews of stock and replenishment of orders are issued at regular intervals kT, where T is the review period and $k = 0, 1, 2, \ldots$. To simplify the notation, kT is denoted by k throughout the paper, and referred to as the "review period". Each order placed at the pth supplier is realized with the lead time $L_p = n_pT$ in each review period k, where n_p is a positive integer, $p = 1, 2, \ldots, m$. The goods decay at a deteriorating rate σ satisfying $0 \leq \sigma \leq 1$ while kept in the on-hand stock. In the specific case, $\sigma \equiv 0$ means that the integrity of the stock is maintained. In what follows, the on-hand stock level, the customer demand placed at the distribution centre,

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and the order quantity determined by suppliers are denoted by $y(k)$, $h(k)$, and $u(k)$, respectively. Suppose that the first order is placed at $k = 0$ and the first order arrives at the distribution centre in period n_p due to the lead time delay, which means that

$$y(k) = 0, \quad k = 0, 1, \ldots, n_p.$$

Then the system dynamics can be described as [5]

$$y(k+1) = (1-\sigma)y(k) + \sum_{p=1}^{m} \xi_p u(k-n_p) - h(k)$$

and the stock level can be written as

$$y(k) = \sum_{p=1}^{m} \lambda_p \sum_{j=0}^{k-n_p-1} (1-\sigma)^{k-n_p-1-j} u(j) - \sum_{j=0}^{k-1} \rho^{k-1-j} h(j).$$

In the total order quantity of the suppliers, rearrange the share order quantity of the suppliers with equal lead time and define $a_j = \Sigma_{p:L_p=jT} \lambda_p$ with $\sum_{j=1}^{n_m} a_j = 1$. Let $n = \max\{n_p \mid p = 1, 2, \ldots, m\} + 1$ and

$$\begin{cases} x_1(k) = y(k), \\ x_j(k) = u(k-n-1+j), \quad j = 2, 3, \ldots, n \end{cases} \tag{2.1}$$

and define $x(k) = [x_1(k) \quad x_2(k) \quad \ldots \quad x_n(k)]^T$; then

$$\begin{cases} x_1(k+1) = (1-\sigma)x_1(k) - h(k) + \sum_{i=2}^{n} a_{n-i+1} x_i(k), \\ x_j(k+1) = x_{j+1}(k), \quad j = 2, 3, \ldots, n-1, \\ x_n(k+1) = u(k). \end{cases}$$

Further, a state-space model of the inventory system is given by [5]

$$x(k+1) = Ax(k) + Bu(k) + Vh(k) \tag{2.2}$$

and

$$y(k) = Cx(k), \tag{2.3}$$

where

$$\begin{cases} A = \begin{bmatrix} 1-\sigma & a_{n-1} & a_{n-2} & \ldots & a_1 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}, \\ V = \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \end{cases} \tag{2.4}$$

In this paper, the order quantity $u(k)$ is designed as

$$u(k) = K_1 x(k) + K_2 x(k-d), \tag{2.5}$$

where K_1 and K_2 are gain matrices and d is an integer to be determined.

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REMARK 2.1. In some existing order compensation schemes [5, 7], it follows from (2.1) that only current stock levels $y(k)$ and several delayed order quantity signals are used to determine $u(k)$, while, from (2.5), to design the control law $u(k)$, the current and delayed stock levels $y(k)$ and $y(k-d)$ and the delayed order quantity signals are utilized. Specifically, let $K_2 \equiv 0$; then the control law (2.5) reduces to a traditional state feedback control law.

Substituting (2.5) into (2.2) yields the closed-loop system

$$x(k+1) = (A + BK_1)x(k) + BK_2x(k-d) + Vh(k), \quad k = 0, 1, 2. \tag{2.6}$$

The initial state is supplemented as $x(k) = x_0, k = -d, -d+1, \ldots, 0$, where x_0 is the initial state of the system.

