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NORMAL SUBGROUPS WHOSE CONJUGACY CLASS GRAPH HAS DIAMETER THREE

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Abstract

Let G be a finite group and let N be a normal subgroup of G. We determine the structure of N when the diameter of the graph associated to the G-conjugacy classes contained in N is as large as possible, that is, equal to three.

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1. Introduction

Let G be a finite group and let N be a normal subgroup of G. If $x \in N$, we denote by $x^G = \{x^g \mid g \in G\}$ the G-conjugacy class of x. Let $\Gamma_G(N)$ be the graph associated to these G-conjugacy classes, which was defined in [2] as follows: its vertices are the G-conjugacy classes of N of cardinality bigger than 1, that is, G-classes of elements lying in $N \setminus (\mathbf{Z}(G) \cap N)$, and two of them are joined by an edge if their sizes are not coprime. It was proved in [2] that $d(\Gamma_G(N)) \le 3$, where $d(\Gamma_G(N))$ denotes the diameter of the graph. In this note we analyse the structure properties of N when $d(\Gamma_G(N)) = 3$.

The above graph extends the ordinary graph, $\Gamma(G)$, which was formally defined in [3], and whose vertices are the noncentral conjugacy classes of G, and two vertices are joined by an edge if their sizes are not coprime. The graph $\Gamma_G(N)$ can be viewed as the subgraph of $\Gamma(G)$ induced by those vertices of $\Gamma(G)$ which are vertices in $\Gamma_G(N)$. This fact does not allow us, however, to obtain directly properties of the graph of G-classes.

Concerning ordinary classes, Kazarin [8] characterised the structure of a group G having two 'isolated classes'. We recall that a group G is said to have isolated classes if there exist elements $x, y \in G$ such that every element of G has a conjugacy class size

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coprime to either $|x^G|$ or $|y^G|$. In particular, Kazarin determined the structure of those groups G with $d(\Gamma(G)) = 3$. On the other hand, the disconnected graph was studied by Bertram *et al.* [3]. It should be noted that similar results have also been studied for other graphs. In [6], Dolfi defined the graph $\Gamma'(G)$ whose vertices are the elements of the set of all primes which occur as divisors of the lengths of the conjugacy classes of G, and two vertices p, q are joined by an edge if there exists a conjugacy class in G whose length is a multiple of pq. In [5], Casolo and Dolfi described all finite groups G for which $\Gamma'(G)$ is connected and has diameter three.

We remark that the primes dividing the G-conjugacy class sizes do not need to divide $|N|$. This especially occurs when N is Abelian and noncentral in G and, consequently, we may have no control over this set of primes. For this reason, we observe that new cases appear when dealing with G-classes which are not contemplated in the ordinary case. The main result of this note is Theorem 1.1 and it is inspired by [8]. From now on, if G is a finite group, we denote by $\pi(G)$ the set of primes dividing $|G|$ and, analogously, if X is a set, then $\pi(X)$ denotes the set of primes dividing $|X|$.

THEOREM 1.1. *Let G be a finite group and $N \trianglelefteq G$. Suppose that x^G and y^G are two noncentral G-conjugacy classes of N such that any G-conjugacy class of N has size coprime with $|x^G|$ or $|y^G|$. Let $\pi_x = \pi(x^G)$, $\pi_y = \pi(y^G)$ and $\pi = \pi_x \cup \pi_y$. Then $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$ with $x, y \in \mathbf{O}_\pi(N)$, which is either a quasi-Frobenius group with Abelian kernel and complement, or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$, and P is a p-group for a prime p.*

Notice that in the conditions of Theorem 1.1, we have two possibilities: either $d(\Gamma_G(N)) \leq 2$ or $d(\Gamma_G(N)) = 3$. In the former case, the graph is disconnected and the structure of N is already determined by [2, Theorem E]. We slightly improve this result in Corollary 1.2. In the second case, the graph is connected. This follows from [2, Theorem B] because, when the graph $\Gamma_G(N)$ is disconnected, each connected component is a complete graph. Therefore, we deduce the following consequences for each of these cases.

