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Abstract

In a previous paper, the potential automorphy of certain Galois representations to GL_n for n even was established, following the work of Harris, Shepherd–Barron and Taylor and using the lifting theorems of Clozel, Harris and Taylor. In this paper, we extend those results to $n=3$ and $n=5$, and conditionally to all other odd n. The key additional tools necessary are results which give the automorphy or potential automorphy of symmetric powers of elliptic curves, most notably those of Gelbert, Jacquet, Kim, Shahidi and Harris.

1. Introduction

In [Bar08], it is established that certain even-dimensional Galois representations become automorphic when one makes a suitably large totally real field extension. In this paper we have two main aims. The first is to extend those results to three- and five-dimensional representations; in particular, we aim to prove the following theorem, which refers to a constant $C(n, N)$ which will be defined in the proof.

Theorem 1. *Suppose that F/F_0 is a Galois extension of CM fields*[1] *and that $n=3$ or 5. Suppose that $N \geqslant n+6$ an even integer, and suppose that $l > C(n, N)$ is a prime which is unramified in F and $l \equiv 1 \bmod N$. Let v_q be a prime of F above a rational prime $q \neq l$ such that $q \nmid N$. Let $\mathcal{L}$ be a finite set of primes of F not containing primes above lq.*

Suppose that we are given a representation

$$r : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_n(\mathbb{Z}_l)$$

with the following properties (we will write $\bar{r}$ for the semisimplification of the reduction of r):

(i) *r ramifies only at finitely many primes;*
(ii) *$r^c \cong r^\vee \epsilon_l^{1-n}$, with Bellaiche–Chenevier sign*[2] *$+1$;*

[1] A number field is called a *CM field* if it may be expressed as a degree 2, totally complex extension of a totally real field.

[2] More concretely, we can think of the isomorphism $r^c \cong r^\vee \epsilon_l^{1-n}$ as giving us a pairing $\langle *, * \rangle$ on $(\mathbb{Z}_l)^n$ satisfying $\langle r(\sigma)v_1, r({}^c\sigma)v_2 \rangle = \epsilon_l^{1-n}(\sigma)\langle v_1, v_2 \rangle$ for each $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and $v_1, v_2 \in (\mathbb{Z}_l)^n$. If r is in addition assumed to be absolutely irreducible, this pairing will either be symmetric or antisymmetric; and whether it is symmetric or antisymmetric turns out to only depend on r. We define the *sign* of r to be $+1$ if the pairing is symmetric and -1 if it is antisymmetric. This is the appropriate generalization of an 'odd' two-dimensional Galois representation over a totally real field; just as even representations are rather badly behaved, so are representations with sign -1, so we would not expect to get good results for such representations.

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(iii) *for each prime* $\mathfrak{w}|l$ *of* F, $r|_{\mathrm{Gal}(\overline{F}_{\mathfrak{w}}/F_{\mathfrak{w}})}$ *is crystalline with Hodge–Tate numbers* $\{0, 1, \ldots, n-1\}$;

(iv) r *is unramified at all the primes of* $\mathcal{L}$;

(v) $(\det \bar{r})^2 \cong \epsilon_l^{n(1-n)} \mod l$;

(vi) *let* r' *denote the extension of* r *to a continuous homomorphism* $\mathrm{Gal}(\overline{F}/F^+) \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ *as described in [CHT05, § 1], then* $\bar{r}'(\mathrm{Gal}(\overline{F}/F(\zeta_l))$ *is 'big' in the sense of 'big image';*

(vii) $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ *does not contain* $F(\zeta_l)$;

(viii) $\bar{r}$ *satisfies, for each prime* $\mathfrak{w}|l$ *of* F,

$$\bar{r}|_{I_{F_w}} \cong 1 \oplus \epsilon_l^{-1} \oplus \cdots \oplus \epsilon_l^{1-n};$$

(ix) $r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})}^{\mathrm{ss}}$ *and* $\bar{r}|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})}$ *are unramified, with* $r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})}^{\mathrm{ss}}$ *having Frobenius eigenvalues* $1, (\#k(v_q)), \ldots, (\#k(v_q))^{n-1}$.

Then there exists a CM field F' *which is Galois over* F_0 *and linearly independent from* $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ *over* F. *Moreover, all primes of* $\mathcal{L}$ *and all primes of* F *above* l *are unramified in* F'. *Finally, there is a prime* w_q *of* F' *over* v_q *such that* $r|_{\mathrm{Gal}(\overline{F}/F')}$ *is automorphic of weight zero and type* $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$.

The second aim is to make, conditional on some conjectures of Harris and co-workers which are expected to become theorems by 2010, a further extension to all remaining odd n, using the work of Harris [Har07] which establishes the potential automorphy of all odd-dimensional symmetric powers of elliptic curves subject to these 'expected theorems'. To properly state these expected theorems and set them in the appropriate context takes most of the first section of [Har07], so a restatement of them will not be given here, but simply note that they can be found as expected Theorems 1.2, 1.4 and 1.7 there. For the remainder of this paper we shall refer to these statements as 'the expected theorems of [Har07]'.

We then have the following theorem.

Theorem 2. *Suppose that we admit the expected theorems of [Har07], and let* F/F_0 *be a Galois extension of CM fields and* n *an odd integer. Suppose further that* $N \geqslant n+6$ *is an even integer, that* $l > C(n, N)$ *is a prime which is unramified in* F, *and that* $l \equiv 1 \mod N$. *Let* $\mathcal{L}$ *be a finite set of primes of* F *not containing primes above* lq.

