A GENUINE TOPOLOGY FOR THE FIELD OF MIKUSIŃSKI OPERATORS

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Let C denote the complex algebra of continuous functions of a non-negative real variable under addition, scalar multiplication and convolution. C has no divisors of zero and its quotient field F is called the field of Mikusiński operators [1]. It is well known that Mikusiński has defined a sequential convergence in F which is not topological [2]. Using a recent result due to T.K. Boehem [3] we shall provide F with a sequential convergence which is topological.

Hereafter, let the complex algebra C be endowed with the topology of compact convergence and for each $p \in C$, let $S(p) = \sup\{\tau : t \le \tau \Rightarrow p(t) = 0\}$ denote the ordinary support number of p. If $f = p/q \in F$, the number $S(p) - S(q)$ is called the support number of the operator f. The fact that this number is uniquely determined by f follows from Titchmarsh's fundamental theorem [1] which implies that $S(p') - S(q') = S(p) - S(q)$ whenever $p/q = p'/q'$. Boehem's theorem states that if $\{q_n\}$ is a sequence in C, then a necessary and sufficient condition that there exists a nonzero $q \in C$ such that each q_n factors q, i.e. $q = q_n r_n$ with $r_n \in C$, is that the support numbers $S(q_n)$ are uniformly bounded on the right. In particular, this implies that every sequence $\{f_n\} = \{p_n/q_n\}$ of operators with support numbers $S(p_n) - S(q_n)$ uniformly bounded on the left possesses a common denominator $q \in C$, i.e., $qf_n = qp_n/q_n = r_n p_n \in C$ for all n.

DEFINITION. A sequence $\{f_n\}$ of operators is said to be convergent in F if and only if the support numbers of the sequence are uniformly bounded on the left and for any subsequence $\{f_i\}$ and for any $q \in C$ satisfying $qf_i \in C$ for all i, the corresponding function sequence $\{qf_i\}$ is convergent in C.

This definition is meaningful since if $q_1 f_i \in C$ and $q_2 f_j \in C$ for all i and j, with $q_1, q_2 \neq 0$, $q_1 f_i \to p_1$ and $q_2 f_j \to p_2$ in C, then[2]

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[2] Note that by Boehem's theorem there exists a nonzero $q \in C$ such that $qf_n \to p$ in C.

$p_1/q_1 = p_2/q_2 \varepsilon F$ and so there is a unique operator limit determined by any convergent operator sequence $\{f_n\}$. According to this definition, a "constant" sequence, i.e., $f_n = f$ for all n, is convergent and converges to the "constant" and every subsequence of a convergent sequence is convergent and converges to the same operator. Moreover, if an operator sequence $\{f_n\}$ fails to converge in F to an operator f, then either the support numbers are not uniformly bounded on the left or for some subsequence $\{f_i\}$ and some nonzero $q \varepsilon C$ the corresponding function sequence $\{qf_i\}$ fails to converge to qf in C. If the support numbers are not uniformly bounded on the left, then there exists a subsequence $\{f_m\}$ with the support numbers tending monotonically to $-\infty$. If a function sequence $\{qf_i\}$ fails to converge to qf in C, then either $qf \not\varepsilon C$, or $qf \varepsilon C$ and the sequence fails to converge to the function qf uniformly on some compact set. Thus, in any event, if an operator sequence $\{f_n\}$ fails to converge in F to an operator f, there exists a subsequence $\{f_m\}$ such that every subsequence of the latter also fails to converge in F to the operator f. These three properties guarantee that convergence in F is topological [4, 5].

This new convergence concept appears to be as useful in the operational calculus as the one originally introduced by Mikusiński. In particular, addition and multiplication in F are continuous in this topology and the "standard" convergence theorems [6] are available. However, F does not become a topological field since the reciprocal mapping is not continuous [6].

EXAMPLE. Let ϕ be a nontrivial C^∞ function with compact support on the positive half-line. For each positive integer n, let $a_n = \sup \{|\phi^{(n)}(t)|:t \geq 0\}$, where $\phi^{(n)}$ denotes the n^{th} derivative of ϕ. Then if s denotes the derivative operator in F, the function sequence $\{\frac{\phi s^n}{n a_n}\} = \{\frac{\phi^{(n)}(t)}{n a_n}\}$ converges in C. On the other hand, if k is a positive constant and the function ψ is defined by $\psi(t) = \phi(kt)$ for $t \geq 0$, then the function sequence $\{\frac{\psi s^n}{n a_n}\} = \{\frac{k^n \phi^{(n)}(kt)}{n a_n}\}$ does not converge in C if $k > 1$. It follows that the operator sequence $\{\frac{s^n}{n a_n}\}$ fails to converge in F and yet it does converge in the original sense defined by Mikusiński [1].

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