COROLLARY 1.2. *Let G be a finite group and $N \trianglelefteq G$. Suppose that $\Gamma_G(N)$ is disconnected and let $x, y \in N$ such that $(|x^G|, |y^G|) = 1$. Set $\pi = \pi(x^G) \cup \pi(y^G)$. Then x, $y \in \mathbf{O}_\pi(N)$, $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$ with $\mathbf{O}_{\pi'}(N) \leq \mathbf{Z}(G)$, and either $\mathbf{O}_\pi(N)$ is a quasi-Frobenius group with Abelian kernel and complement, or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(G)$, and P is a p-group for a prime p.*

COROLLARY 1.3. *Let G be a finite group and $N \trianglelefteq G$. Suppose that $\Gamma_G(N)$ is connected with $d(\Gamma_G(N)) = 3$. Let $x, y \in N$ such that $d(x^G, y^G) = 3$. Set $\pi = \pi(x^G) \cup \pi(y^G)$. Then x, $y \in \mathbf{O}_\pi(N)$, $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$, where either $\mathbf{O}_\pi(N)$ is a quasi-Frobenius group with Abelian kernel and complement, or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$, and P is a p-group for a prime p.*

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2. Proofs

First, we state three elementary results which are needed to prove the main result.

Lemma 2.1 [1, Lemma 8]. *Let G be a π-separable group. Then the conjugacy class size of every π-element of G is a π-number if and only if $G = H \times K$, where H and K are a Hall π-subgroup and a π-complement of G, respectively.*

In the particular case in which $\pi = p'$, the complement of some prime p, Lemma 2.1 is true without assuming p-separability (or equivalently p-solvability). We recall that the class size of an element is also sometimes called the index of the element.

Lemma 2.2 [4, Lemma 1]. *If every p'-element of a group G has index prime to p, for some prime p, then the Sylow p-subgroup of G is a direct factor of G.*

Lemma 2.3 [2, Lemma 2.1]. *Let G be a finite group and $N \trianglelefteq G$. Let $B = b^G$ and $C = c^G$ be two noncentral G-conjugacy classes of N. If $(|B|, |C|) = 1$, then:*

(i) $\mathbf{C}_G(b)\mathbf{C}_G(c) = G$;
(ii) *$BC = CB$ is a noncentral G-class of N and $|BC|$ divides $|B|\,|C|$;*
(iii) *suppose that $d(B, C) \geq 3$ and $|B| < |C|$. Then $|BC| = |C|$ and $CBB^{-1} = C$. Furthermore, $C\langle BB^{-1}\rangle = C$, $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ and $|\langle BB^{-1}\rangle|$ divides $|C|$.*

Proof of Theorem 1.1. We proceed by induction on $|N|$. Notice that the hypotheses are inherited by every normal subgroup in G which is contained in N and contains x and y. By using the primary decomposition, we can assume that both x and y have order a power of a prime, say p and q, respectively.

Step 1. We have $q = p$ if and only if $xy = yx$.

Suppose that $xy = yx$ and that $p \neq q$. Observe that $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and, consequently, both $|x^G|$ and $|y^G|$ divide $|(xy)^G|$. Thus, we obtain a G-conjugacy class connected with x^G and y^G, which contradicts the hypotheses. Conversely, suppose that $p = q$. We know that p cannot divide either $|x^G|$ or $|y^G|$. Furthermore, the hypotheses imply that $(|x^G|, |y^G|) = 1$. Therefore, we have $G = \mathbf{C}_G(x)\mathbf{C}_G(y)$ and $|x^G| = |G : \mathbf{C}_G(x)| = |\mathbf{C}_G(y) : \mathbf{C}_G(x) \cap \mathbf{C}_G(y)|$. Now, since y is a p-element in $\mathbf{Z}(\mathbf{C}_G(y))$, we deduce that $y \in \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and hence $xy = yx$.

Step 2. We have $p \in \pi_y$ and $q \in \pi_x$ and hence $p, q \in \pi$.