Suppose that we are given a representation

$$r : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Z}_l)$$

with the following properties:

(i) r *ramifies only at finitely many primes;*

(ii) $r^c \cong r^\vee \epsilon_l^{1-n}$, *with Bellaiche–Chenevier sign* $+1$;

(iii) *for each prime* $\mathfrak{w}|l$ *of* F, $r|_{\mathrm{Gal}(\overline{F}_{\mathfrak{w}}/F_{\mathfrak{w}})}$ *is crystalline with Hodge–Tate numbers* $\{0, 1, \ldots, n-1\}$;

(iv) r *is unramified at all the primes of* $\mathcal{L}$;

(v) $(\det \bar{r})^2 \cong \epsilon_l^{n(1-n)} \mod l$;

(vi) *let* r' *denote the extension of* r *to a continuous homomorphism* $\mathrm{Gal}(\overline{F}/F^+) \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ *as described in [CHT05, § 1], then* $\bar{r}'(\mathrm{Gal}(\overline{F}/F(\zeta_l))$ *is 'big' in the sense of 'big image';*

(vii) $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ *does not contain* $F(\zeta_l)$;

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(viii) *$\bar{r}$ satisfies, for each prime $\mathfrak{w}|l$ of F,*

$$\bar{r}|_{I_{F_w}} \cong 1 \oplus \epsilon_l^{-1} \oplus \cdots \oplus \epsilon_l^{1-n}.$$

Then there exists a CM field F' which is Galois over F_0 and linearly independent from $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ over F. Moreover, all primes of $\mathcal{L}$ and all primes of F above l are unramified in F'. Finally, $r|_{\operatorname{Gal}(\overline{F}/F')}$ is automorphic of weight zero.

It is also worth noting that it is possible to use the freedom to vary N to deduce corollaries where the congruence condition on l is removed and replaced with a condition that l avoid a certain set of primes of Dirichlet density zero. See Appendix A for the details. While these are strict weakenings of the theorems above, they may be useful for applications.

COROLLARY 3. *Suppose that n is 3 or 5. There is a set Λ of rational primes whose complement has Dirichlet density zero and a function $N : \Lambda \to \mathbb{Z}$ with the following property. Suppose that F/F_0 is a Galois extension of CM fields and $l \in \Lambda$ is a prime which is unramified in F. Let v_q be a prime of F above a rational prime $q \neq l$, and $q \nmid N(l)$. Let $\mathcal{L}$ be a finite set of primes of F not containing primes above lq.*

Finally, suppose that we are given a representation $r : \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_n(\mathbb{Z}_l)$ enjoying the properties (i)–(ix) of Theorem 1, and such that property (vi) is 'robust' in the sense that it remains true when $\bar{r}$ is restricted to any subgroup of $\operatorname{Gal}(\overline{F}/F)$ with cyclic quotient.

Then there is a totally real field F' which is Galois over F_0 and linearly independent from $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ over F. Moreover, all primes of $\mathcal{L}$ and all primes of F above l are unramified in F'. Finally, there is a prime w_q of F' over v_q such that $r|_{\operatorname{Gal}(\overline{F}/F')}$ is automorphic of weight zero and type $\{\operatorname{Sp}_n(1)\}_{\{w_q\}}$.

COROLLARY 4. *Suppose that we admit the expected theorems of [Har07], and let n be an integer. There is a set Λ of rational primes whose complement has Dirichlet density zero and a function $N : \Lambda \to \mathbb{Z}$ with the following property. Suppose that F/F_0 is a Galois extension of CM fields and $l \in \Lambda$ is a prime which is unramified in F. Let $\mathcal{L}$ be a finite set of primes of F not containing primes above l.*

Finally, suppose that we are given a representation $r : \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_n(\mathbb{Z}_l)$ enjoying the properties (i)–(viii) of Theorem 2, and such that property (vi) is 'robust' in the sense that it remains true when $\bar{r}$ is restricted to any subgroup of $\operatorname{Gal}(\overline{F}/F)$ with cyclic quotient.

Then there is a totally real field F' which is Galois over F_0 and linearly independent from $\overline{F}^{\ker \operatorname{ad} \bar{r}}$ over F. Moreover, all primes of $\mathcal{L}$ and all primes of F above l are unramified in F'. Finally, $r|_{\operatorname{Gal}(\overline{F}/F')}$ is automorphic of weight zero.

We close the introduction with a simplified overview of the methods necessary to prove the two theorems above, with particular attention given to what is novel in the proofs. The basic structure of all potential automorphy proofs follows the seminal work of Taylor [Tay03]. Two main ingredients are necessary. The first is a good supply of representations with certain properties, most notably that they are known *a priori* to be automorphic. The second is a family $\mathfrak{F}$ of varieties whose cohomology is 'very flexible'. By *very flexible*, we mean that given specified l-adic and l'-adic representations r and r', and subject to certain conditions, we can find (over some suitably large totally real field) an element V of the family whose mod l cohomology looks like the residual representation of r, and whose mod l' cohomology looks like the residual representation of r'.

These two ingredients are then applied to prove the theorem as follows. We then apply the very flexible cohomology property taking r to be the Galois representation which we would like to show is potentially modular and r' to be one of our good supply of representations that are already known to be modular. Over the field of definition of this variety (which may well be a very large extension of the field we started with) r and the cohomology of V (respectively r' and the cohomology of V) agree mod l (respectively l'). We can then apply a modularity lifting theorem twice: once, to deduce that the cohomology of V is modular (since it agrees mod l' with r'), then again to deduce that r is modular (since it agrees mod l with the cohomology of V).

In practice, the actual argument involves a few more steps, since one has to accommodate various conditions that the lifting theorems have, most notably conditions that the representations we work with must be Steinberg at certain places. (We say that an n dimensional automorphic representation π of $\mathrm{GL}_n(A_K)$ *is Steinberg* at a finite place v if the local representation π_v is an unramified twist of the Steinberg representation of $\mathrm{GL}_n(K_v)$.) For an overview explaining slightly more of the details, we refer the reader to the introduction to [Bar08]; this, however, is more-or-less the flavor of the argument.

The reason that the earlier paper had to restrict itself to even-dimensional representations was not a restriction in the part of the proof constructing the family with very flexible cohomology, but rather in the supply of 'good' Galois representations r' already known to be modular. In particular, [Bar08] followed the paper [HST06] in taking these r' to be induced from a character of a CM extension of $\mathbb{Q}$. Such an r' must necessarily be even-dimensional, and so the results are automatically restricted to even-dimensional representations.

The principal new idea in the present paper is to use a different source of 'already automorphic' Galois representations; in particular, we will use the symmetric powers of the cohomology of a suitably chosen elliptic curve. For $n=3$ and $n=5$, we will be able to take the symmetric square and fourth power, which are known unconditionally to be automorphic, and cuspidal in the appropriate cases, through work of Gelbert and Jacquet and Kim and Shahidi respectively [GJ78, Kim02, KS02]. For larger odd n, we use the result of [Har07] which, as has already been mentioned, proves, subject to the expected theorems, that all remaining symmetric powers are potentially automorphic. (Potential automorphy, rather than true automorphy, is good enough for our purposes.)

