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ORDERED GROUPS SATISFYING THE MAXIMAL CONDITION LOCALLY

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1. Introduction. Let $\mathfrak{X}$ denote the class of all (fully) ordered groups satisfying the maximal condition on subgroups, and let $L\mathfrak{X}$ denote the class of all locally $\mathfrak{X}$ groups. In this paper we investigate the family of convex subgroups of $L\mathfrak{X}$ groups.

It is well known (see [**1**, pp. 51, 54]) that every convex subgroup of an $\mathfrak{X}$ is normal in G, and for any jump $D \prec C$ in the family of convex subgroups, $[G', C] \subseteq D$. We observe that these properties are also true for any $L\mathfrak{X}$ group and record, without proof, the following.

THEOREM 1. *Any convex subgroup of an $L\mathfrak{X}$ group G is normal in G, and for any jump $D \prec C$ in the family of convex subgroups, $[G', C] \subseteq D$.*

As a consequence of the above theorem, a subgroup H of an $L\mathfrak{X}$ group G is convex under some order on G if and only if H is normal in G and $G/H \in L\mathfrak{X}$. In particular, if G is a torsion-free locally nilpotent group, then necessary and sufficient conditions that G admits an order with respect to which H is convex are that H be normal and isolated in G. This answers, in part, a question of Fuchs [**1**, p. 209, Problem 9(a)].

From Theorem 1, we may also conclude that for any $L\mathfrak{X}$ group G, the derived subgroup G' has a central system, and if $G \in \mathfrak{X}$, then G' has a descending central system. In particular, every ordered polycyclic group is nilpotent-by-abelian; however, such a group need not be nilpotent as is demonstrated by the following result.

THEOREM 2. *Any ordered locally supersolvable group is torsion-free locally nilpotent. An ordered polycyclic group need not have a non-trivial centre nor a descending central system.*

The above results correct the assertions made by Ree in [**3**].†

Teh [**5**] has shown that a torsion-free abelian group of rank one admits exactly two different orders, whereas a torsion-free abelian group whose rank exceeds one admits uncountably many different orders. It is shown here that a non-abelian torsion-free locally nilpotent group admits infinitely many orders. We also study the structure of $L\mathfrak{X}$ groups which admit only finitely many different orders and conclude the following.

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†[**4**, Theorem 2] is also false. Ree used the results in [**3**] in the proof of this theorem. A counter-example is given in the Ph.D. Thesis of R. J. Hursey, Jr., to be submitted to the University of Alberta.

THEOREM 3. *If a non-abelian $L\mathfrak{X}$ group G admits only finitely many different orders, then the Fitting subgroup N of G exists and coincides with the isolator $I(G')$ of G'; G/N is non-trivial and locally cyclic; and N is an absolutely convex subgroup of G. Moreover, if $G \in X$, then G is polycyclic.*

The existence of a non-abelian polycyclic group admitting only finitely many different orders is demonstrated by the example following the proof of Theorem 2.

We conclude this section with the following remark.

Remark. If G is an ordered polycyclic group and $l(G)$ is the number of infinite cyclic factors in any cyclic series of G, then G is nilpotent if and only if the number of subgroups of G convex with respect to some order on G is precisely $1 + l(G)$.

2. Definitions and Notation. If G is a group on which there can be defined a (full) order relation $\leqq$ with the property that $a, b, x, y \in G$ and $a \leqq b$ imply $xay \leqq xby$, then G is said to be an *ordered group* and $\leqq$ is said to be an *order on* G. Associated with an order $\leqq$ on G is the *positive cone* $P(G)$ of G, $P(G) = \{x | x \in G \text{ and } 1 \leqq x\}$. It follows that the subset $P(G)$ of the ordered group G has the following four properties:

(i) $P(G) \cap P^{-1}(G) = 1$;
(ii) $P(G)P(G) \subseteq P(G)$;
(iii) $x^{-1}P(G)x \subseteq P(G)$ for each $x \in G$; and
(iv) $P(G) \cup P^{-1}(G) = G$.