We define $K = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$. First, we assume that $p \neq q$ and $xy \neq yx$. Then $|G : K| = |G : \mathbf{C}_G(x)|\,|\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap \mathbf{C}_G(y)| = |x^G|\,|y^G|$, which is a π-number. Since $x \in \mathbf{Z}(\mathbf{C}_G(x))$ and x is a p-element but $x \notin K$, we know that p divides $|\mathbf{C}_G(x) : K| = |y^G|$. This means that $p \in \pi_y$. Similarly, q divides $|x^G|$, that is, $q \in \pi_x$. As a result, $p, q \in \pi$.

Suppose now that $p = q$ and $xy = yx$. Let us see that $p \in \pi$. We write $X = x^G$ and $Y = y^G$ and we assume that $|X| > |Y|$. By hypothesis, the distance between X and Y in $\Gamma_G(N)$ is bigger than or equal to 3. We can apply Lemma 2.3(iii) and we get $X\langle YY^{-1}\rangle = X$, $\langle YY^{-1}\rangle \subseteq \langle XX^{-1}\rangle$ and $|\langle YY^{-1}\rangle|$ divides $|X|$. On the other hand, since

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$G = \mathbf{C}_G(x)\mathbf{C}_G(y)$, we have $X \subseteq \mathbf{C}_G(y)$. As a result, $\langle YY^{-1}\rangle \subseteq \langle XX^{-1}\rangle \subseteq \mathbf{C}_G(y)$. In particular, if we take $z = y^g \neq y$, for some $g \in G$, we have $w = zy^{-1} \in \langle YY^{-1}\rangle \subseteq \mathbf{C}_G(y)$, so $[z, y] = 1$. Consequently, w is a nontrivial p-element and, since p divides $|\langle YY^{-1}\rangle|$, which divides $|X|$, we conclude that $p \in \pi_x$. If $|Y| > |X|$, we can argue similarly to get $p \in \pi_y$.

Step 3. We can assume that $N/\mathbf{Z}(N)$ is neither a p-group nor a q-group. In particular, we can assume that N is not Abelian.

Suppose that $N/\mathbf{Z}(N)$ is a p-group. The argument is analogous if we suppose that it is a q-group. Hence, we can write $N = P \times A$, where $A \leq \mathbf{Z}(N)$ and A is a p'-group. If $p \neq q$, it follows that $x \in P$ and $y \in A$, which leads to a contradiction with Step 1. Thus, $p = q$ and $x, y \in P$, so the theorem is proved.

Step 4. We can suppose that N is not a π-group.

Let us see that if N is a π-group, then N is a quasi-Frobenius group with Abelian kernel and complement, or $N = P \times A$ with $A \leq \mathbf{Z}(N)$ and A a p'-group. Assume that N is a π-group. As N is non-Abelian by Step 3, there exists a conjugacy class z^N such that $|z^N| \neq 1$. Since $|z^N|$ divides $|z^G|$, then either $(|z^N|, |x^G|) = 1$ or $(|z^N|, |y^G|) = 1$. As N is a π-group, then $|z^N|$ is either a π_x-number or a π_y-number. If $\Gamma(N)$ is disconnected, we know by Theorem 2 of [3] that N is a quasi-Frobenius group with Abelian kernel and complement. Moreover, $\Gamma(N)$ cannot be empty because by Step 3, N can be assumed to be non-Abelian. Consequently, we can assume that $\Gamma(N)$ is connected and this forces either $|x^N| = 1$ or $|y^N| = 1$. Suppose for instance that $|x^N| = 1$, that is, $x \in \mathbf{Z}(N)$. Again by Step 3, we can find an s-element w of $N\backslash\mathbf{Z}(N)$ with $s \neq p$. Observe that $|w^N|$ must be a π_y-number, so w^G is connected to y^G in $\Gamma_G(N)$. As x and w have coprime orders and $x \in \mathbf{Z}(N)$, we have that $|w^G|$ and $|x^G|$ both divide $|(wx)^G|$. As a consequence, we have a contradiction because $|(wx)^G|$ has primes in π_x and π_y. Thus, we can suppose that N is not a π-group.

