A GENERAL TURÁN EXPRESSION FOR THE ZETA FUNCTION

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1. Introduction. In 1948 Gabor Szegő [9] gave four proofs of a remarkable inequality communicated to him by Paul Turán, who later published an original proof [10]. The Turán theorem states that if $P_n(x)$ is the Legendre polynomial, then

$$(1.1) \qquad P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geq 0, \quad \text{for } n \geq 1, \quad |x| \leq 1,$$

with equality holding only when $|x| = 1$.

Since then many similar inequalities have been found for various special functions, particularly for the Legendre and Hermite polynomials. Reference may be had to the recent work of Danese [2] and Chatterjea [1]. Danese gives an extensive bibliography.

Not only have inequalities been found, but also explicit forms and transformations of Turán expressions have been given.

Let us define a general Turán functional operator T by

$$(1.2) \qquad T_{x,a,b}(f) = T_x f(x) = f(x+a)f(x+b) - f(x)f(x+a+b) .$$

The writer [3, 4, 5] has found that this general operator enjoys a number of remarkable properties, and we might mention a few simple examples:

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$$T(x) = ab,$$

$$T(\sin x) = T(\cos x) = \sin a \sin b,$$

(1.3)

$$T(\sinh x) = -T(\cosh x) = \sinh a \sinh b,$$

$$T(\exp(c+dx)) = 0 \quad (c, d \text{ constants}),$$

and in a problem in the American Mathematical Monthly [12] it was shown that for the Fibonacci numbers

$$(1.4) \qquad T_n(f_n) = (-1)^n f_a f_b ,$$

where $f_1 = 1 = f_2$ and $f_{n+2} = f_{n+1} + f_n$.

The writer [3] found that the relation for the hyperbolic trigonometric sine and cosine can be generalized to give

$$(1.5) \qquad T_x(p\,u^{x+c} + q\,v^{x+c}) = -\,pq\,(uv)^{x+c}\,(u^a - v^a)(u^b - v^b),$$

where p, q, u, v, c are constants. This also generalizes (1.4).

In view of the elegance of the various explicit relations which may be found for the Turán operator T, it may be of interest to show what may be done for the Zeta function, which we shall define by

$$(1.6) \qquad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1.$$

There is no difficulty in extending the results to complex x, so that we shall suppose x to be real. Our results depend on some very simple number-theoretic calculations.

2. General Turán expression for the Zeta function. To transform the general Turán expression

$$(2.1) \qquad T\zeta(x) = \zeta(x+a)\,\zeta(x+b) - \zeta(x)\,\zeta(x+a+b),$$

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we shall need the fact [6, p. 250] that

$$(2.2) \qquad \zeta(x)\,\zeta(x-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^x}, \quad x > 1,\ x - a > 1,$$

and where

$$\sigma_a(n) = \sum_{d \mid n} d^a, \qquad \sigma_0(n) = \sum_{d \mid n} 1 = \tau(n).$$

THEOREM 1. Let $x > 2$. Then

$$(2.3) \qquad T_{x,1,-1}\,\zeta(x) = \zeta(x+1)\,\zeta(x-1) - \zeta^2(x)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{x+1}} \sum_{\substack{d \mid n \\ d < \sqrt{n}}} \left(\frac{n}{d} - d\right)^2.$$

Proof. From (2.2) it is readily verified that

$$T_{x,1,-1}\,\zeta(x) = \sum_{n=1}^{\infty} \frac{\sigma_2(n) - n\tau(n)}{n^{x+1}},$$

and

$$\sigma_2(n) - n\tau(n) = \sum_{d \mid n} (d^2 - n)$$

$$= \sum_{\substack{d \mid n \\ d > \sqrt{n}}} (d^2 - n) + \sum_{\substack{d \mid n \\ d < \sqrt{n}}} (d^2 - n)$$

$$= \sum_{\substack{d \mid n \\ d < \sqrt{n}}} \left(\left(\frac{n}{d}\right)^2 - n\right) + \sum_{\substack{d \mid n \\ d < \sqrt{n}}} (d^2 - n)$$

$$= \sum_{\substack{d \mid n \\ d < \sqrt{n}}} \left(\frac{n}{d} - d\right)^2.$$

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Since the coefficients are all positive we have the corollary that $T_{x,1,-1}\zeta(x) > 0$ for $x > 2$.