Conversely, if $P(G)$ is a subset of a group G possessing properties (i)–(iv), then G is an ordered group under the relation $\leqq$ given by

$$a \leqq b \Leftrightarrow a^{-1}b \in P(G).$$

A subgroup C of a group G ordered with respect to $\leqq$ is *convex* if $g \in G$, $c \in C$, and $1 \leqq g \leqq c$ imply $g \in C$. A subgroup C of an ordered group G is *absolutely convex* if C is a convex subgroup of G with respect to each order on G. A subgroup A of a group G is *isolated* in G if $g \in G$, n a positive integer, and $g^n \in A$ imply $g \in A$. The *isolator* in G of a subgroup A of G is the intersection of all isolated subgroups of G containing A.

If $\leqq$ is an order relation on G, D is a subgroup of G, and $x \in G$, then we write $x < D$ $(x > D)$ to mean that $x < d$ $(x > d)$ for every $d \in D$. If $D \subset C$ are convex subgroups of an ordered group with the property that no convex subgroup of G lies strictly between D and C, then $D \prec C$ is a jump in the system of all convex subgroups of G.

3. Proofs. It is well known that if a relation $\leqq$ determines an order on a group G, then the set Σ of all convex subgroups of G forms a chain with respect to set inclusion, including $\{1\}$ and G, and is closed under arbitrary unions, intersections, and conjugations by elements of G. Also, for any jump $D \prec C$

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in Σ, the normalizer of D in G coincides with the normalizer of C in G, and C/D is order-isomorphic to a subgroup of R^+, the additive group of real numbers, so that any order-preserving automorphism of C/D is essentially a multiplication by a positive real. Thus the automorphism either fixes only the identity element of C/D or it is the trivial automorphism. In particular, if G is an ordered locally nilpotent group, then every inner automorphism of G induces the trivial automorphism on C/D. To prove this, suppose, if possible, that for some $\bar{g} \in G/D$ and for some $\bar{c} \in C/D$, $[\bar{g}, \bar{c}] \neq \bar{1}$. Let $\bar{K} = \langle \bar{g}, \bar{c} \rangle$. Then under the restriction to $\bar{K}$ of the full order on G/D, $\bar{K} \cap \bar{C}$ is a normal convex Archimedean subgroup of $\bar{K}$. Since $\bar{K}$ is a finitely generated nilpotent group, $\bar{K} \cap \bar{C} \cap Z(\bar{K}) \neq \bar{1}$ so that $\bar{K} \cap \bar{C} \subseteq Z(\bar{K})$ giving us the required contradiction. We record this result as follows.

LEMMA 1. *If G is a torsion-free locally nilpotent group and $D \prec C$ is a jump in the family of convex subgroups with respect to some order on G, then $[G, C] \subseteq D$.*

Proof of Theorem 2. Let G be an ordered supersolvable group with $1 = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n = G$ the family of convex subgroups under some order on G. By Theorem 1, C_i is normal in G for all $i = 1, \ldots, n$. G also has an invariant cyclic series $1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_m = G$. Let i be the smallest integer for which $G_i \cap C_1 \neq 1$. Then $1 \neq G_i \cap C_1$ is normal in G and $G_i \cap C_1 \cong G_{i-1}(G_i \cap C_1)/G_{i-1}$. Also $G_i \cap C_1$ is infinite since G is torsion-free. Thus $G_i \cap C_1$ is an infinite cyclic, normal subgroup of G, and therefore lies in the centre of G since the automorphisms of $G_i \cap C_1$ induced by elements of G by conjugation are order-preserving and hence trivial. Thus $Z(G) \cap C_1 \neq 1$ and we conclude that $C_1 \subseteq Z(G)$. Now $l(G/C_1) < l(G)$, and G/C_1 is an ordered supersolvable group. By induction on $l(G)$, we have that G/C_1 is nilpotent. But $C_1 \subseteq Z(G)$, whence G is nilpotent.