The organization of the remainder of this paper is as follows. In § 2 we explain the very simple modifications to the arguments of [Bar08] that are necessary to extend the construction of motives with very flexible cohomology to the odd-dimensional case. In § 3 we briefly review the literature [GJ78, Kim02, KS02] on symmetric powers of two-dimensional Galois representations and the results of Harris in [Har07], and use these results to construct the supply of 'good' representations which we need. Finally, in § 4 we put these pieces together to get a proof of the main theorem.

2. Extending the analysis of the Dwork family

In this section we describe how the geometric arguments in [Bar08, § 2] can be extended to cover the case of odd n. The changes necessary are very straightforward. We follow the first page of [Bar08, § 2], which sets up the notation, completely unaltered. The first difference between the analysis here and that in [Bar08] comes when we reach the point just before Proposition 4 where a choice of a certain even-dimensional piece of the cohomology $\mathrm{Prim}_{l,[v]}$ is made, corresponding to a choice of an element $[v] \in (\mathbb{Z}/N\mathbb{Z})_0/\langle W\rangle$. At this point, we assume that we have an *odd*

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number n, $n>1$; we then write $n=2k+1$ and we choose a different v, viz

$$v=(0,0,\ldots,0,1,k+2,k+3,\ldots,N/2-2,N/2,N/2+1,\ldots,N-k-3,N-k-2,N-2),$$

where we include every number once, except that we *omit* the ranges $2,3,\ldots,k+1$ and $N-k-1,\ldots,N-3$, together with the singletons $N/2-1$ and $N-1$, and where the number of zeroes at the beginning is $n+1$, calculated to ensure that there are N numbers in total. Note that these numbers add up to 0 mod N. Note also that the ranges 'make sense' as long as $N>n+3$; for instance, if $n=3$, $N=8$, we take $v=(0,0,0,0,1,4,5,6)$.

We then have the following analogues of [Bar08, Proposition 3, Corollary 4 and Lemma 6]. (As usual, we write $\mathrm{Prim}_{l,[v],t}$ for the stalk of $\mathrm{Prim}_{l,[v]}$ at t, etc.)

PROPOSITION 5. *We have the following facts about the varieties Y_t and the sheaves $\mathrm{Prim}[l]_{[v]}$, $\mathrm{Prim}_{l,[v]}$ and $\mathcal{F}^i[l]$. Let F be a number field. Recall that we are assuming that $l\equiv 1 \bmod N$ throughout.*

(i) *If $t\in T_0^{(l)}(F)$ and $\mathfrak{q}$ is a place of F such that $v_{\mathfrak{q}}(1-t^N)=0$, then Y_t has good reduction at $\mathfrak{q}$.*

(ii) *Suppose that $t\in T_0^{(l)}(F)$. The Galois representation*

$$\mathrm{Prim}_{l,[v],t}:\mathrm{Gal}(\overline{F}/F)\to \mathrm{GL}_n(\mathbb{Q}_l)$$

satisfies $\mathrm{Prim}^c_{l,[v],t}\cong \mathrm{Prim}^\vee_{l,[v],t}\epsilon_l^{2-N}$. Similarly, $\mathrm{Prim}[l]^c_{[v],t}\cong \mathrm{Prim}[l]^\vee_{[v],t}\epsilon_l^{2-N}$. (Indeed these isomorphisms patch for different t to give a sheaf isomorphism.)

(iii) *The sheaf $\mathrm{Prim}_{l,[v]}$ has rank n. There is a tuple $\vec{h}=(h(\sigma))_{\sigma\in\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ such that the Hodge–Tate numbers of $\mathrm{Prim}_{l,[v]}$ at the embedding σ are $\{h(\sigma),h(\sigma)+1,\ldots,h(\sigma)+n-1\}$.*

(iv) *Let $\vec{h}$ continue to denote the tuple defined in the previous part. Suppose $\mathfrak{w}|l$, and let $\sigma\in\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)$ denote the corresponding embedding. Then*

$$\mathrm{Prim}_{l,[v],0}|_{I_{\mathfrak{w}}}\cong \epsilon_l^{-h(\sigma)}\oplus\epsilon_l^{-h(\sigma)-1}\oplus\cdots\oplus\epsilon_l^{1-h(\sigma)-n},$$

and

$$\mathrm{Prim}[l]_{[v],0}|_{I_{\mathfrak{w}}}\cong \epsilon_l^{-h(\sigma)}\oplus\epsilon_l^{-h(\sigma)-1}\oplus\cdots\oplus\epsilon_l^{1-h(\sigma)-n}.$$

(v) *Let $\mathfrak{q}$ be a prime of F above a rational prime which does not divide N. If $\lambda_{\mathfrak{q}}\in T_0^{(l)}(F_{\mathfrak{q}})$ has $v_{\mathfrak{q}}(\lambda_{\mathfrak{q}})<0$, then $(\mathrm{Prim}_{l,[v],\lambda_{\mathfrak{q}}})^{\mathrm{ss}}$ is unramified, and $(\mathrm{Prim}_{l,[v],\lambda_{\mathfrak{q}}})^{\mathrm{ss}}(\mathrm{Frob}_{\mathfrak{q}})$ has eigenvalues*

$$\{\alpha,\alpha\#k(\mathfrak{q}),\alpha(\#k(\mathfrak{q}))^2,\ldots,\alpha(\#k(\mathfrak{q}))^{n-1}\}$$

for some α.

(vi) *Let $\mathfrak{q}$ be a prime of F above a rational prime which does not divide N. If $\lambda_{\mathfrak{q}}\in T_0^{(l)}(F_{\mathfrak{q}})$ has $v_{\mathfrak{q}}(\lambda_{\mathfrak{q}})<0$ and $l|v_{\mathfrak{q}}(\lambda_{\mathfrak{q}})$, then $(\mathrm{Prim}[l]_{[v],\lambda_{\mathfrak{q}}})$ is unramified (even without semisimplification).*

(vii) *The monodromy of $\mathrm{Prim}_{l,[v]}$ is Zariski dense in $\{A\in\mathrm{GL}_n \mid \det A=\pm 1\}$.*

Proof. The proof of the corresponding proposition of [Bar08] only depends on v insofar as it requires v to possess the following properties:

- v satisfies point (1) of the equivalent conditions in [Kat07, Lemma 10.1], viz that the value 0 occurs more than once and no other value does;
- $-v$ is not a permutation of v;
- v omits precisely n congruence classes mod N.

