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QUANTUM SOLITONS

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Abstract. Two themes are considered in this paper. First of all in the Introduction a comment is made about the Tzitzéica equation which occurred in the author's work as reported in [**1**]. Much recent work has referred to this equation and a representative bibliography is given in this paper with brief comments. The second theme of this paper is concerned with work surrounding Quantum Solitons and the author's talk at the ISLAND 2 meeting under the title 'Quantum solitons' is briefly summarised. An extended review of this subject matter has now appeared in [**2**] and as there the quantum soliton of the quantum attractive NLS model is seen as a 'qubit' for quantum information purposes. It is hoped that this summary and/or the reference [**2**] can help to stimulate further interest in this 'quantum' aspect of our subject in the solitons community. The actual *observation* of a quantum soliton is also reported and a proper theoretical description of it given. Reference is made to q-boson lattices first as a simple example of the quantum inverse method in the Introduction and then as a subject matter in its own right at the end of the paper.

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1. Introduction. Circumstances have prevented the author from meeting the final deadline for this paper and the present brief note must serve instead. In this note I attempt quickly to summarise the material I was able actually to present at Auchrannie Spa Resort on the Isle of Arran, Scotland, where the ISLAND 2 meeting was held during June 22–27, 2003. The Organisers are to be congratulated (and thanked!) for this excellent meeting.

A second theme of this note, actually referred to first in this Introduction, is a comment on Tzitzéica's equation which, following Hirota's presentation [**3**] in particular (and see the paper by Kaptsov and Shan'ko [**4**]) we can take in the form

$$(\log h)_{xy} = h - h^{-2} \tag{1}$$

obtained by the Rumanian mathematician G. Tzitzéica [**5**, **6**] as the compatibility condition for the Gauss equations

$$\begin{aligned} r_{xx} &= (h_x r_x + \lambda r_y)/h \\ r_{xy} &= hr \\ r_{yy} &= (h_y r_y + \lambda^{-1} r_x)/h. \end{aligned} \tag{2}$$

It is trivial to point out that when $h = e^{\phi(x,y)}$ equation (1) becomes

$$\phi_{xy} = e^{\phi} - e^{-2\phi} \tag{3}$$

an equation which emerged in the work of the writer [**1**] on the polynomial conserved densities (p.c.d.s) of the nonlinear 'sine-Gordon' equations $\phi_{xy} = F(\phi)$ in 1977. Because of a previous result [**7**] on the Bäcklund transformations (BTs), namely that the system $\phi_{xy} = F(\phi)$ has an auto-BT iff $\ddot{F}(\phi) = \alpha^2 F(\phi)$ for some complex valued constant α not excluding $\alpha = 0$ where dot denotes differentiation with respect to the argument (a situation which includes the sine-Gordon and sinh-Gordon equations and the famous Liouville equation $\phi_{xy} = e^{\phi}$) it was concluded (incorrectly!) that equation (3) could not have a countable infinity of p.c.d.s.[1] But Mikhailov [**8**] gave a recursion formula for an infinite set of non-trivial polynomial conserved densities for an equivalent form of equation (3) – refuting the Theorem 9 of [**1**]. However, although a classical r-matrix is now known for equation (3), and that equation's complete integrability as a Hamiltonian system to that extent demonstrated [**9**], it seems to have been solved only very recently for its soliton solutions [**4**] and this by finding a trilinear form [**3**, **4**] for the equation. Thus although the quantum R-matrix is known [**10**] for the *quantum integrable* field theory with Hamiltonian[2]

$$\mathcal{H} = \int dx \left\{ \frac{\gamma}{2}\Pi^2 + \frac{1}{2\gamma}(\partial_x u)^2 + \frac{m^2}{\gamma}\left[e^u + \frac{1}{2}e^{-2u} - \frac{3}{2} \right] \right\} \tag{4}$$

with $\Pi(x) = \gamma^{-1}\partial_t u(x)$ and equal time commutation relations (with $\hbar = 1$)

$$[\Pi(x), u(y)] = -i\delta(x - y) \tag{5}$$

and consequentially with equations of motion

$$\left(\partial_t^2 - \partial_x^2\right)u + m^2[e^u - e^{-2u}] = 0 \tag{6}$$

independent of the coupling constant $\gamma \in \mathbb{R}$ ($\gamma > 0$) (and note that, after the canonical transformation $\sqrt{\gamma}\Pi \to \Pi$, $\phi \to \sqrt{\gamma}\phi$ in $\mathcal{H}$, γ serves to couple in the nonlinear part of the system) nothing is known (I believe) about its quantum solitons the theory of which is still to be worked out. But because of the results in [**13**, **14**] which show that the quantum and classical correlation functions for optical solitons can be quite similar and because all of this mathematical structure *is* known anyway, it must be certain that these quantum solitons actually exist, while comparable results for the classical equations have already been found as has been indicated above. A possible caveat only being that [**4**] suggests that the *classical* soliton solutions are actually quite hard to find. Concerning 'quantum solitons' there is also the small point that it is really necessary to describe first of all what we actually *mean* by a 'quantum soliton' and this was the subject matter of my actual talk on 23 June in Arran. For those interested in the Tzitzéica equation as such a very short additional biography is in [**15**, **16**, **17**].

