Supplement to a Note on Recurrent Sequences

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A lemma used in the above paper[1] can be extended to cover the case of a sequence S_n determined by a recurrence relation of the form

(1) $$S_n = a_n^1 S_{n-1} + a_n^2 S_{n-2} + \dots + a_n^r S_{n-r}$$

r being any positive integer.

For convenience denote $|a_n^r|$ by b_n^r everywhere. Let us suppose that an inequality of the form

(2) $$|S_p| \leqq K k_p \prod_{t=1}^{p-1} l_t$$

holds for $p = 2, 3, \dots, n-1$. The constant K can be determined from the initial values of S_n, when k_n and l_n have been found.

Taking moduli in (1), the inequality (2) is seen to be true for $p = n$ also, if

$$b_n^1 \, k_{n-1} \prod_{t=1}^{n-2} l_t + b_n^2 \, k_{n-2} \prod_{t=1}^{n-3} l_t + \dots + b_n^r \, k_{n-r} \prod_{t=1}^{n-r-1} l_t \leqq k_n \, . \prod_{t=1}^{n-1} l_t \, .$$

Choosing b_n^1 as k_n everywhere, and dividing across by $b_n^1 \prod_{t=1}^{n-2} l_t$, we obtain

$$k_{n-1} + \frac{b_n^2 \, k_{n-2}}{k_n \, l_{n-2}} + \frac{b_n^3 \, k_{n-3}}{k_n \, l_{n-2} \, l_{n-3}} + \dots + \frac{b_n^r \, k_{n-r}}{k_n \, l_{n-2} \, l_{n-3} \dots l_{n-r}} \leqq l_{n-1}.$$

This will clearly be satisfied if $k_n \leqq l_n$ always, and

(3) $$k_{n-1} + \frac{b_n^2}{k_n} + \frac{b_n^3}{k_n \, k_{n-2}} + \dots + \frac{b_n^r}{k_n \, k_{n-2} \, k_{n-3} \dots k_{n-r+1}} = l_{n-1}.$$

But equation (3) contains the other condition $k_n \leqq l_n$. Hence (2) is satisfied for all p if $k_n = b_n^1$ always, and l_{n-1} is then found from (3), by substitution there for k_n.

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[1] *Proc. Edinburgh Math. Soc.* (2), 3 (1932), 147-150.

The theorems of the previous note can now be extended, using the lemma in this wider form. Theorems of another type may perhaps be indicated here.

From a recurrence relation of the form

$$(4) \qquad u_n = a_n^1\, u_{n+1} + a_n^2\, u_{n+2} + \ldots . + a_n^r\, u_{n+r} + c_n\, \theta_n$$

where $\lim \theta_n = 0$, we can derive the relation

$$(5) \qquad u_n = A_m^n\, u_{m+1} + B_m^n\, u_{m+2} + \ldots . + L_m^n\, u_{m+r} + \sum_{t=n}^{m} \lambda_n^t\, \theta_t$$

and $n \leqq m+1$. When $n = m+1$, $A_m^n = 1$, while B_m^n etc., and λ_n all vanish.

Because of (4), a recurrence relation

$$A_m^n = a_n^1\, A_m^{n+1} + a_n^2\, A_m^{n+2} + \ldots . + a_n^r\, A_m^{n+r}$$

exists for A_m^n. In addition A_m^{m+1}, which we regard as the first term of the sequence A_m^n, is unity. There is a similar recurrence relation for the other coefficients B_m^n etc., and for the λ's. The lemma may thus be used to fix limits for these coefficients, having imposed suitable conditions on the a's and on c_n.

When a condition

$$\sum_{t=n}^{m} |\lambda_n^t| < R$$

holds, R being independent of m and n, it follows that

$$\sum_{t=n}^{m} \lambda_n^t\, \theta_t \to 0$$

as $n \to \infty$, uniformly with respect to m, since $\lim \theta_n = 0$. If in addition each of the other terms $A_m^n\, u_{m+1}$ etc. on the right hand side of (5) tends to zero as m tends to infinity, n being fixed, then $\lim u_n = 0$.

The following well known theorem[1] is a special case of this kind.

If $\lim \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$, $b_n \geqq b_{n+1}$ *always, and* $\lim a_n = \lim b_n = 0$, *then* $\lim a_n/b_n = l$.

[1] Bromwich, *Infinite Series* (1926), 413.

For, writing $a_n = b_n (l + u_n)$, the hypotheses yield the relation

$$u_n = \frac{b_{n+1}}{b_n} u_{n+1} + \left(1 - \frac{b_{n+1}}{b_n}\right) \theta_n$$

where θ_n, b_n, $b_n u_n$ all tend to zero. The sequence u_n is easily seen to obey the conditions of the preceding paragraph, and so $\lim u_n = 0$, as required.

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