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Since our new v is readily seen to possess these properties too, we can carry over the proof unchanged. $\square$

Corollary 6. *There is a constant $C(n, N)$ such that if M is an integer divisible only by primes $p > C(n, N)$ and if $t \in T_0^{(M)}$, then the map*

$$\pi_1(T_0^{(M)}, t) \to \mathrm{GL}_n(\mathrm{Prim}[M]_{[v],t})$$

surjects onto $\mathrm{SL}_n^{\pm}(\mathrm{Prim}[M]_{[v],t})$. (Here $SL_n^{\pm}(\mathrm{Prim}[M]_{[v],t})$ denotes the group of automorphisms of $\mathrm{Prim}[M]_{[v],t}$ with determinant ± 1.) (We may, and shall, additionally assume that $C(n, N) > n$.)

Proof. The argument is identical to the proof of [Bar08, Corollary 4] or [HST06, Lemma 1.11], deducing the result from part (7) of the previous proposition and from [MVW84, Theorem 7.5 and Lemma 8.4] (or [Nor87, Theorem 5.1]). $\square$

Lemma 7. *Define a character $G_{\mathbb{Q}(\mu_N)} \to \mathbb{Q}_l^{\times}$:*

$$\phi_l := \Lambda_{v,W} \prod_i (\lambda^{\mathrm{can}}_{G_{\mathbb{Q}(\mu_N)}}(\{\chi_i\}, \{1\}))^2$$

(where $\Lambda_{v,W}$ is the Galois character defined in [Kat07, Theorem 5.3] and the χ_i are the maps $\mu_N \to \mu_N$ naturally associated to the elements $v_i \in \mathbb{Z}/N/\mathbb{Z}$); we have that

$$(\det \mathrm{Prim}_{l,[v],t=2})^2 = \phi_l^{2n} \epsilon_l^{n(1-n)}.$$

Proof. The proof is exactly as for [Bar08, Lemma 7]. $\square$

We can use this result to define a notation which will be useful to us in the remainder of this paper. Looking at the Hodge–Tate number of either side of the equation above at a prime $\mathfrak{l}$ over l, and writing $\mathrm{HT}_{\mathfrak{l}}(\phi_l)$ for the Hodge–Tate number of ϕ_l at that place, we get

$$2 \times (\vec{h}(\mathfrak{l}) + (\vec{h}(\mathfrak{l}) + 1) + \cdots + (\vec{h}(\mathfrak{l}) + n - 1)) = 2n\,\mathrm{HT}_{\mathfrak{l}}(\phi_l) + n(n-1),$$
$$(2\vec{h}(\mathfrak{l}) + n - 1)n = 2n\,\mathrm{HT}_{\mathfrak{l}}(\phi_l) + n(n-1),$$
$$(2\vec{h}(\mathfrak{l}))n = 2n\,\mathrm{HT}_{\mathfrak{l}}(\phi_l),$$

and we deduce that $\mathrm{HT}_{\mathfrak{l}}(\phi_l) = \vec{h}(\mathfrak{l})$. Thus we can use twisting by ϕ_l to shift the Hodge–Tate numbers of an arbitrary representation by $\vec{h}$.

Definition 8. Given an l-adic Galois representation r, we will write $r(-\vec{h})$ for the twist of r by the character ϕ_l introduced above, and $r(\vec{h})$ for the twist by the inverse. (Thus, for example, the Hodge–Tate numbers of $r(\vec{h})$ are those of r shifted by *minus* $\vec{h}$.)

We can finally deduce an analogue of [Bar08, Proposition 7]. (We have slightly changed the notation.)

Proposition 9. *The family Y_t and the piece of its cohomology corresponding to $\mathrm{Prim}_{l,[v],t}$ have the following property.*

Suppose K'/K is a Galois extension of CM fields, with totally real subfields K'^+, K^+, n is a positive odd integer, $l_1, l_2 \ldots l_r$ are distinct primes which are unramified in K, and that we are given residual representations

$$\bar{\rho}_i : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F}_{l_i}).$$

Suppose further that we are given $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_s$, distinct primes of K above rational primes $q_1, \ldots, q_s$ respectively, and $\mathcal{L}$ a set of primes of F not including the $\mathfrak{q}_j$ or any primes above

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the l_i. Suppose that each q_j satisfies $q_j \nmid N$. Finally, suppose that the following conditions are satisfied for each i.

(i) $l_i > C(n, N)$.

(ii) $l_i \equiv 1 \bmod N$.

(iii) *$\bar{\rho}_i$ is unramified at each prime of $\mathcal{L}$ and the l_k, $k \neq i$.*

(iv) *For each prime $\mathfrak{w}$ above l_i, we have that*

$$\bar{\rho}_i|_{I_{\mathfrak{w}}} \cong 1 \oplus \epsilon_{l_i}^{-1} \oplus \cdots \oplus \epsilon_{l_i}^{1-n}.$$

(v) *We have that there exists a polarization $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \epsilon_{l_i}^{1-n}$; given this, we can associate to $\bar{\rho}_i$ a sign in the sense of Bellaiche–Chenevier and we require that this sign is $+1$. Finally, we require that $(\det \bar{\rho}_i)^2 \cong \epsilon_{l_i}^{n(1-n)}$.*

Then we can find a CM field F/K, linearly disjoint from K'/K, a finite-order character $\chi_i : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{Q}_{l_i}$ for each i, and a $t \in F$ such that:

(i) *all primes of K above the $\{l_i\}_{i=1,\ldots,r}$ and all the $\mathcal{L}$ are unramified in F;*

(ii) *for all i, Y_t has good reduction at each prime lying above l_i, and each prime above the primes of $\mathcal{L}$;*

(iii) *for all i and $\mathfrak{w}|l_i$, $\mathrm{Prim}_{\mathfrak{w},t}(\vec{h}) \otimes \chi_i$ is crystalline with Hodge–Tate numbers $\{0, 1, \ldots, n-1\}$;*