[1] With hindsight it is easy to see that equation (3) could have an auto BT by introducing a second field auxiliary to the single field $\phi(x, y)$, and see also both of [**1**] and the recent paper [**15**] below for BT's of the Tzitzéica equation 1.

[2] For what may be the simplest example of the use of the *quantum inverse method* [**9**, **10**], which otherwise I cannot review in this short article, see *e.g.* the two papers [**11**, **12**]. Incidentally these two papers concern q-deformation (of a completely integrable Bose gas model) and "q-boson" algebras mentioned otherwise at the end of this paper [**23**, **24**, **25**, **26**, **27**].

2. Quantum solitons. According to the popular article by Izo Abram [**18**] quantum solitons have been with us for a long time and their physical manifestations could be ideal for the purposes of modern quantum information theory (including quantum computing). Briefly, for the quantum integrable attractive Nonlinear Schrödinger model describing optical pulse transmission in optical fibres, single soliton solutions are evidently 'bits' of information which because they must ultimately have to be quantum in character are actually quantum 'bits' or 'qubits' for quantum information storage and transmission purposes. As such they will be the classical soliton solutions with an additional quantum 'jitter' imposed upon them so that for conjugate dynamical pairs linear momentum and linear position, or indeed for particle number and conjugate phase, this jitter is a manifestation of Heisenberg's uncertainty principle $\Delta p \Delta q \geq \frac{1}{2}$ (for $\hbar = 1$) (or $\Delta n \Delta \phi \geq \frac{1}{2}$) (*e.g.* the 'fluctuation' $\Delta p = [\langle |(p - \bar{p})^2| \rangle]^{\frac{1}{2}}$ about the mean of linear momentum p in the state $|\rangle$ *i.e.* $\bar{p} = \langle |p| \rangle$, satisfies the Heisenberg lower bound). Moreover the quantum state $|\rangle$ is a 'coherent state' and these quantum pulses *are* coherent. Of the large number of different quantum coherent states the best known example is the Glauber coherent state $|\alpha\rangle$ which satisfies $a|\alpha\rangle = \alpha|\alpha\rangle$, $\alpha \in \mathbb{C}$, and the operator a with its adjoint $a^\dagger$ satisfies the Heisenberg-Weyl Lie algebra $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$, $[a, a^\dagger] = 1$; N is the number operator (whose eigenvalues could be the number of photons *e.g.*).

Pursuing these ideas I have shown with Miki Wadati, Tokyo, how the 'quantum soliton' of the quantum attractive Nonlinear Schrödinger (NLS) model is the quantum mechanical matrix element (for large enough values of the positive integers n)

$$\langle n, P'|\hat{\psi}(x)|n+1, P\rangle = \pi |c|^{-\frac{1}{2}} \exp[i(P - P')x] \operatorname{sech}\left[\pi |c|^{-\frac{1}{2}} \left(\frac{1}{n}P' - \frac{1}{n+1}P\right)\right] \tag{7}$$

(This particular result was first found by Wadati in 1984 in the paper [**19**] and was subsequently found by Lai and Haus in the paper [**20**] who also give results for multisolitons.) In this expression (7) $\hat{\psi}(x)$ is the quantum field which satisfies the attractive Bose gas equations

$$\begin{aligned} -i\hbar\partial_t\hat{\psi} &= \frac{\hbar^2}{2m}\partial_x^2\hat{\psi} - 2c\hat{\psi}^\dagger\hat{\psi}\hat{\psi} \\ +i\hbar\partial_t\hat{\psi}^\dagger &= \frac{\hbar^2}{2m}\partial_x^2\hat{\psi}^\dagger - 2c\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi} \end{aligned} \tag{8}$$

with $c \in \mathbb{R}$ and $c < 0$ and with Bose commutation relations $[\hat{\psi}^\dagger(x), \hat{\psi}(x')] = -i\hbar\delta(x - x')$, $[\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')] = [\hat{\psi}(x), \hat{\psi}(x')] = 0$. It is usual to choose units for (8) so that $\hbar = 1$ and $m = \frac{1}{2}$. The states $|n, P\rangle$ in (7) are simultaneous eigenstates of $\hat{N} \equiv \int \hat{\psi}^\dagger\hat{\psi}\,dx$, $\hat{P} \equiv -\frac{1}{2i}\int[\hat{\psi}^\dagger\hat{\psi}_x - (\hat{\psi}^\dagger_x)\hat{\psi}]dx = +i\int \hat{\psi}^\dagger\hat{\psi}_x dx$, $\hat{\mathcal{H}} = \int[\hat{\psi}^\dagger_x\hat{\psi}_x + c\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}]dx = \int[-\hat{\psi}^\dagger\hat{\psi}_{xx} + c\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}]dx$ and, of course, of a further countable infinity of mutually commuting operators. They (the states $|n, p\rangle$) are bound states (with $c < 0$) which occur for the "n strings" in the complex momentum plane