We now construct an example of an ordered polycyclic group with trivial centre. Let H be the subgroup of R^+ given by

$$H = \langle a, b \rangle, \qquad \text{where } a = 1 \text{ and } b = \tfrac{1}{2}(1 + \sqrt{5}).$$

Let θ be an automorphism of H given by $a^\theta = b$; $b^\theta = a + b$. It is easily seen that the automorphism θ is the same as multiplication by $\frac{1}{2}(1 + \sqrt{5})$ for

$$1 \cdot \tfrac{1}{2}(1 + \sqrt{5}) = \tfrac{1}{2}(1 + \sqrt{5}); \quad \tfrac{1}{2}(1 + \sqrt{5}) \cdot \tfrac{1}{2}(1 + \sqrt{5}) = 1 + \tfrac{1}{2}(1 + \sqrt{5}).$$

Thus θ is an order-preserving automorphism of H. Let G be the semidirect product of H by $\langle \theta \rangle$, so that G is an ordered polycyclic group with G, H, and the identity group as the convex subgroups with respect to the order on G with positive cone $P(G) = P(H) \cup \{x \mid x \in (G \backslash H) \text{ and } x \in \theta^k H, k \geqq 1\}$, where $P(H)$ denotes the set of all positive real numbers in H. For any $h \neq 1$ in H, $h = ma + nb$ for some integers m and n. $\theta^{-1} h \theta = na + (m + n)b \neq ma + nb$ unless $m - n = 0$. If $g \in G \backslash H$, then $a^{-1} g a \neq g$ since $a^{-1} \theta^r a \neq \theta^r$ for any $r \neq 0$. Thus $Z(G) = 1$. Note also that $[a, \theta] = b - a$ and $[a, \theta^2] = b$

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so that $[\theta, a][a, \theta^2] = a$. Thus G has no descending central system. This completes the proof of Theorem 2.

It follows by a straightforward argument that under any order on G, the set of convex subgroups consists of G, H, and the identity group. H is therefore an absolutely convex subgroup of G. There are only four different orders that can be defined on G. These are obtained by interchanging either or both of the sets of positive elements and negative elements in H and G/H.

If $G \in L\mathfrak{X}$ and Σ is the set of convex subgroups under some order $\leqq$ on G with positive cone $P(G)$, and $D \prec C$ is a jump in Σ, then D and C are both normal in G by Theorem 1, and we can define a different order $P_C(G)$ on G by

$$P_C(G) = (P(G) \cap D) \cup \{x \mid x \in C\backslash D \text{ and } x < D\} \\ \cup \{x \mid x \in G\backslash C \text{ and } x > C\}.$$

It is clear that $P_C(G) \neq P(G)$. In order to show that $P_C(G)$ defines a full order on G, we must show that

(i) $P_C(G) \cap P_C^{-1}(G) = 1$,
(ii) $P_C(G) \cup P_C^{-1}(G) = G$,
(iii) $P_C(G)$ is a semigroup, and
(iv) $P_C(G)$ is invariant under conjugation by elements of G.

Any element $y \neq 1$ in G satisfies precisely one of the following:

(I) $y > C$,
(II) $y \in C$ and $y < D$,
(III) $y \in D$ and $y > 1$,
(IV) $y < C$,
(V) $y \in C$ and $y > D$,
(VI) $y \in D$ and $y < 1$.

$y \in P_C(G)$ if (I), (II), or (III) holds and $y \in P_C^{-1}(G)$ if (IV), (V), or (VI) holds. This verifies (i) and (ii). For any $g \in G$, $g^{-1}yg$ satisfies (I), (II), or (III) if and only if y does so, since D and C are both normal in G. This yields (iv). Finally, let y and z be any two elements in $P_C(G)$, and assume, without loss of generality, that $y \geqq z$. Then by inspection of the possible cases for y and z, it follows that $yz \in P_C(G)$. This proves the assertion that $P_C(G)$ defines a full order on G. If the set Σ is infinite, then using the above construction we obtain infinitely many different orders on G, one for each jump $D \prec C$ in Σ. Thus a necessary condition that an $L\mathfrak{X}$ group G admit only finitely many different full orders is that the number of convex subgroups under any order on G be finite. But this together with Theorem 1 imply that G' is nilpotent.