This paper intends to design a delayed feedback H_∞ control law (2.5) such that the system (2.6) is asymptotically stable and, under the zero initial condition, the H_∞ performance

$$\|y(k)\| \leq \gamma\|h(k)\| \tag{2.7}$$

of the system is guaranteed for nonzero $h(k)$ and a prescribed $\gamma > 0$.

3. Proposed inventory policy

To obtain the main results, the following lemmas are needed.

LEMMA 3.1 [3] (Schur complement). *For a given symmetric matrix*

$$\Upsilon = \Upsilon^T = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ * & \Upsilon_{22} \end{bmatrix},$$

where $\Upsilon_{11} \in \mathbb{R}^{r\times r}$, the following conditions are equivalent:

(1) $\Upsilon < 0$;
(2) $\Upsilon_{11} < 0,\ \Upsilon_{22} - \Upsilon_{12}^T\Upsilon_{11}^{-1}\Upsilon_{12} < 0$;
(3) $\Upsilon_{22} < 0,\ \Upsilon_{11} - \Upsilon_{12}\Upsilon_{22}^{-1}\Upsilon_{12}^T < 0$.

LEMMA 3.2 [11]. *For a constant matrix $R \in \mathbb{R}^{n\times n}$ with $R = R^T > 0$, integers r_1 and r_2 with $r_2 - r_1 > 0$, and a vector function $w : \{r_1, r_1+1, \ldots, r_2\} \longmapsto \mathbb{R}^n$, the inequality*

$$\sum_{j=r_1}^{r_2-1} \eta^T(j)R\eta(j) \geq \frac{1}{r_2 - r_1}v_1^TRv_1 + \frac{3(r_2 - r_1 - 1)}{(r_2 - r_1)(r_2 - r_1 + 1)}v_2^TRv_2,$$

holds, where $\eta(j) = w(j+1) - w(j)$ and

$$v_1 = w(r_2) - w(r_1), v_2 = w(r_2) + w(r_1) - \frac{2}{r_2 - r_1 - 1}\sum_{j=r_1+1}^{r_2-1} w(j).$$

A sufficient condition for the existence of the order policy $u(k)$ is provided as follows.

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PROPOSITION 3.3. *For a given scalar $d > 1$, the closed-loop system (2.6) is asymptotically stable and the H_∞ performance (2.7) is guaranteed for nonzero $h(k)$ and a prescribed $\gamma > 0$ if there exist $n \times n$ real matrices $P > 0$, $Q > 0$, and $R > 0$ and $1 \times n$ real matrices K_1 and K_2 such that*

$$\begin{bmatrix} \Lambda_{11} & d_2R & d_3R & 0 & \Lambda_{15} & \Lambda_{15} - I & C^T \\ * & \Lambda_{22} & d_3R & 0 & K_2^TB^T & K_2^TB^T & 0 \\ * & * & -2d_3R & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & V^T & V^T & 0 \\ * & * & * & * & -P^{-1} & 0 & 0 \\ * & * & * & * & * & -(d^2R)^{-1} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.1}$$

where

$$d_1 = \frac{4d-2}{d+1}, \quad d_2 = \frac{4-2d}{d+1}, \quad d_3 = \frac{6(d-1)}{d+1}, \quad d_4 = \frac{4d-2}{d+1}$$

and $\Lambda_{11} = -d_1R - P + Q$, $\Lambda_{15} = A^T + K_1^TB^T$, and $\Lambda_{22} = -d_4R - Q$.

PROOF. We aim to construct a Lyapunov–Krasovskii functional candidate [11] as

$$V(k) = x^T(k)Px(k) + \sum_{j=k-d}^{k-1} x^T(j)Qx(j) + d\sum_{j=-d}^{-1}\sum_{i=k+j}^{k-1} \eta^T(i)R\eta(i),$$

where $P > 0$, $Q > 0$, and $R > 0$. Calculating the forward difference of $V(k)$ yields

$$\begin{aligned} \Delta V(k) = {} & x^T(k)(Q-P)x(k) - x^T(k-d)Qx(k-d) + d^2\eta^T(k)R\eta(k) \\ & + x^T(k+1)Px(k+1) - d\sum_{j=k-d}^{k-1}\eta^T(j)R\eta(j), \end{aligned} \tag{3.2}$$