$$k_j = p + \frac{1}{2}\{n - (2j - 1)\}ic; \quad j = 1, 2, \ldots, n \tag{9}$$

with $\hat{N}|n, P\rangle = n|n, P\rangle$ and $P = np$. By introducing the time t by going to the Schrödinger picture (of wave mechanics) and taking appropriate Fourier transforms on P, P' with kernels e^{-iPX}, $e^{-iP'X'}$ one finds that the matrix element (for large enough n)

$$M_n = \langle n, X', t|\hat{\psi}(x)|n+1, X, t\rangle$$
$$= 2\eta|c|^{-\frac{1}{2}} e^{4i\eta^2 t}\mathrm{sech}[2\eta(x - X')]\delta\left(X' - \frac{n+1}{n}X + \frac{1}{n}x\right) \qquad (10)$$

where X is identified with a point x_0 and $\eta \equiv \frac{1}{4}|c|(n+1)$. The factor on the δ-function is *exactly* the usual one soliton solution of the classical NLS equation corresponding to the discrete eigenvalue of the associated linear problem which is $\zeta = \xi + i\eta$, $\eta > 0$ with $\xi = 0$ the classical soliton at rest. A Galilean boost operator can be used to obtain the more general result which is the classical soliton moving at a finite velocity as shown in [**19**].

Unfortunately we cannot yet say anything comparable about the quantum and classical multi-soliton solutions (despite the results in [**20**]), but the classical oscillatory solutions are exemplified by quantum 'phonon' solutions. The whole of this analysis together with the comparable analysis for the quantum sine-Gordon equation is given in the joint paper by R. K. Bullough and Miki Wadati [**2**] which has now appeared in the Journal of Physics B. Quantum and Semiclassical Optics in a Special Issue on Optical Solitons edited by P. D. Drummond *et al.*

Thus for further information and in the hope that the present summary will stimulate interest of the soliton community as represented at ISLAND 2 in quantum solitons the reader is referred to this article [**2**].

An additional point made at ISLAND 2 concerns actual observation of a quantum soliton by L. Khaykovich *et al.* [**21**] in ultracold ^7Li at temperatures $T \lesssim 10^{-6}K$ in 2002. At comparable very low temperatures the metal vapours ^{87}Rb, ^{23}Na, and ^7Li display features of the Bose-Einstein condensation predicted by Einstein in 1924. Modelled by the quantum NLS model equations (8) in one space dimension only, both of ^{87}Rb and ^{23}Na are repulsive with $c > 0$. However ^7Li can be put into a quantum state of negative scattering length a and since coupling constants $g = 2c = 4\pi\hbar^2 a/m$ $a < 0$ means $c < 0$. This result is a necessary condition for any quantum soliton solution of the quantum NLS model equations (8) and in practice Khaykovich *et al.* [**21**] report observation of a single moving pulse whose soliton density expectation value $\langle n+1, P|\hat{\psi}^\dagger(x)\hat{\psi}(x)|n+1, P\rangle$ is consistent with the theory advanced in our joint paper [**2**]. The situation is complicated by the fact that the experimental situation is necessarily three dimensional in space. But it is argued that the ultracold ^7Li atoms first of all undergo a transverse 'collapse' of Zakharov type in two space dimensions already studied in our paper [**22**] and this collapse produces a strictly one dimensional system, the one dimensional quantum soliton!

Two other quantum systems with n-string solutions (and so with quantum soliton solutions) were reported in Arran and these bear directly on the particular theme of the meeting which was 'Discrete Systems and Geometry'. These two quantum systems involve "q-boson" algebras and "q-spin operators" belonging to the quantum group $su_q(2)$ ($= U_q[su(2)]$), U means universal enveloping algebra (of the Lie algebra $su(2)$) and $q \in \mathbb{R}$ is a 'deformation parameter') – see the author's paper [**23**] where quantum Bethe equations are found and solved. Classical 'q-solitons' and 'q-multi-solitons' were first reported in [**24**]. At the level of the Bethe equations quantum soliton solutions were found for the Maxwell-Bloch equations on a lattice in [**25**]. See also [**26**]. Perhaps the simplest q-boson system is the quantum Ablowitz-Ladik equation [**27**].

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