*Bull. Aust. Math. Soc.* **82** (2010), 274–281
doi:10.1017/S0004972710000286

# CHARACTER AMENABILITY AND CONTRACTIBILITY OF ABSTRACT SEGAL ALGEBRAS

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(Received 17 December 2009)

**Abstract**

Let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$. For a nonzero character $\phi$ on $\mathcal{A}$, we study $\phi$-amenability, and $\phi$-contractibility of $\mathcal{A}$ and $\mathcal{B}$. We then apply these results to abstract Segal algebras related to locally compact groups.

2000 *Mathematics subject classification*: primary 46H05; secondary 43A07.

*Keywords and phrases*: character amenable, abstract Segal algebra.

## 1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$, consisting of all nonzero characters on $\mathcal{A}$. Kaniuth *et al.* [7, 8] introduced and studied the concept of $\phi$*-amenability* for Banach algebras as a generalization of left amenability of Lau algebras. In fact, $\mathcal{A}$ is called $\phi$*-amenable* if there exists a bounded net $(a_\alpha)$ in $\mathcal{A}$ such that

$$\phi(a_\alpha) \to 1 \quad \text{and} \quad \|aa_\alpha - \phi(a)a_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. Any such net is called a bounded approximate $\phi$-mean.

Moreover, the notion of (right) character amenability was introduced and studied by Monfared [9]. Character amenability of $\mathcal{A}$ is equivalent to $\mathcal{A}$ being $\phi$-amenable for all $\phi \in \sigma(\mathcal{A})$ and $\mathcal{A}$ having a bounded right approximate identity.

For $\phi \in \sigma(\mathcal{A})$, the notion of $\phi$*-contractibility* of $\mathcal{A}$ was recently introduced and studied by Hu *et al.* [4]. In fact, $\mathcal{A}$ is called $\phi$*-contractible* if there exists a (right) $\phi$-diagonal for $\mathcal{A}$; that is, an element $\mathbf{m}$ in the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\phi(\pi(\mathbf{m})) = 1 \quad \text{and} \quad a \cdot \mathbf{m} = \phi(a)\mathbf{m}$$

for all $a \in \mathcal{A}$, where $\pi$ denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$ given by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$.

Several authors have studied various notions of amenability for abstract Segal algebras; see, for example, Samea [13] and Tewari and Parthasarathy [14]. Here, we characterize character amenability and character contractibility of abstract Segal algebras. We then give some applications of our results to abstract Segal algebras on a locally compact group $G$.

## 2. Character amenability and contractibility of abstract Segal algebras

Let $\mathcal{A}$ be a Banach algebra with the norm $\|\cdot\|_{\mathcal{A}}$. Then a Banach algebra $\mathcal{B}$ with the norm $\|\cdot\|_{\mathcal{B}}$ is an abstract Segal algebra with respect to $\mathcal{A}$ if:

(1) $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$;
(2) there exists $M > 0$ such that $\|b\|_{\mathcal{A}} \le M\|b\|_{\mathcal{B}}$ for all $b \in \mathcal{B}$;
(3) there exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \le C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ for all $a, b \in \mathcal{B}$.

We begin this section with the following result.

THEOREM 2.1. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then the following statements are equivalent.*

(a) *$\mathcal{B}$ is character amenable.*
(b) *$\mathcal{A} = \mathcal{B}$, and $\mathcal{A}$ is character amenable.*
(c) *$\mathcal{A}$ is Banach algebra isomorphic to $\mathcal{B}$ and $\mathcal{A}$ is character amenable.*