Step 5. Conclusion in case $p \neq q$.

Let z be a π'-element of $K \cap N$ and let us prove that $|z^G|$ is a π'-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume for instance that $s \in \pi_y$, otherwise we proceed analogously. Since $|z^G|$ divides $|(zx)^G|$, we deduce that s divides $|(zx)^G|$. Also, we know by Step 2 that $q \in \pi_x$. This forces $|(zx)^G|$ to be divisible by primes in π_x and π_y, which is a contradiction. Consequently, $s \notin \pi$ and $|z^G|$ is a π'-number, as asserted.

Let M be the subgroup generated by all π'-elements of $K \cap N$. We prove that M is a nontrivial normal subgroup of G. If $M \neq 1$, then $K \cap N$ is a π-group and, since $|N : K \cap N| = |KN : K|$ divides $|G : K|$, which is also a π-number, N is a π-group, contrary to Step 4. Let α be a generator of M, so $|\alpha^G|$ is π'-number. As $(|G : K|, |\alpha^G|) = 1$, we have $G = K\mathbf{C}_G(\alpha)$ and, hence, $\alpha^G = \alpha^K \subseteq K \cap N$. Therefore, $\alpha^G \subseteq M$.

Let $D = \langle x^G, y^G\rangle$. Notice that $D \trianglelefteq G$ and $D \subseteq N$. Let α be a generator of M. As we have proved in the previous paragraph, $|\alpha^G|$ is a π'-number and then $(|\alpha^G|, |x^G|) = 1$, so

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$G = \mathbf{C}_G(x)\mathbf{C}_G(\alpha)$. Thus, $x^G = x^{\mathbf{C}_G(\alpha)} \subseteq \mathbf{C}_G(\alpha)$ because $\alpha \in K$. The same happens for y, that is, $y^G \subseteq \mathbf{C}_G(\alpha)$, so we conclude that $[M, D] = 1$.

We define $L = MD$ and we distinguish two cases. Assume first that $L < N$. Note that $x, y \in L \trianglelefteq G$ and L trivially satisfies the hypotheses of the theorem. By applying induction to L, we have $L = \mathbf{O}_\pi(L) \times \mathbf{O}_{\pi'}(L)$. Observe that the fact that $M \neq 1$ implies that $\mathbf{O}_{\pi'}(L) > 1$. Now, by the definition of M, $|K \cap N : M|$ is a π-number. As $|N : K \cap N|$ is also a π-number, so is $|N : \mathbf{O}_{\pi'}(L)|$. Then $\mathbf{O}_{\pi'}(L) = \mathbf{O}_{\pi'}(N)$ is a Hall π'-subgroup of N. We can apply Lemma 2.1 to conclude that $N = \mathbf{O}_\pi(N) \times \mathbf{O}_{\pi'}(N)$ with $x, y \in \mathbf{O}_\pi(N)$. Since $\mathbf{O}_{\pi'}(N) > 1$, we apply the inductive hypotheses to $\mathbf{O}_\pi(N) < N$ and we deduce that $\mathbf{O}_\pi(N)$ is a quasi-Frobenius group with Abelian kernel and complement, or $\mathbf{O}_\pi(N) = P \times A$ with $A \leq \mathbf{Z}(N)$, and P is a p-group, so the proof is finished.