THEOREM 2. Let $x > 1$, $x+a > 1$, $x+b > 1$, $x+a+b > 1$. Then

$$(2.4) \qquad T_{x,a,b}\,\zeta(x) = -\sum_{n=1}^{\infty} \frac{1}{n^{x+a+b}} \sum_{\substack{d|n \\ d<\sqrt{n}}} \left[\left(\frac{n}{d}\right)^{b} - d^{b}\right]\left[\left(\frac{n}{d}\right)^{a} - d^{a}\right].$$

Proof. Let $a \geq b$. Then it follows again from (2.2) that

$$(2.5) \qquad T_{x,a,b}\,\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^{x+a+b}} \left[n^{b} \cdot \sigma_{a-b}(n) - \sigma_{a+b}(n)\right],$$

and

$$n^{b} \cdot \sigma_{a-b}(n) - \sigma_{a+b}(n) = \sum_{d|n} d^{a}\left[\left(\frac{n}{d}\right)^{b} - d^{b}\right]$$

$$= \sum_{\substack{d|n \\ d<\sqrt{n}}} d^{a}\left[\left(\frac{n}{d}\right)^{b} - d^{b}\right] + \sum_{\substack{d|n \\ d>\sqrt{n}}} d^{a}\left[\left(\frac{n}{d}\right)^{b} - d^{b}\right]$$

$$= \sum_{\substack{d|n \\ d<\sqrt{n}}} d^{a}\left[\left(\frac{n}{d}\right)^{b} - d^{b}\right] + \sum_{\substack{d|n \\ d<\sqrt{n}}} \left(\frac{n}{d}\right)^{a}\left[d^{b} - \left(\frac{n}{d}\right)^{b}\right]$$

$$= \sum_{\substack{d|n \\ d<\sqrt{n}}} \left[d^{a} - \left(\frac{n}{d}\right)^{a}\right]\left[\left(\frac{n}{d}\right)^{b} - d^{b}\right],$$

from which the desired result follows. It is interesting to compare this with the relation (1.5) above. The theorem also provides a number of inequalities depending on the values of a and b.

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It is also interesting to compare (2.4) with the corresponding result for binomial coefficients

$$(2.6) \qquad T_{x,a,b}\binom{x}{n} = \binom{x+a+b}{n} \sum_{k=1}^{n} \binom{x}{n-k} \frac{\binom{a}{k}\binom{b}{k}}{\binom{x+a+b}{k}},$$

which is developed in [5] along with expressions for Hermite polynomials, the Bessel function, and various others. There the operator T as defined by (1.2) is considered as merely one special case of the more general operator

$$(2.7) \qquad T_{x,a,b} f(x) = f(x \cup a) \cap f(x \cup b) * f(x) \cap f(x \cup a \cup b).$$

This general operator is suggested by (1.2) if we replace + by $\cup$ and $\cdot$ by $\cap$. The abstract operations $\cup$, $\cap$ could be defined in various ways: set union and intersection, l.u.b. and g.l.b., l.c.m. and g.c.d., etc. This general operator suggested the present application to the Zeta function, and by taking $*$ to be ordinary division instead of subtraction we shall define an operator R which is related to some work of Ramanujan.