PROOF. (a) $\Rightarrow$ (b). Suppose that $\mathcal{B}$ is character amenable. Then $\mathcal{B}$ has a bounded right approximate identity $(e_\alpha)$. Since $\mathcal{B}$ is an abstract Segal algebra, there exists $C > 0$ such that

$$\|ab\|_{\mathcal{B}} \le C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$$

for all $a, b \in \mathcal{B}$. So, for each $b \in \mathcal{B}$,

$$\begin{aligned}\|a\|_{\mathcal{B}} = \lim_{\alpha} \|ae_\alpha\|_{\mathcal{B}} &\le C\|a\|_{\mathcal{A}} \liminf_{\alpha} \|e_\alpha\|_{\mathcal{B}} \\ &\le C\Big(\sup_{\alpha} \|e_\alpha\|_{\mathcal{B}}\Big)\|a\|_{\mathcal{A}}.\end{aligned}$$

On the other hand, there exists $M > 0$ such that $\|a\|_{\mathcal{A}} \le M\|a\|_{\mathcal{B}}$. Thus the norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent on $\mathcal{B}$. Since $\mathcal{B}$ is dense in $\mathcal{A}$, it follows that $\mathcal{A} = \mathcal{B}$ and $\mathcal{A}$ is character amenable.

(b) $\Rightarrow$ (c). Since $\mathcal{B}$ is an abstract Segal algebra with respect to $\mathcal{A}$, there exists $M > 0$ such that

$$\|a\|_{\mathcal{A}} \le M\|a\|_{\mathcal{B}}$$

for all $a \in \mathcal{A}$. Thus $\mathcal{B}$ is Banach algebra isomorphic to $\mathcal{A}$ by the open mapping theorem. The implication (c) $\Rightarrow$ (a) is trivial. □

Before we give our next result, let us present an elementary lemma.

LEMMA 2.2. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\sigma(\mathcal{B}) = \{\phi|_{\mathcal{B}} : \phi \in \sigma(\mathcal{A})\}$.*

PROOF. Since $\mathcal{B}$ is dense in $\mathcal{A}$, it follows that $\phi|_{\mathcal{B}} \neq 0$ and so $\phi|_{\mathcal{B}} \in \sigma(\mathcal{B})$. Now suppose that $\psi \in \sigma(\mathcal{B})$. Then there exists $b_0 \in \mathcal{B}$ such that $\psi(b_0) = 1$. Thus for each $b \in \mathcal{B}$,

$$\psi(b) = \psi(bb_0).$$

https://doi.org/10.1017/S0004972710000286 Published online by Cambridge University Press

By assumption, there exists $C > 0$ such that $\|bb_0\|_{\mathcal{B}} \leq C\|b\|_{\mathcal{A}}\|b_0\|_{\mathcal{B}}$ for all $b \in \mathcal{B}$ and consequently

$$|\psi(b)| = |\psi(bb_0)| \leq C\|b\|_{\mathcal{A}}\|b_0\|_{\mathcal{B}}.$$

This shows that $\psi$ is continuous on $\mathcal{B}$ with respect to the norm $\|\cdot\|_{\mathcal{A}}$. Since $\mathcal{B}$ is dense in $\mathcal{A}$, it follows that $\psi$ has a unique extension $\phi \in \sigma(\mathcal{A})$. □

PROPOSITION 2.3. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$ and $\phi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is $\phi$-amenable if and only if $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-amenable.*

PROOF. Suppose that $\mathcal{A}$ is $\phi$-amenable. Then there is a bounded approximate $\phi$-mean $(a_\alpha)$ in $\mathcal{A}$. Fix $b_0 \in \mathcal{B}$ such that $\phi(b_0) = 1$ and set

$$b_\alpha := a_\alpha b_0^2 \in \mathcal{B}$$

for all $\alpha$. Since $\mathcal{B}$ is an abstract Segal algebra with respect to $\mathcal{A}$, there exist $C > 0$ and $M > 0$ such that for each $b \in \mathcal{B}$,

$$\begin{aligned}\|bb_\alpha - \phi(b)b_\alpha\|_{\mathcal{B}} &\leq C\|ba_\alpha - \phi(b)a_\alpha\|_{\mathcal{A}}\|b_0\|_{\mathcal{A}}\|b_0\|_{\mathcal{B}}\\ &\leq MC\|ba_\alpha - \phi(b)a_\alpha\|_{\mathcal{A}}\|b_0\|_{\mathcal{B}}^2 \to 0\end{aligned}$$

and

$$\phi(b_\alpha) = \phi(a_\alpha) \to 1.$$

Since $(a_\alpha)$ is $\|\cdot\|_{\mathcal{A}}$-bounded, it follows that $(b_\alpha)$ is $\|\cdot\|_{\mathcal{B}}$-bounded. Thus $\mathcal{B}$ is $\phi_{\mathcal{B}}$-amenable.