where $\eta(k) = x(k+1) - x(k)$. By Lemma 3.2 with $r_1 = k-d$ and $r_2 = k$,

$$-d\sum_{j=k-d}^{k-1}\eta^T(j)R\eta(j) \le -v_1^TRv_1 - \frac{3(d-1)}{d+1}v_2^TRv_2, \tag{3.3}$$

where $v_1 = x(k) - x(k-d)$ and $v_2 = x(k) + x(k-d) - 2z(k)$ with $z(k) = \{1/(d-1)\} \sum_{j=k-d+1}^{-1} x(j)$.

To prove the asymptotic stability of the system (2.6), let $h(k) \equiv 0$ and denote $\alpha^T(k) = [x^T(k) \quad x^T(k-d) \quad z^T(k)]$. Then, from (2.6), (3.2), and (3.3),

$$\Delta V(k) \le \alpha^T(k)\{\Xi_1 + \Xi_2^TP\Xi_2 + \Xi_3^T(d^2R)\Xi_3\}\alpha(k),$$

where

$$\Xi_1 = \begin{bmatrix} \Lambda_{11} & d_2R & d_3R \\ * & \Lambda_{22} & d_3R \\ * & * & -2d_3R \end{bmatrix}, \quad \Xi_2^T = \begin{bmatrix} \Lambda_{15} \\ K_2^TB^T \\ 0 \end{bmatrix}, \quad \Xi_3^T = \begin{bmatrix} \Lambda_{15} - I \\ K_2^TB^T \\ 0 \end{bmatrix}.$$

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Note that if (3.1) holds, then, by Schur complement, we have $\Xi_1 + \Xi_2^T P \Xi_2 + \Xi_3^T (d^2 R) \Xi_3 < 0$, which means that the supply chain system (2.6) is asymptotically stable.

To prove that the H_∞ performance (2.7) is guaranteed under the zero initial condition, let $\beta^T(k) = [x^T(k) \quad x^T(k-d) \quad z^T(k) \quad h^T(k)]$ and

$$\Omega_1 = \begin{bmatrix} \Lambda_{11} & d_2 R & d_3 R & 0 \\ * & \Lambda_{22} & d_3 R & 0 \\ * & * & -2d_3 R & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}, \quad \Omega_2^T = \begin{bmatrix} \Lambda_{15} \\ K_2^T B^T \\ 0 \\ V^T \end{bmatrix}, \quad \Omega_3^T = \begin{bmatrix} \Lambda_{15} - I \\ K_2^T B^T \\ 0 \\ V^T \end{bmatrix}.$$

Then, from (2.3), (2.6), (3.2), and (3.3),

$$\Delta V(k) + y^T(k) y(k) - \gamma^2 h^T(k) h(k) = \beta^T(k)(\Omega_1 + \Omega_2^T P \Omega_2 + d^2 \Omega_3^T R \Omega_3 + \Omega_4^T \Omega_4)\beta(k), \tag{3.4}$$

where $\Omega_4 = [C \quad 0 \quad 0 \quad 0]$.

Note that by Schur complement in Lemma 3.1, (3.1) is equivalent to the inequality

$$\Omega_1 + \Omega_2^T P \Omega_2 + d^2 \Omega_3^T R \Omega_3 + \Omega_4^T \Omega_4 < 0.$$

Then, from (3.4),

$$y^T(k) y(k) - \gamma^2 h^T(k) h(k) \le 0,$$

which shows that the H_∞ performance index is satisfied. This completes the proof. □