Four other operators considered in [5] and which are special cases of (2.7) are

$$(2.8) \qquad S(f) = f(x+a) + f(x+b) - f(x) - f(x+a+b),$$

$$(2.9) \qquad A(f) = f(x+a) + f(x+b) + f(x) + f(x+a+b),$$

$$(2.10) \qquad P(f) = f(x+a)f(x+b)f(x)f(x+a+b),$$

$$(2.11) \qquad R(f) = \frac{f(x+a)f(x+b)}{f(x)f(x+a+b)}.$$

It may be of interest to point out that the two linear operators S and A may be extended in a natural way, using the Bernoulli and Euler polynomials of higher order as developed by Nörlund [7]. Indeed we could define (following Nörlund)

$$(2.12) \qquad S^n_{x;a_1,a_2,\ldots,a_n}(f) = \left(\prod_{i=1}^{n} a_i \underset{x,a_i}{\Delta}\right) f(x)$$

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and

$$(2.13) \qquad A^n_{x;a_1,a_2,\ldots,a_n}(f) = 2^n \left(\prod_{i=1}^{n} \nabla_{x,a_i} \right) f(x)$$

where

$$\mathop{\Delta}_{x,h} f(x) = \frac{f(x+h) - f(x)}{h} \text{ and } \mathop{\nabla}_{x,h} f(x) = \frac{f(x+h) + f(x)}{2},$$

which are studied by Nörlund using generalized Bernoulli and Euler polynomials of higher order; he develops inverse operators and solves functional equations involving these operators.

The five operators T, S, A, P, R are related by various formulas developed in [5], and we wish to mention in particular the fact that

$$(2.14) \qquad P(f) = -\frac{T(f)}{T(1/f)}.$$

This relation suggests that having found $T(\zeta)$ it might be of values to determine $T(1/\zeta)$ and $P(\zeta)$.

Ramanujan [8] has found what is equivalent to a Turán expression for $P(\zeta)$, and we may state his result in the form of the

THEOREM 3 (Ramanujan). Let $x > 1$, $x+a > 1$, $x+b > 1$, $x+a+b > 1$. Then

$$(2.15) \qquad P(\zeta) = \zeta(x)\,\zeta(x+a)\,\zeta(x+b)\,\zeta(x+a+b)$$

$$= \zeta(2x+a+b) \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^{x+a+b}}.$$

This relation is also given in Hardy and Wright [6, p. 256], as well as in many standard references on the Zeta function.

We may readily establish a formula for $T(1/\zeta)$ by recalling a formula for the reciprocal of the Zeta function,

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using the Möbius function $\mu(n)$. We have

$$(2.16) \qquad \frac{1}{\zeta(x)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^x}, \quad x > 1,$$

which yields

$$(2.17) \qquad \frac{1}{\zeta(x+a)\,\zeta(x+b)} = \sum_{n=1}^{\infty} \frac{1}{n^{x+a}} \sum_{d|n} d^{a-b}\,\mu(d)\,\mu(n/d),$$

and we then find

$$T(1/\zeta) = \sum_{n=1}^{\infty} \frac{1}{n^{x+a+b}} \sum_{d|n} \mu(d)\,\mu(n/d)\,d^a \left[\left(\frac{n}{d}\right)^b - d^b\right].$$

Thus we find

THEOREM 4. Let $x > 1$, $x+a > 1$, $x+b > 1$, $x+a+b > 1$. Then

$$(2.18) \qquad T(1/\zeta) = -\sum_{n=1}^{\infty} \frac{1}{n^{x+a+b}} \sum_{\substack{d|n \\ d < \sqrt{n}}} \mu(d)\,\mu\left(\frac{n}{d}\right)\left[\left(\frac{n}{d}\right)^a - d^a\right]\left[\left(\frac{n}{d}\right)^b - d^b\right].$$

Ramanujan [8] remarked that he had found the various relations (among them the one equivalent to (2.15) above) incidentally in the course of his other investigations. He said that "none of them seem to be of particular importance, nor does their proof involve the use of any new ideas, but some of them are so curious that they seem to be worth printing." It is in the same spirit that we offer the relations developed here. It is interesting to note that Ramanujan also gave an elegant formula for $P(\eta)$ where

$$\eta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}.$$

The relation is very similar to (2.15) and involves alternating

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signs. The Turán expressions for $\eta(x)$ could be developed in much the same way as we have found them for $\zeta(x)$.

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