Conversely, suppose that $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-amenable. Then there is a bounded approximate $\phi|_{\mathcal{B}}$-mean $(b_\alpha)$ in $\mathcal{B}$. Fix $b_0 \in \mathcal{B}$ such that $\phi(b_0) = 1$ and set

$$a_\alpha := b_0 b_\alpha$$

for all $\alpha$. Since $\mathcal{B}$ is an abstract Segal algebra, there exists $M > 0$ such that for each $a \in \mathcal{A}$,

$$\begin{aligned}\|aa_\alpha - \phi(a)a_\alpha\|_{\mathcal{A}} &= \|ab_0b_\alpha - \phi(a)b_0b_\alpha\|_{\mathcal{A}}\\ &\leq \|ab_0b_\alpha - \phi(a)\phi|_{\mathcal{B}}(b_0)b_\alpha\|_{\mathcal{A}} + \|\phi(a)\phi|_{\mathcal{B}}(b_0)b_\alpha - \phi(a)b_0b_\alpha\|_{\mathcal{A}}\\ &\leq M(\|ab_0b_\alpha - \phi|_{\mathcal{B}}(ab_0)b_\alpha\|_{\mathcal{B}} + |\phi(a)|\ \|\phi|_{\mathcal{B}}(b_0)b_\alpha - b_0b_\alpha\|_{\mathcal{B}})\\ &\to 0\end{aligned}$$

and

$$\phi(a_\alpha) = \phi|_{\mathcal{B}}(b_\alpha) \to 1.$$

Since $\|\cdot\|_{\mathcal{A}} \leq M\|\cdot\|_{\mathcal{B}}$, it follows that $(a_\alpha)$ is a $\|\cdot\|_{\mathcal{A}}$-bounded approximate $\phi$-mean in $\mathcal{A}$, and therefore $\mathcal{A}$ is $\phi$-amenable. □

As a consequence of Proposition 2.3 we have the following result.

COROLLARY 2.4. *Let $\mathcal{B}$ be an abstract Segal algebra with respect to a character amenable Banach algebra $\mathcal{A}$. Then $\mathcal{B}$ is $\phi$-amenable for all $\phi \in \sigma(\mathcal{B})$. Moreover, $\mathcal{B}$ is character amenable if and only if $\mathcal{A} = \mathcal{B}$.*

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PROPOSITION 2.5. *Let $\mathcal{B}$ be an abstract Segal algebra with respect to a Banach algebra $\mathcal{A}$ and $\phi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is $\phi$-contractible if and only if $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-contractible.*

PROOF. Suppose that $\mathcal{A}$ is $\phi$-contractible. Then there is a $\phi$-diagonal $\mathbf{m} \in \mathcal{A} \widehat{\otimes} \mathcal{A}$ for $\mathcal{A}$. Thus, $\phi(\pi(\mathbf{m})) = 1$ and $a\pi(\mathbf{m}) = \phi(a)\pi(\mathbf{m})$ for all $a \in \mathcal{A}$. Since $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$, there exists $b_0 \in \mathcal{B}$ such that $\phi(b_0) = 1$ and so

$$b_1 := b_0\pi(\mathbf{m}) \in \mathcal{B}.$$

Now, for each $b \in \mathcal{B}$ we have

$$bb_1 = \phi(b)b_1 \quad \text{and} \quad \phi(b_1) = 1.$$

It is clear that $b_1 \otimes b_1 \in \mathcal{B} \widehat{\otimes} \mathcal{B}$ is a $\phi|_{\mathcal{B}}$-diagonal for $\mathcal{B}$.