(iv) *for each $\mathfrak{Q}$ above some $\mathfrak{q}_j$, we have that $(\mathrm{Prim}_{l_i,[v],t})^{\mathrm{ss}}$ and χ_i are unramified at $\mathfrak{Q}$, with $(\mathrm{Prim}^{\mathrm{ss}}_{l_i,[v],t}(\vec{h}) \otimes \chi)(\mathrm{Frob}_{\mathfrak{Q}})$ having eigenvalues*

$$\{1, \#k(\mathfrak{Q}), \#k(\mathfrak{Q})^2, \ldots, \#k(\mathfrak{Q})^{n-1}\};$$

(v) $\mathrm{Prim}[l_i]_{[v],t}(\vec{h}) \otimes \bar{\chi}_i = \bar{\rho}_i$ *for all i.*

Proof. The proof is identical to that of [Bar08, Proposition 7], except that all concerns about the sign of the pairing determinant are eliminated, since for odd-dimensional representations one can turn a pairing with square determinant into one with non-square determinant, and *vice versa*, by multiplying the whole pairing by a non-square. □

3. Review of certain automorphy and potential automorphy theorems

As was mentioned in the introduction, the key difficulty in proving a potential automorphy theorem for odd-dimensional representations is finding a good starting point: a supply of odd-dimensional representations known to be (at least potentially) modular. We will use two sources here. On the one hand, we have the functoriality theorems for the symmetric square and fourth power of a two-dimensional representation, as proved by Gelbert, Jacquet, Shahidi and Kim. (The symmetric square will be three-dimensional and the fourth power will be five-dimensional.) Let us review these theorems.

THEOREM 10 (Gelbert and Jacquet; [GJ78, part of Theorem 9.3]). *Let F be a number field, and let π be a unitary irreducible representation of $\mathrm{GL}_2(\mathbb{A}_F)$ which is automorphic cuspidal. Assume that for any character χ of $\mathbb{A}^\times/F^\times$, $\chi \neq 1$, we have that π and $\pi \otimes \chi$ are inequivalent. Then $\mathrm{Sym}^2\pi$ is automorphic cuspidal.*[3]

[3] $\mathrm{Sym}^2\pi$ may be defined by putting together local pieces; the local definition comes from the local Langlands correspondence of [HT01].

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Theorem 11 (Kim [Kim02, Theorem B]). *Suppose that F is a number field, and π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then $\mathrm{Sym}^4\pi$ is an automorphic representation*[4] *of $\mathrm{GL}_5(\mathbb{A}_F)$. If $\mathrm{Sym}^3(\pi)$ is cuspidal, then $\mathrm{Sym}^4(\pi)$ is either cuspidal or induced from cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ and $\mathrm{GL}_3(\mathbb{A}_F)$.*

Theorem 12 (Kim and Shahidi [KS02, Theorem 3.3.7]). *Suppose that F is a number field, and π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then $\mathrm{Sym}^4\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_5(\mathbb{A}_F)$, unless:*

- *π is monomial (this is equivalent to π being of dihedral type);*
- *π is not monomial and $\mathrm{Sym}^3\pi$ is not cuspidal; this occurs when there exists a non-trivial grössencharacter μ such that $\mathrm{Ad}(\pi) \sim \mathrm{Ad}(\pi) \otimes \mu$ (this is equivalent to π being of tetrahedral type);*
- *$\mathrm{Sym}^3\pi$ is cuspidal, but there exists a non-trivial quadratic character η such that $\mathrm{Sym}^3(\pi) \sim \mathrm{Sym}^3(\pi) \otimes \eta$ (this is equivalent to π being of octahedral type).*

Since a representation corresponding to a Galois representation with distinct Hodge–Tate weights will never be dihedral, tetrahedral or octahedral, we may conclude the following.

Corollary 13. *Suppose that F is a CM or totally real field, and π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to an irreducible Galois representation r with Hodge–Tate numbers $\{0, 1\}$ (for example, the cohomology of an elliptic curve), and such that there exists some place v_q of F such that π has type $\{Sp(1)\}_{\{v_q\}}$.*

Then $\mathrm{Sym}^2 r$ and $\mathrm{Sym}^4 r$ are automorphic of type $\{Sp(1)\}_{\{v_q\}}$; that is, there are RAESDC representations,[5] *Steinberg at v_q, viz. $\mathrm{Sym}^2\pi$ and $\mathrm{Sym}^4\pi$, to which the Galois representations $\mathrm{Sym}^2 r$ and $\mathrm{Sym}^4 r$ are attached by the procedure of Harris and Taylor (see [CHT05, Proposition 4.3.1], which is based closely on [HT01, Theorem VII.1.9]).*

Proof. Given the theorems above, we merely check the trivialities that π being regular, being algebraic, being essentially self-dual, and being Steinberg at v_q, imply the same for $\mathrm{Sym}^2\pi$ and $\mathrm{Sym}^4\pi$. □

This is our first source of 'starting points' for proving odd-dimensional cases of potential automorphy. The second source is potential automorphy theorems for odd-dimensional symmetric powers of elliptic curves, as proved by Harris.

Theorem 14 (Harris [Har07, Theorem 4.4]). *Let n be an odd positive integer. Assume the expected theorems of [Har07]. Let $F^{*,+}/F^+$ be an extension of totally real fields, L a quadratic imaginary field, and $\mathcal{L}$ a finite set of places of F^+. There is a field M which depends only on L and n such that whenever $l > C'(n)$ is a rational prime unramified in F^+ and split in M, and E an elliptic curve over F^+ with good reduction at l and the primes of $\mathcal{L}$, then there is a finite totally real Galois extension $F^{\prime,+}/F^+$, linearly disjoint from $F^{*,+}/F^+$ and unramified at primes of $\mathcal{L}$, and a RAESDC automorphic representation π of $\mathrm{GL}_n(F^{\prime,+})$, whose associated Galois representation is isomorphic to $(\mathrm{Sym}^n H^1(E, \mathbb{Z}_l))|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F^{\prime,+})}$. In other words, $\mathrm{Sym}^n H^1(E, \mathbb{Z}_l)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F^{\prime,+})}$ is automorphic.*

(Here $C'(n)$ is a constant depending only on n.)

[4] $\mathrm{Sym}^4\pi$ is defined by putting together local pieces; the local definition comes from local Langlands.

