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BETTER BOUNDS IN CHEN'S INEQUALITIES FOR THE EULER CONSTANT

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Abstract

In this paper we improve the inequalities obtained by Chen in 2009 for the Euler–Mascheroni constant.

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1. Introduction

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n, \quad n \geq 1,$$

is convergent to a limit $\gamma = 0.5772\ldots$, now known as the Euler–Mascheroni constant. Many authors have obtained estimates for $\gamma_n - \gamma$, for example the following increasingly better bounds:

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2(n-1)}, \quad n \geq 2 \quad [8],$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, \quad n \geq 1 \quad [11],$$

$$\frac{1-\gamma}{n} < \gamma_n - \gamma < \frac{1}{2n}, \quad n \geq 1 \quad [2],$$

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \quad n \geq 1 \quad [6, 7],$$

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \quad n \geq 1 \quad [9],$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \quad n \geq 1 \quad [1, 9].$$

The convergence of the sequence γ_n to γ is very slow. In 1993, DeTemple [5] studied a modified sequence which converges faster, and he proved that

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad n \geq 1,$$

where $R_n = 1 + \frac{1}{2} + \cdots + 1/n - \ln(n + \frac{1}{2})$. In 2010, Chen [4] proved that for all integers $n \geq 1$,

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2},$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(\frac{3}{2})]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

In 1999, Vernescu [10] found a fast convergent sequence for γ, by replacing the last term of the harmonic sum. He proved that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \quad n \geq 2,$$

is strictly increasing and convergent to γ. Moreover,

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}, \quad n \geq 2.$$

Recently, Chen [3] obtained the following bounds for $\gamma - x_n$:

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n < \frac{1}{12(n+b)^2}, \quad n \geq 1,$$

with the best possible constants

$$a = \frac{1}{\sqrt{12\gamma - 6}} - 1 = 0.038859\ldots \quad \text{and} \quad b = 0.$$

In this paper we obtain a better estimation for the left-hand inequality, and for the right-hand inequality we remark that $b = 0$ is the best constant using an elementary sequence method.

THEOREM 1.1.

(i) *For every integer $n \geq 2$,*

$$\gamma - x_n < \frac{1}{12n^2}.$$

(ii) *For every $a > 0$, there exists $n_a \in N$, $n_a \geq 2$, such that*

$$\frac{1}{12(n+a)^2} < \gamma - x_n \quad \textit{for all } n \geq n_a.$$

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Proof. For $a \geq 0$, we define the sequence $(a_n)_{n\geq 2}$ by

$$a_n = \gamma - x_n - \frac{1}{12(n+a)^2} = \gamma - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n-1} - \frac{1}{2n} + \ln n - \frac{1}{12(n+a)^2}.$$

Thus, $a_{n+1} - a_n = f(n)$ where

$$f(n) = -\frac{1}{2n} - \frac{1}{2n+2} + \ln(n+1) - \ln n - \frac{1}{12(n+a+1)^2} + \frac{1}{12(n+a)^2}.$$

The derivative of the function f is

$$f'(n) = \frac{P(n)}{6n^2(n+1)^2(n+a)^3(n+a+1)^3},$$

where

$$\begin{aligned} P(n) &= 3(n+a)^3(n+a+1)^3 - n^2(n+1)^2[3n^2 + 3n(2a+1) + 3a^2 + 3a + 1] \\ &= 12an^5 + (42a^2 + 30a - 1)n^4 + 2(30a^3 + 42a^2 + 12a - 1)n^3 \\ &\quad + (45a^4 + 90a^3 + 51a^2 + 6a - 1)n^2 + 9(2a^5 + 5a^4 + 4a^3 + a^2)n \\ &\quad + 3(a^6 + 3a^5 + 3a^4 + a^3). \end{aligned}$$

(i) If $a = 0$, then $P(n) = -n^4 - 2n^3 - n^2 < 0$ for all $n \geq 1$ and f is strictly decreasing. Since $f(\infty) = 0$, $f(n) > 0$ for all $n \geq 1$ and $(a_n)_{n\geq 2}$ is strictly increasing. Since (a_n) converges to zero, $a_n < 0$ for all $n \geq 2$, whence

$$\gamma - x_n < \frac{1}{12n^2} \quad \text{for all } n \geq 2.$$

(ii) If $a > 0$ then there exists $n_a \in N$, $n_a \geq 2$, such that $P(n) > 0$ for all $n \geq n_a$ and then f is strictly increasing on $[n_a, \infty)$. Since $f(\infty) = 0$, $f(n) < 0$ for all $n \geq n_a$ and $(a_n)_{n\geq n_a}$ is strictly decreasing. Since (a_n) converges to zero, $a_n > 0$ for all $n \geq n_a$ and

$$\frac{1}{12(n+a)^2} < \gamma - x_n \quad \text{for all } n \geq n_a. \qquad \square$$

Now we find the constant n_a in some particular cases. For example, if

$$a = 0.03 = \frac{3}{100} < \frac{1}{\sqrt{12\gamma - 6}} - 1 = 0.038859\ldots,$$

then

$$P(n) = \frac{9}{25}n^5 - \frac{311}{5000}n^4 - \frac{60139}{50000}n^3 - \frac{15432671}{20000000}n^2 + \frac{45544437}{5000000000}n + \frac{88510887}{1000000000000} > 0$$

for all $n \geq 3$, and so

$$\frac{1}{12(n + \frac{3}{100})^2} < \gamma - x_n < \frac{1}{12n^2} \quad \text{for all } n \geq 3.$$

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Let us remark that a direct calculation shows that these inequalities hold for $n = 2$, whence

$$\frac{1}{12(n+\frac{3}{100})^2} < \gamma - x_n < \frac{1}{12n^2} \quad \text{for all } n \geq 2.$$

Next, if $a = 0.01 = \frac{1}{100}$, then

$$P(n) = \frac{3}{25}n^5 - \frac{3479}{5000}n^4 - \frac{87577}{50000}n^3 - \frac{18696191}{20000000}n^2 + \frac{4682259}{5000000000}n + \frac{3090903}{1000000000000} > 0$$

for all $n \geq 8$, and so

$$\frac{1}{12(n+\frac{1}{100})^2} < \gamma - x_n < \frac{1}{12n^2} \quad \text{for all } n \geq 8.$$

A direct calculation shows that these inequalities hold for $n \in \{5, 6, 7\}$, and so

$$\frac{1}{12(n+\frac{1}{100})^2} < \gamma - x_n < \frac{1}{12n^2} \quad \text{for all } n \geq 5.$$

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