For the converse, suppose that $\mathbf{m} \in \mathcal{B} \widehat{\otimes} \mathcal{B}$ is a $\phi|_{\mathcal{B}}$-diagonal for $\mathcal{B}$. Then $\pi(\mathbf{m}) \cdot \mathbf{m} = \mathbf{m}$ and $a\pi(\mathbf{m}) \in \mathcal{B}$ for all $a \in \mathcal{A}$. We conclude that $\phi(\pi(\mathbf{m})) = 1$ and

$$a \cdot \mathbf{m} = a \cdot (\pi(\mathbf{m}) \cdot \mathbf{m}) = a\pi(\mathbf{m}) \cdot \mathbf{m} = \phi(a)\mathbf{m}$$

for all $a \in \mathcal{A}$; that is, $\mathbf{m}$ is a $\phi$-diagonal for $\mathcal{A}$. □

## 3. Applications to group algebras

Let $G$ be a locally compact group with left Haar measure $\lambda_G$ and let $L^p(G)$, $1 \le p \le \infty$, be the usual Lebesgue space with respect to $\lambda_G$ as defined in [2]. The convolution product of two measurable functions $f$ and $g$ at $x \in G$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\lambda_G(y),$$

whenever the integral exists. Then $L^1(G)$ endowed with the norm $\|\cdot\|_1$ and the convolution product $*$ is a Banach algebra, called the group algebra of $G$. Let $\widehat{G}$ denote the dual group of $G$ consisting of all continuous homomorphisms $\rho$ from $G$ into the circle group $\mathbb{T}$, and define $\phi_\rho \in \sigma(L^1(G))$ to be the character induced by $\rho$ on $L^1(G)$; that is,

$$\phi_\rho(h) = \int_G \overline{\rho(x)} f(x)\, d\lambda_G(x) \quad (f \in L^1(G)).$$

It is known that there is no other character on $L^1(G)$; that is,

$$\sigma(L^1(G)) = \{\phi_\rho : \rho \in \widehat{G}\};$$

see, for example, [6, Theorem 2.7.2] or [2, Theorem 23.7].

Recall that $G$ is called amenable if $L^1(G)$ is $\phi_1$-amenable; or equivalently, there is a bounded approximate $\phi_1$-mean in $L^1(G)$.

Before we give our first result in the section, recall from [5] that every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$ but not conversely; see also [10, 11]. The linear subspace $S^1(G)$ of the convolution group algebra $L^1(G)$ is said to be a Segal algebra on $G$ if it satisfies the following conditions.

(1) $S^1(G)$ is dense in $L^1(G)$.

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(2) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_s$ and for each $f \in S^1(G)$

$$\|f\|_1 \leq \|f\|_s.$$

(3) $S^1(G)$ is left transition invariant and the map $x \mapsto \delta_x * f$ of $G$ into $S^1(G)$ is continuous.

(4) $\|\delta_x * f\|_s = \|f\|_s$ for all $f \in S^1(G)$ and $x \in G$.

PROPOSITION 3.1. *Let $G$ be an amenable locally compact group. Then any abstract Segal algebra $S(G)$ with respect to $L^1(G)$ is $\phi$-amenable for all $\phi \in \sigma(S(G))$. Moreover, $S(G)$ is character amenable if and only if $S(G) = L^1(G)$.*

PROOF. Since $G$ is amenable, it follows that $L^1(G)$ is amenable, and so $L^1(G)$ is character amenable by [9, Corollary 2.4]. The proof is now complete by Corollary 2.4. □