[5] RAESDC stands for regular algebraic, essentially self dual, and cuspidal (for a full definition, see [CHT05, § 4.3, p. 125]).

Remark 15. The field L and the conditions on l do not occur in the statement of [Har07, Theorem 4.4], since they reflect ongoing assumptions throughout that paper. For the convenience of the reader, we will point out exactly where the assumptions on l arise. In the 'set up' in § 2.3, we assume that l splits in L and a certain field $\mathbb{Q}(\chi)$ depending only on L, and assume that $l > 2n + 1$. Then, in applying Corollary 2.5, we additionally require that l is larger than some $C(n)$ and that it splits in a certain cyclotomic extension. Finally, in applying Lemma 3.2 we assume that $l > 4n - 1$. These conditions can be combined, as we do above, by saying that l exceeds some $C'(n)$ and splits in some M.

Remark 16. The set $\mathcal{L}$ of places does not appear in the statement of [Har07, Theorem 4.4], but the field $F'^{,+}$ is chosen by an application of Theorem 2.1 of that paper; this theorem allows us to ensure that the field we choose does not ramify at some finite set of primes. The condition that E must have good reduction at the primes of $\mathcal{L}$ means we can just add the primes of $\mathcal{L}$ to that set.[6] Similarly, the field $F^{*,+}$ does not occur in the statement of Theorem 4.4, but Theorem 2.1 allows us to avoid any fixed field; see Harris' remarks immediately after the statement of Theorem 2.1.

We are now in a position to address the main theorems.

4. Arguments for the main theorems

We will give a proof of both of our main theorems simultaneously; let us restate them in a convenient form to do so. (We have also relabelled the set $\mathcal{L}$ of primes as $\mathcal{L}'$; this will avoid confusion from a clash of notation with the different $\mathcal{L}$ that arise in the statements of the propositions and theorems which we will apply in the proof.)

Theorem (Restatement of Theorems 1 and 2). *Suppose that F/F_0 is a Galois extension of CM fields and that n is an odd integer. Suppose $N \geqslant n + 6$ an even integer. Suppose that $l > C(n, N)$ is a prime which is unramified in F and $l \equiv 1 \bmod N$. Let $\mathcal{L}'$ be a finite set of primes of F not containing primes above lq.*

Suppose that we are given a representation

$$r : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{Z}_l)$$

with the following properties (we will write $\bar{r}$ for the semisimplification of the reduction of r):

- (i) *r ramifies only at finitely many primes;*
- (ii) *$r^c \cong r^\vee \epsilon_l^{1-n}$, with sign $+1$;*
- (iii) *r is unramified at all the primes of $\mathcal{L}'$;*
- (iv) *$(\det \bar{r})^2 \cong \epsilon_l^{n(1-n)} \bmod l$;*
- (v) *let r' denote the extension of r to a continuous homomorphism*

$$\mathrm{Gal}(\overline{F}/F^+) \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

  as described in [CHT05, § 1], then $\bar{r}'(\mathrm{Gal}(\overline{F}/F(\zeta_l))$ is 'big' in the sense of 'big image';
- (vi) *$\overline{F}^{\ker \mathrm{ad}\, \bar{r}}$ does not contain $F(\zeta_l)$;*
- (vii) *for each prime $w|l$ of F, $r|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is crystalline with Hodge–Tate numbers $\{0, 1, \dots, n-1\}$; also*

$$\bar{r}|_{I_{F_w}} \cong 1 \oplus \epsilon_l^{-1} \oplus \cdots \oplus \epsilon_l^{1-n}.$$

[6] We might also have to be careful in our choice of the character χ, so that it is unramified at the primes of $\mathcal{L}$, but this is also easily achieved.

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Suppose further that one of the following holds.

Case X. *We have that $n=3$ or $n=5$. Moreover, there exists v_q, a prime of F above a rational prime $q \neq l$ such that $q \nmid N$, such that $(r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\mathrm{ss}}$ and $\bar{r}|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})}$ are unramified and $(r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\mathrm{ss}}$ has Frobenius eigenvalues $1, (\#k(v_q)), \ldots, (\#k(v_q))^{n-1}$.*

Case Y. *We admit the expected theorems of [Har07].*

Then there is a CM field F' which is Galois over F_0 and linearly independent from $\overline{F}^{\ker \mathrm{ad}\, \bar{r}}$ over F. Moreover, all primes of $\mathcal{L}'$ and all primes of F above l are unramified in F'. Finally, there is a prime w_q of F' over v_q such that $r|_{\mathrm{Gal}(\overline{F}/F')}$ is automorphic of weight zero and type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$.

Proof. Step A. We begin by choosing a quadratic imaginary field L linearly disjoint from F over $\mathbb{Q}$. Let M be the field given in Theorem 14. We then choose a prime l' with the following properties, which is clearly possible:

(A1) l' is unramified in F;

(A2x) $l' > C(2)$ (this is the constant $C(n)$ defined in [HST06, Theorem 3.2]);

(A2y) $l' > C'(n)$ (the constant from Theorem 14);

(A3) $l' > C(n,N)$ (the constant from Proposition 9);

(A4) l' splits in $\mathbb{Q}(\zeta_N)$ and in M;

(A5) r is unramified at l', $l' \neq l$ and $l' \neq q$.

Step B. Choose an elliptic curve E over $\mathbb{Q}$ with the following properties:

(B1) E has good ordinary reduction at l', with $H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})$ semisimple (or, in other words, tamely ramified);

(B2) E has good ordinary reduction at l and the primes of $\mathcal{L}'$;

(B3) *(if we are in case X)* E has multiplicative reduction at q;

(B4) the Galois representation coming from the cohomology $H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})$ is surjective.

It is possible to do this using the form of Hilbert irreducibility with weak approximation (see [Eke90]). We can impose the condition over q by insisting that the j invariant satisfies $v_q(j) < 0$; we can impose the condition at the prime l' by taking the Serre–Tate canonical lift of an ordinary elliptic curve.

We will write r_E for the n-dimensional Galois representation given by $\mathrm{Sym}^{n-1} H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})$.