Now let G be an $L\mathfrak{X}$ group admitting only finitely many different full orders. Then G' is nilpotent as was shown above. Let J be the isolator of G', so that G/J is a torsion-free abelian group. Therefore, there exists an order on G with J as a convex subgroup. If the rank of G/J is greater than 1, then by Teh's theorem [**5**] there are uncountably many different orders of G/J and each of these gives a different order on G, contradicting the hypothesis that G admits only finitely many different orders. Thus we conclude that G/J is locally cyclic. This implies that J is absolutely convex, for if $\leqq$ denotes an order on G with

$$1 = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{n-1} \subseteq C_n = G$$

the corresponding set of convex subgroups of G, then G/C_{n-1} is torsion-free abelian. Hence $G/(J \cap C_{n-1})$ is torsion-free abelian, and $G/(J \cap C_{n-1})$ is locally cyclic for the same reason as G/J is locally cyclic. Since J and C_{n-1} are both isolated, we conclude that $J = C_{n-1}$. Thus J is absolutely convex. Now let N be the locally nilpotent radical of G (i.e., N is the unique largest normal locally nilpotent subgroup of G), and let $I(N)$ be the isolator of N in G. Clearly, $I(N) \supseteq C_{n-1} = J$ and $[I(N), G] \subseteq C_{n-1}$. We assert that

$$[I(N), C_i] \subseteq C_{i-1} \quad \text{for all } i = 1, \ldots, n,$$

for let $s < n$ be the smallest integer such that $[I(N), C_s] \not\subseteq C_{s-1}$. Then

$$[C_{s-1}, \underbrace{I(N), \ldots, I(N)}_{s-1}] = 1.$$

Also $[C_s, C_s] \subseteq C_{s-1}$ and $C_s \subseteq I(N)$ so that

$$[\underbrace{C_s, C_s, \ldots, C_s}_{s+1}] = 1,$$

whence $C_s \subseteq N$. Now restrict the order on G to N, making N an ordered group with convex subgroups $N \cap C_i$, $i = 0, 1, \ldots, n$. Since $C_{s-1} \subseteq C_s \subseteq N$, $C_{s-1} \prec C_s$ is a jump. Since N is locally nilpotent, $[N, C_s] \subseteq C_{s-1}$ by Lemma 1. Since $[I(N), C_s] \not\subseteq C_{s-1}$, $[x, c] \notin C_{s-1}$ for some $x \in I(N)$ and some $c \in C_s$. But $I(N)/N$ is periodic, so that $[x^r, c] \in C_{s-1}$ for some positive integer r. But G/C_{s-1} is an ordered group and, by a result of Neumann [**2**, p. 2], $[x^r, c] \in C_{s-1}$ implies $[x, c] \in C_{s-1}$. This contradicts our hypothesis and, therefore, $I(N) = N$ is the locally nilpotent radical of G. Note that, since there exists an order on N whose corresponding family of convex subgroups is finite, N is actually nilpotent. We now summarize our results as follows.

LEMMA 2. *If an $L\mathfrak{X}$ group G admits only finitely many different orders, then the Fitting subgroup N of G is isolated and contains G'. Also $G/I(G')$ is locally cyclic, where $I(G')$ is the isolator of G' in G.*

Note that if $I(G') \neq N$, then $N = G$ and G is nilpotent. But in this case, $G/I(G')$ is locally cyclic only if G is abelian of rank one. Thus, if G is non-abelian, then $N = I(G')$. The last assertion of Theorem 3 is immediate.

As a concluding remark, let us note that we have shown in the proof of Theorem 3 that any $L\mathfrak{X}$ group G admitting only finitely many different orders has the properties that the locally nilpotent radical N of G coincides with $I(G')$ and that $G/I(G')$ is non-trivial, unless G is abelian of rank one. Therefore, any non-abelian, torsion-free, locally nilpotent group admits infinitely many different orders.

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