From now on, we assume that $L = N$ and we show that $\mathbf{Z}(N) = 1$ and $N = M \times D$ with $x, y \in D$. If $\mathbf{Z}(N) \neq 1$, we take $\overline{N} = N/\mathbf{Z}(N)$ and $\overline{G} = G/\mathbf{Z}(N)$. If $|\overline{x}^{\overline{G}}| = 1$, then $[\overline{x}, \overline{y}] = 1$ and thus $[x, y] \in \mathbf{Z}(N)$. Since $(o(x), o(y)) = 1$, it is easy to prove that $[x, y] = 1$, which is a contradiction. Analogously, we have $|\overline{y}^{\overline{G}}| \neq 1$. Consequently, $\overline{N}$ satisfies the assumptions of the theorem. By induction, we have $\overline{N} = \mathbf{O}_{\pi'}(\overline{N}) \times \mathbf{O}_\pi(\overline{N})$ with $\overline{x}, \overline{y} \in \mathbf{O}_\pi(\overline{N})$ and $\mathbf{O}_\pi(\overline{N})$ is either a quasi-Frobenius group with Abelian kernel and complement, or $\overline{N} = \overline{P} \times \overline{A}$ with $\overline{A} \leqslant \mathbf{Z}(\overline{N})$, and $\overline{P}$ a p-group. In the latter case, $[\overline{y}, \overline{x}] = 1$, which leads to a contradiction as we have seen before. So, we are in the former case. It follows that $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$ with $x, y \in \mathbf{O}_\pi(N)$ and, by applying induction to $\mathbf{O}_\pi(N) < N$, we have the result. Therefore, $\mathbf{Z}(N) = 1$. On the other hand, we have proved that $[M, D] = 1$. Hence, $M \cap D \subseteq \mathbf{Z}(N) = 1$ and $N = M \times D$ with $x, y \in D$.

Since $M \neq 1$, we can apply induction to D and get $D = \mathbf{O}_{\pi'}(D) \times \mathbf{O}_\pi(D)$ with $x, y \in \mathbf{O}_\pi(D)$ and $\mathbf{O}_\pi(D)$ is a Frobenius group with Abelian kernel and complement (notice that $\mathbf{Z}(\mathbf{O}_\pi(D)) = 1$ because $\mathbf{Z}(N) = 1$). The p-group case cannot occur because x and y do not commute. Notice that if M is a π'-group, then the theorem is proved. We assume that M is not a π'-group and seek a contradiction. Let $s \in \pi$ such that s divides $|M|$. We can assume that $s \in \pi_x$ (we proceed analogously if $s \in \pi_y$). Suppose that there exists an s'-element $z \in M$ such that $|z^M|$ is divisible by s. Since N is the direct product of M and D, $(zy)^N = z^N y^N$ is a nontrivial class of N whose size is divisible by s and by some prime of $|y^N| \neq 1$. This is not possible because $|(zy)^G|$ would have primes in π_x and π_y. Thus, the class size of every s'-element of M is an s'-number. By Lemma 2.2, we have $M = M_1 \times S$ with $S \in \mathrm{Syl}_s(M)$. In this case, $\mathbf{Z}(S) \subseteq \mathbf{Z}(N) = 1$, which is a contradiction.

Step 6. Conclusion in case $p = q$.

Let $K = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ as in Step 2. Let z be a p'-element of $K \cap N$ and let us prove that $|z^G|$ is a π'-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume that $s \in \pi_y$. Since $|z^G|$ divides $|(zx)^G|$, we see that s divides $|(zx)^G|$. On the other hand, we know by the proof of Step 2 that $q \in \pi_x$. Therefore, $|(zx)^G|$ is divisible by primes in π_x and π_y, which is a contradiction. As a consequence, $s \notin \pi$ and $|z^G|$ is a π'-number.

Let T be the subgroup generated by all p'-elements of $K \cap N$. We prove that T is a nontrivial normal subgroup of G. In fact, $T \neq 1$ because otherwise $K \cap N$ would be a π-group and this implies that N is a π-group by arguing as in Step 5, and this contradicts Step 4. If α is a generator of T, we know that $|\alpha^G|$ is a π'-number. Then $(|G : K|, |\alpha^G|) = 1$, so $G = K\mathbf{C}_G(\alpha)$ and $\alpha^G = \alpha^K \subseteq K \cap N$. This proves that $\alpha^G \subseteq T$.