To solve the gain matrices K_1 and K_2 in (2.5), we pre- and post-multiply the matrix on the left-hand side of the inequality (3.1) by $\mathrm{diag}\{P^{-1}, P^{-1}, P^{-1}, I, I, I, I\}$, and its transpose, respectively. Then, setting $\bar{P} = P^{-1}$, $\bar{Q} = P^{-1} Q P^{-1}$, $\bar{R} = P^{-1} R P^{-1}$, $\bar{K}_i = K_i P^{-1}, i = 1, 2$, and noting that $-\bar{P}\bar{R}^{-1}\bar{P} \le \nu^2 \bar{R} - 2\nu \bar{P}$ for $\nu > 0$,

$$\begin{bmatrix} \Theta_{11} & d_2 \bar{R} & d_3 \bar{R} & 0 & \Theta_{15} & \Theta_{15} - I & \bar{P} C^T \\ * & \Theta_{22} & d_3 \bar{R} & 0 & \bar{K}_2^T B^T & \bar{K}_2^T B^T & 0 \\ * & * & -2d_3 \bar{R} & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & V^T & V^T & 0 \\ * & * & * & * & -\bar{P} & 0 & 0 \\ * & * & * & * & * & -\dfrac{2\nu}{d^2}\bar{P} + \dfrac{\nu^2}{d^2}\bar{R} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{3.5}$$

where

$$\Theta_{11} = -d_1 \bar{R} - \bar{P} + \bar{Q}, \quad \Theta_{15} = \bar{P} A^T + \bar{K}_1^T B^T, \quad \Theta_{22} = -d_4 \bar{R} - \bar{Q}.$$

If the linear matrix inequality (3.5) is feasible, then the gain matrices of (2.5) are determined by $K_i = \bar{K}_i \bar{P}^{-1}, i = 1, 2$.

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4. Simulation results

To show the proposed order policy of the inventory system, a simulation example is given in this section. The main parameters of the system are mainly derived from [7], where five suppliers ($m = 5$) are utilized to replenish the stock level, the review period $T = 1$ (day) and $L_1 = 3T, L_2 = L_3 = 6T, L_4 = 9T$, and $L_5 = 11T$. In this case, $n = 12$. Let $\lambda_1 = \lambda_2 = 1/8$ and $\lambda_3 = \lambda_4 = \lambda_5 = 1/4$. Then, in (2.4), we have $a_i = 0, i = 1, 2, 4, 5, 7, 8, 10$, $a_3 = 1/8$, $a_6 = 3/8$, and $a_9 = a_{11} = 1/4$. Suppose that the inventory deteriorating rate σ is set as 0.05 and the initial state of the system is taken as a zero vector. The initial stock level is given as $y(0) = 300$.

For comparison purposes, a traditional H_∞ controller (HIC: H infinity controller) is designed first. Let $\gamma = 25$; then the gain matrix K of HIC can be computed as

$$K = [\Pi_1 \quad \Pi_2],$$

where

$$\Pi_1 = -[2.6741 \quad 0.7058 \quad 0.7304 \quad 1.4297 \quad 1.4300 \quad 1.2703],$$
$$\Pi_2 = -[2.2191 \quad 2.0174 \quad 1.5291 \quad 1.7266 \quad 1.1596 \quad 0.9516].$$

Let $d = 2$. Solving the linear matrix inequality (3.5) yields the gain matrices

$$K_1 = [\Pi_3 \quad \Pi_4] \quad \text{and} \quad K_2 = [\Pi_5 \quad \Pi_6],$$

where

$$\Pi_3 = -[0.1844 \quad 0.0487 \quad 0.0517 \quad 0.1019 \quad 0.1059 \quad 0.1428],$$
$$\Pi_4 = -[0.1645 \quad 0.1025 \quad 0.6097 \quad -0.5143 \quad 1.2384 \quad -1.0290],$$
$$\Pi_5 = -[0.0010 \quad 0.0003 \quad 0.0002 \quad 0.0005 \quad 0.0004 \quad 0.0003],$$
$$\Pi_6 = -[0.0011 \quad 0 \quad 0.0014 \quad -0.0005 \quad 0.0012 \quad 0.0002].$$

Thus, a delayed H_∞ controller (DHIC) is obtained.