Step C. We now apply Proposition 9 with $K=F$, $r=2$, $l_1=l$, $l_2=l'$, $K'=\overline{F}^{\ker \mathrm{ad}\, \bar{r} \cap \ker \mathrm{ad}\, \bar{r}_E}$, $\bar{\rho}_1 = r \bmod l$, $\bar{\rho}_2 = r_E|_{G_F} \bmod l'$, and $\mathcal{L} = \mathcal{L}'$. In case X we take $s=1$ and $\mathfrak{q}_1 = v_q$; in case Y we take $s=0$. Let us verify the conditions of this proposition in turn.

(i) *The l_i are large enough.* For l, this is a hypothesis of the theorem. For l', it is point (A3) above.

(ii) *The l_i split in $\mathbb{Q}(\zeta_N)$.* Again, for l this is a hypothesis of the theorem; for l' it is point (A4) above.

(iii) *The $\bar{\rho}_i$ are unramified at the primes of $\mathcal{L}'$ and the l_i.* This follows from point (B2), hypothesis (vi), and point (A5).

(iv) *The $\bar{\rho}_i$ have the right restriction to inertia.* For $\bar{\rho}_1$ this is hypothesis (vii) of our theorem; for $\bar{\rho}_2$ it follows from point (B1) above.

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(v) *Essentially self-dual with correct determinant and sign.* For $\bar{\rho}_1$ this is hypotheses (i) and (ii) of the theorem being proved; for $\bar{\rho}_2$ it follows from the fact that r_E is symplectic with multiplier $\epsilon_{l'}^{1-n}$ and $r_E = r_E^c$.

This constructs a CM field F_0/F, characters $\chi_1 : G_F \to Q_l$ and $\chi_2 : G_F \to Q_{l'}$, and a $t \in F_0$ such that:

(C1) all primes of F above l, l' and all the primes of $\mathcal{L}'$ are unramified in the field F_0;

(C2) Y_t has good reduction at all primes of F_0 above l, l' and all the primes of $\mathcal{L}'$;

(C3) for all $\mathfrak{w}|ll'$, $\mathrm{Prim}_{\mathfrak{w},t}(\vec{h}) \otimes \chi_i$ is crystalline with Hodge–Tate numbers $\{0, 1, \ldots, n-1\}$;

(C4) *(if we are in case X)* For each $\mathfrak{Q}$ above v_q, we have that $(\mathrm{Prim}_{l_i,[v],t})^{\mathrm{ss}}$ and χ_i are unramified at $\mathfrak{Q}$, with $(\mathrm{Prim}^{\mathrm{ss}}_{l_i,[v],t}(\vec{h}) \otimes \chi)(\mathrm{Frob}_{\mathfrak{Q}})$ having eigenvalues

$$\{1, \#k(\mathfrak{Q}), \#k(\mathfrak{Q})^2, \ldots, \#k(\mathfrak{Q})^{n-1}\};$$

(C5) we have that $\mathrm{Prim}[l]_{[v],t}(\vec{h}) \otimes \bar{\chi}_i = r|_{G_{F_0}} \bmod l$;

(C6) we have that $\mathrm{Prim}[l']_{[v],t}(\vec{h}) \otimes \bar{\chi}_i = r_E|_{G_{F_0}} \bmod l'$;

(C7) F_0 is linearly disjoint from $\overline{F}^{\ker \mathrm{ad}\, \bar{r} \cap \ker \mathrm{ad}\, \bar{r}_E}$ over F.

Step D. The objective of this step is to find a totally real Galois field extension F_1^+/F_0^+, with the following properties:

(D1) $r_E|_{G_{F_1^+}}$ is automorphic (in case X, automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$);

(D2) none of the primes of $\mathcal{L}'$ ramify in F_1;

(D3) l does not ramify in F_1;

(D4) F_1^+ is linearly disjoint from $(\overline{F}^{\ker \mathrm{ad}\, \bar{r} \cap \ker \mathrm{ad}\, \bar{r}_E})^+$ over F^+.

We will use slightly different arguments in case X and case Y.

Case X. We apply Theorem 14, taking $\mathcal{L} = \mathcal{L}' \cup \{l\}$, $F^{*,+} = (\overline{F}^{\ker \mathrm{ad}\, \bar{r} \cap \ker \mathrm{ad}\, \bar{r}_E})^+$ and with the l in the theorem being our l'. We observe that l' is split in M (by condition (A4)), and that E has good reduction at l, l' and the primes of $\mathcal{L}'$ (by conditions (B1), (B2) above), that l' is large enough (by point (A2x)), and finally that the primes of $\mathcal{L}'$, l and l' do not ramify in F_1^+ by conditions (C1), (A1) above. Thus we meet the conditions of the theorem, which immediately gives us what we want.

Case Y. We apply [HST06, Theorem 3.2], *in the case* $n = 2$, to the *two-dimensional* Galois representation $H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})$. In fact, we will apply a modified version of the theorem with a collection $\mathcal{L}$ of primes which may be chosen not to ramify in the field extension the theorem will produce, and where we are allowed to assume that this extension is linearly disjoint from any other fixed extension $F^{*,+}$; the requisite arguments are just as in Remark 16 above. We will take $\mathcal{L} = \mathcal{L}' \cup \{l\}$, $F^{*,+} = (\overline{F}^{\ker \mathrm{ad}\, \bar{r} \cap \ker \mathrm{ad}\, \bar{r}_E})^+$, and the l in the theorem is our l'.

It is immediate that the representation $H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})$ is odd and that it has Hodge–Tate numbers $\{0, 1\}$. It is also surjective (by condition (B4)) and Steinberg at primes above q (by condition (B3)). Finally, note that the primes of $\mathcal{L}'$, l and l' do not ramify in F_1^+ by conditions (C1), (A1) above and l' is large enough to apply the theorem by condition (A2y). Whence we satisfy the conditions of the theorem, and we can construct a field F_1^+/F_0^+

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with $H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})|_{G_{F_1^+}}$ automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$. Applying Corollary 13, and the fact that $n = 3$ or $n = 5$, we deduce that $r_E|_{G_{F_1^+}}$ $(= \mathrm{Sym}^{n-1} H^1(E \times \overline{\mathbb{Q}}, \mathbb{Z}_{l'})|_{G_{F_1^+}})$ is automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$.