EXAMPLE 3.2. Let $G$ be a compact group endowed with the normalized Haar measure. Then the convolution Banach algebra $L^\infty(G)$ is a symmetric abstract Segal algebra with respect to $L^1(G)$. Since $G$ is amenable, the convolution Banach algebra $L^\infty(G)$ is $\phi$-amenable for all $\phi \in \sigma(L^\infty(G))$ by Proposition 3.1. We can show that $L^\infty(G)$ has a right approximate identity if and only if $G$ is finite. To see this, suppose that $L^\infty(G)$ has a right approximate identity. Thus $L^\infty(G) * L^\infty(G)$ is dense in $L^\infty(G)$, but it is well known that

$$L^\infty(G) * L^\infty(G) \subseteq L^\infty(G) * L^1(G) \subseteq C(G),$$

where $C(G)$ is the set of all continuous functions on $G$. This will yield that $C(G)$ is dense in $L^\infty(G)$ with the uniform norm. Thus $C(G) = L^\infty(G)$ and consequently $G$ is finite by [3, Lemma 37.3]. The converse is clear.

THEOREM 3.3. *Let $G$ be a locally compact group and let $\rho \in \widehat{G}$. Then the following statements are equivalent.*

(a) *$G$ is compact.*
(b) *All abstract Segal algebras on $G$ are $\phi_\rho$-contractible.*
(c) *There is an abstract Segal algebra on $G$ which is $\phi_\rho$-contractible.*

PROOF. (a) ⇒ (b). Fix $\rho \in \widehat{G}$. Since $G$ is compact, it follows that $\rho \in L^1(G)$ and $f * \rho = \phi_\rho(f)\rho$ for all $f \in L^1(G)$ and

$$\phi_\rho(\rho) = \int_G \rho\overline{\rho}\, d\lambda = \int_G |\rho|^2\, d\lambda = 1.$$

It is clear that $\mathbf{m} = \rho \otimes \rho$ is a $\phi_\rho$-diagonal for $L^1(G)$. Thus $L^1(G)$ is $\phi_\rho$-contractible and hence any abstract Segal algebra on $G$ is $\phi_\rho$-contractible by Proposition 2.5.

(a) ⇒ (c). This implication is trivial.

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(c) $\Rightarrow$ (a). Suppose that there is an abstract Segal algebra with respect to $L^1(G)$ which is $\phi_\rho$-contractible. Then $L^1(G)$ is $\phi_\rho$-contractible by Proposition 2.5. It follows that there is a $\phi_\rho$-diagonal for $L^1(G)$, say $\mathbf{m}$. Thus, $\phi_\rho(\pi(\mathbf{m})) = 1$ and

$$f * \pi(\mathbf{m}) = \phi_\rho(f)\pi(\mathbf{m})$$

for all $f \in L^1(G)$. So, if we put

$$g := \overline{\rho}\pi(\mathbf{m}) \in L^1(G),$$

then $\phi_1(g) = 1$ and $f * g = \phi_1(f)g$ for all $f \in L^1(G)$; indeed,

$$\begin{aligned}\rho f * \pi(\mathbf{m}) &= (\rho f * \pi(\mathbf{m})) * \pi(\mathbf{m}) \\ &= \phi_\rho(\rho f * \pi(\mathbf{m}))\pi(\mathbf{m}) \\ &= \phi_1(f)\pi(\mathbf{m}).\end{aligned}$$

It follows that $G$ is compact; see, for example, [12, Exercise 1.1.7]. □

COROLLARY 3.4. *Let $G$ be a locally compact group and let $S(G)$ be an abstract Segal algebra with respect to $L^1(G)$ and $\rho \in \widehat{G}$. Then $G$ is amenable if and only if $S(G)$ is $\phi_\rho$-amenable.*