Applying HIC and DHIC to the inventory system (2.6), the curves of order quantity and stock level of the system are depicted in Figures 2 and 3, where the customer demand is set as deterministic and random disturbance signals, respectively. The figures show that compared with HIC, the designed DHIC is more effective to control the inventory system. The order quantity and stock levels of the system under DHIC are smaller than the ones under HIC, which means that DHIC can reduce the bullwhip effect of the inventory system significantly.

5. Conclusions

A periodic review of supply chain management strategy for a perishable inventory system has been proposed based on a delayed feedback H_∞ control scheme. Some sufficient conditions of the existence of the order policy of the system have been derived. Simulation results have demonstrated that by using both current and delayed states of the inventory system, the proposed inventory management strategy is capable of significantly reducing the bullwhip effect and the risk of super-abundance or shortage of the stock.

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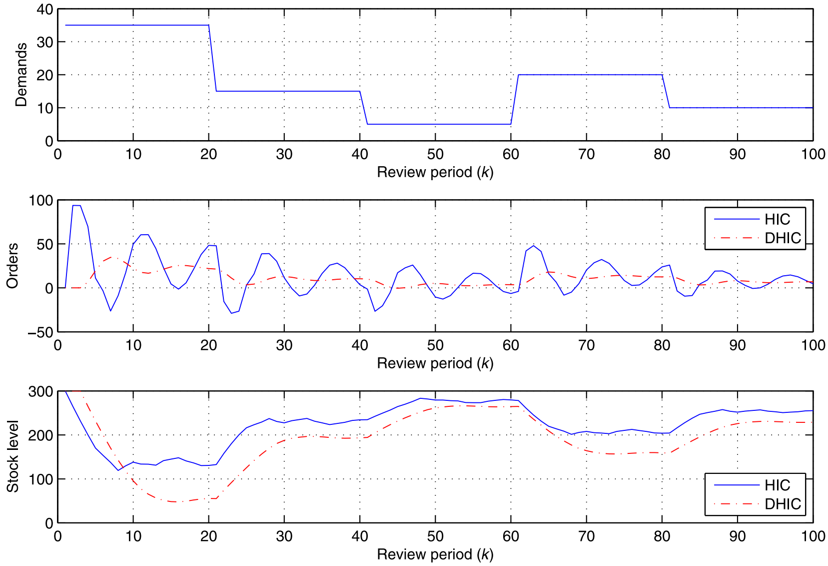

FIGURE 2. Order quantity and stock level of the system with deterministic customer demand.

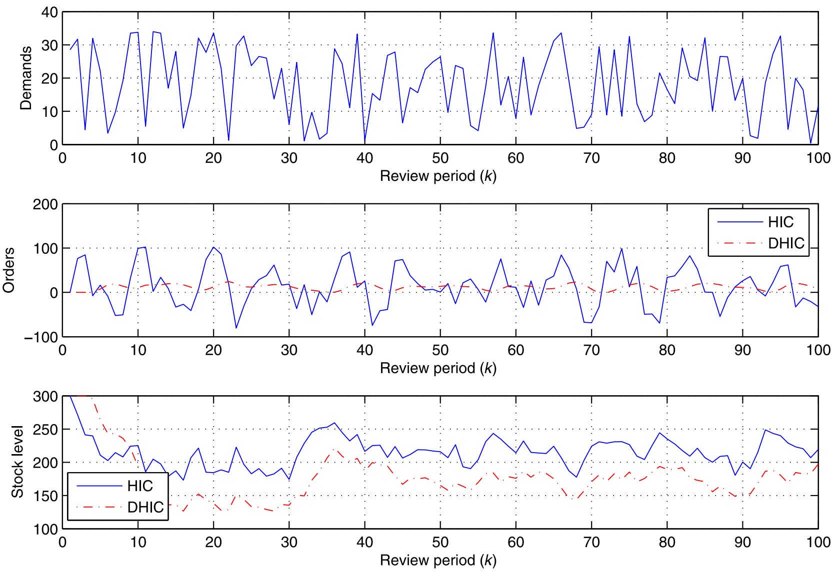

FIGURE 3. Order quantity and stock level of the system with random customer demand.

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