As the class size of every p'-element of T is a p'-number, by using Lemma 2.2, we have $T = \mathbf{O}_p(T) \times \mathbf{O}_{p'}(T)$. However, by definition of T, we have $\mathbf{O}_p(T) = 1$ or equivalently $M = \mathbf{O}_{p'}(T)$. Notice that if $s \in \pi$ and $s \neq p$, then the class size of every element of T is an s'-number, so it is elementary that T has a central Sylow s-subgroup and we can write $T = \mathbf{O}_\pi(T) \times \mathbf{O}_{\pi'}(T)$. On the other hand, $|N : T| = |N : K \cap N||K \cap N : T|$, where $|N : K \cap N| = |KN : K|$ is a π-number and $|K \cap N : T|$ is a power of $p \in \pi$. Therefore, $\mathbf{O}_{\pi'}(T) = \mathbf{O}_{\pi'}(N)$ and $\mathbf{O}_{\pi'}(N)$ is a Hall π'-subgroup of N. We have proved that the class size of every p'-element of N is a π'-number, so, by Lemma 2.1, we have $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_\pi(N)$. We apply induction to $\mathbf{O}_\pi(N) < N$ and the proof is finished. □

Proof of Corollary 1.2. The corollary follows immediately from Theorem 1.1. We only have to notice that if $x, y \in \mathbf{O}_\pi(N)$ and $z \in \mathbf{O}_{\pi'}(N) \backslash \mathbf{Z}(G)$, then there is a path connecting x^G and y^G because $(xz)^G$ is connected to x^G and $(yz)^G$, which is connected to y^G. This contradicts the hypotheses of the theorem. Thus, $\mathbf{O}_{\pi'}(N) \leq \mathbf{Z}(G)$. By the same argument, we obtain $A \leq \mathbf{Z}(G)$ when $\mathbf{O}_\pi(N) = P \times A$. □

Proof of Corollary 1.3. The corollary follows trivially from Theorem 1.1. □

We give an example showing that the converse of Theorem 1.1 is not true.

Example 2.4. We take the special linear group $H = \mathrm{SL}(2, 5)$, which is a group of order 120 that acts Frobeniusly on $K = \mathbb{Z}_{11} \times \mathbb{Z}_{11}$. Let $P \in \mathrm{Syl}_5(H)$ and consider $\mathbf{N}_H(P)$. Define $N := KP$, which trivially is a normal subgroup of $G := K\mathbf{N}_H(P)$. The set of the G-conjugacy class sizes of N is $\{1, 20, 242\}$. The graph $\Gamma_G(N)$ consists of exactly two vertices joined by an edge. Obviously, N is a Frobenius group with Abelian kernel and complement and there do not exist two noncentral G-classes in N such that any noncentral G-class of N has size coprime with one of both of them.

Finally, we give two examples illustrating each case in Theorem 1.1.

Example 2.5. We take the following groups from the library SmallGroups of GAP [7]. Let $G_1 = \mathrm{Id}(324, 8)$ and $G_2 = \mathrm{Id}(168, 44)$ (in fact, G_2 is the semilinear affine group of order 168) whose normal subgroups are the Abelian 3-subgroup $P = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. It is easy to check that P has four G_1-classes whose sizes are $1, 2, 3$ and 3, and A has two G_2-classes of sizes 1 and 7. We construct $N = P \times A$ and $G = G_1 \times G_2$. Then N is a normal subgroup of G and the set of G-conjugacy class sizes of N is $\{1, 2, 3, 7, 14, 21\}$. Therefore, $d(\Gamma_G(N)) = 3$ and N is the direct product of a 3-group and $A \leq \mathbf{Z}(N)$. Notice that, in this example, $\mathbf{O}_{\pi'}(N) = 1$ and $\pi = \{2, 3, 7\}$.

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EXAMPLE 2.6. The quasi-Frobenius case in Theorem 1.1 is the natural extension of the ordinary case. It is enough to consider any group G and $N = G$ such that $\Gamma(G) = \Gamma_G(N)$ has two connected components. By the main theorem of [3], we know that G is a quasi-Frobenius group with Abelian kernel and complement.

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