Step E. Define $F_1 = F_0 F_1^+$, a CM field with totally real subfield F_1^+. We now apply a modularity lifting theorem to deduce that the Galois representation $(\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1)|_{G_{F_1}}$ is automorphic (in case X, automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$.) In case X, the theorem we apply is [Tay06, Theorem 5.2]; in case Y we apply the strengthening of that theorem made possible by admitting the expected theorems of [Har07] and given as Theorem 1.7 of that paper, in which the Steinberg condition is removed. Let us check the conditions of these theorems (following the numbering in [Tay06]) in turn.

(i) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *is conjugate self-dual.* This is immediate from Proposition 5, part 2.

(ii) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *is unramified almost everywhere.* This is trivial.

(iii) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *is crystalline at all places above l'.* This is from point (C3) above.

(iv) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *has the right Hodge–Tate numbers all places above l'.* This is true for the same reason as the previous point.

(v) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *is Steinberg at places above v_q. (This condition is only present in case X.)* This is point (C4) above.

(vi) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *has big image.* By point (C6), it suffices to show that $\bar{r}_E|_{G_{F_1}}$ has big image; by points (C7) and (D4) it then suffices to show that $\bar{r}_E$ has big image. For this we use the simplicity of $\mathrm{PSL}_2(\mathbb{F}_l)$ for $l > 3$, [CHT05, Corollary 2.5.4], and point (B4) above.

(vii) *Let $M' = \ker \mathrm{ad}\ \mathrm{Prim}[l']_{[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$; then $\overline{F}^{M'}$ does not contain $F(\zeta_l)$.* Same argument as previous point.

(viii) $\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1|_{G_{F_1}}$ *is residually automorphic. (In case X we must additionally have 'automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$'.)* By point (D1) above, we have that $r_E|_{G_{F_1^+}}$ is automorphic; by abelian base change, we therefore have that $r_E|_{G_{F_1}}$ is automorphic. By point (C6) above, this gives us what we need.

Step F. We now apply a modularity lifting theorem to deduce that $r|_{\mathrm{Gal}(\bar{F}/F_1)}$ is automorphic (in case X, automorphic of type $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$.) Again we use [Tay06, Theorem 5.2] in case X and [Har07, Theorem 2.7] otherwise. Let us check the conditions.

(i) $r|_{G_{F_1}}$ *is conjugate self-dual.* This is condition (ii) of the theorem currently being proved.

(ii) $r|_{G_{F_1}}$ *is unramified almost everywhere.* This is condition (i) of the theorem currently being proved.

(iii) $r|_{G_{F_1}}$ *is crystalline at all places above l.* This is by condition (vii) of the theorem currently being proved.

(iv) $r|_{G_{F_1}}$ *has the right Hodge–Tate numbers at all places above l.* This is also by condition (vii) of the theorem currently being proved.

(v) $r|_{G_{F_1}}$ *is Steinberg at places above v_q. (This condition is only present in case X.)* This is one of the hypotheses in case X.

(vi) $\bar{r}|_{G_{F_1}}$ *has big image.* Condition (v) of the theorem currently being proved gives that this is true before restriction to G_{F_1}; points (C7) and (D4) then show it remains true after this restriction.

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(vii) $\overline{F}^{\ker \operatorname{ad} \bar{r}|_{G_{F_1}}}$ *does not contain* $F(\zeta_l)$. This is condition (vi) of the theorem currently being proved.

(viii) $r|_{G_{F_1}}$ *is residually automorphic. (In case X we must additionally have 'automorphic of type* $\{\mathrm{Sp}_n(1)\}_{\{w|v_q\}}$*'.)* By point (C5) above, $\mathrm{Prim}[l]_{[v],t}(\vec{h}) \otimes \chi_1 = \bar{r}$, so certainly $\mathrm{Prim}[l]_{[v],t}(\vec{h})|_{G_{F_1}} \otimes \chi_1 = \bar{r}|_{G_{F_1}}$ and it suffices to show that $\mathrm{Prim}_{l,[v],t}(\vec{h})|_{G_{F_1}}$ is automorphic. For this, note that we concluded in step E that $(\mathrm{Prim}_{l',[v],t}(\vec{h}) \otimes \chi_1)|_{G_{F_1}}$ is automorphic. Additionally, we know that Y_t has good reduction at l, l', by point (C2) above. Thus we can deduce that $(\mathrm{Prim}_{l,[v],t}(\vec{h}) \otimes \chi_1)|_{G_{F_1}}$ is also automorphic.

This concludes the proof of the theorem, since we can take $F' = F_2$. (Note that the primes of $\mathcal{L}'$ are unramified in F_2 by condition (D2), and the primes above l are unramified by condition (D3).) □

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Appendix A. Some simple analysis

The purpose of this section is to do some very simple analysis to allow the deduction of Corollaries 3 and 4 from our main theorems. It is trivial that it suffices to prove the following statement.

Proposition 17. *Fix an integer n. Let Λ be a set of rational primes such that for all even $N > n$ there exists a constant $C(N)$ such that all primes l congruent to* 1 mod N *and larger than $C(N)$ lie in Λ. Then Λ has Dirichlet density one.*

Proof. Let $\epsilon > 0$ be given. Then we can find a finite list of even integers $N_1, \dots, N_k$ each of which is greater than n such that the set of primes congruent to 1 mod at least one N_i has Dirichlet density greater than $1 - \epsilon$. (For instance we could take the N_i to simply be twice an increasing list of consecutive primes above n; then the fact that $\prod(1 - (1/p))$ diverges to 0 gives us what we want.)

Then, writing $D^+(S)$ (respectively $D^-(S)$) for the upper (respectively lower) Dirichlet density of a set of primes S, $D(S)$ for the Dirichlet density if it exists, S_1 for the set of primes congruent to 1 mod at least one N_i, and S_0 for the set of primes congruent to 1 mod at least one N_i and larger than the maximum of the $C(N_i)$, we have

$$\begin{aligned} 1 \geqslant D^+(\Lambda) &\geqslant D^-(\Lambda) \\ &= D^-(S_0) \quad (\text{since } S_0 \subset \Lambda) \\ &= D^-(S_1) \quad (\text{finite sets do not affect density}) \\ &= D(S_1) \quad (\text{since } S_1 \text{ has a density}) \\ &> 1 - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have that $1 \geqslant D^+(\Lambda) \geqslant D^-(\Lambda) \geqslant 1$, whence we are done. □

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