PROOF. Suppose that $S(G)$ is $\phi_\rho$-amenable. Then $L^1(G)$ is $\phi_\rho$-amenable by Proposition 2.3. Thus $L^1(G)$ has a bounded approximate $\phi_\rho$-mean, say $(f_\alpha)$. Now, we define $h_\alpha := \overline{\rho} f_\alpha$ for all $\alpha$. It follows that

$$\phi_1(h_\alpha) = \phi_\rho(f_\alpha) \to 1$$

and

$$f * h_\alpha = f * \overline{\rho} f_\alpha = \overline{\rho}(\rho f * f_\alpha)$$

for all $f \in L^1(G)$. Consequently,

$$\begin{aligned}\|f * h_\alpha - \phi_1(f)h_\alpha\|_1 &= \|f * \overline{\rho} f_\alpha - \phi_1(f)\overline{\rho} f_\alpha\|_1 \\ &= \|\rho f * f_\alpha - \phi_1(f) f_\alpha\|_1 \\ &= \|\rho f * f_\alpha - \phi_\rho(\rho f) f_\alpha\|_1 \to 0.\end{aligned}$$

Therefore, $(h_\alpha)$ is a bounded approximate $\phi_1$-mean in $L^1(G)$, and so $G$ is amenable. The converse is trivial by Proposition 3.1. □

Let $G$ be a locally compact group and let $A(G)$ be the Fourier algebra of $G$ as defined in [1]. Then $\sigma(A(G))$ consists of all point evaluations $\phi_x$ ($x \in G$) defined by $\phi_x(f) = f(x)$ for all $f \in A(G)$.

THEOREM 3.5. *Let $G$ be a locally compact group and let $SA(G)$ be an abstract Segal algebra with respect to $A(G)$. Then the following statements are equivalent.*

(a) *$G$ is discrete.*
(b) *$SA(G)$ is $\phi_x$-contractible for all $x \in G$.*
(c) *$SA(G)$ is $\phi_x$-contractible for some $x \in G$.*

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PROOF. Suppose that $G$ is discrete. Then $\chi_{\{x\}} \in A(G)$ for all $x \in G$, where $\chi_{\{x\}}$ is the characteristic function of $\{x\}$. Choose $f_0 \in SA(G)$ such that

$$\|f_0 - \chi_{\{x\}}\|_{A(G)} < 1/2.$$

Then $|f_0(x) - 1| < 1/2$, and so $f_0(x) \neq 0$. Thus,

$$\chi_{\{x\}} = \frac{1}{f_0(x)} f_0 \chi_{\{x\}} \in SA(G).$$

Moreover, $\phi_x(\chi_{\{x\}}) = \chi_{\{x\}}(x) = 1$ and, for each $f \in SA(G)$,

$$f\chi_{\{x\}} = f(x)\chi_{\{x\}} = \phi_x(f)\chi_{\{x\}}.$$

Hence, it is clear that $\chi_{\{x\}} \otimes \chi_{\{x\}}$ is a $\phi_x$-diagonal for $SA(G)$, and consequently, $SA(G)$ is $\phi_x$-contractible. That is, (a) implies (b). That (b) implies (c) is trivial.

For (c) $\Rightarrow$ (a), suppose that $SA(G)$ is $\phi_x$-contractible. Then there is a $\phi_x$-diagonal for $SA(G)$, say $\mathbf{m}$. Thus

$$\pi(\mathbf{m})(x) = \phi_x(\pi(\mathbf{m})) = 1$$

and

$$f\pi(\mathbf{m}) = \phi_x(f)\pi(\mathbf{m}) = f(x)\pi(\mathbf{m})$$

for all $f \in SA(G)$. Now let $y \in G$ and choose $g \in A(G)$ such that $g(y) = 0$ and $g(x) = 1$ and also take $h \in SA(G)$ such that $h(x) = 1$. Therefore, $hg \in SA(G)$ satisfies $\phi_x(hg) = h(x)g(x) = 1$, and so

$$0 = hg\pi(\mathbf{m})(y) = \phi_x(hg)\pi(\mathbf{m})(y) = \pi(\mathbf{m})(y).$$

It follows that $\chi_{\{x\}} = \pi(\mathbf{m})$. Since $\pi(\mathbf{m})$ is a continuous function on $G$, we conclude that $G$ is discrete. □

## Acknowledgements

The authors would like to thank the referee for invaluable comments. They also thank the Center of Excellence for Mathematics at the Isfahan University of Technology for kind support.

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