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BOUNDED AND FULLY BOUNDED MODULES

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Abstract

Generalizing the concept of right bounded rings, a module M_R is called bounded if $\mathrm{ann}_R(M/N) \leq_e R_R$ for all $N \leq_e M_R$. The module M_R is called fully bounded if (M/P) is bounded as a module over $R/\mathrm{ann}_R(M/P)$ for any $\mathcal{L}_2$-prime submodule $P \lhd M_R$. Boundedness and right boundedness are Morita invariant properties. Rings with all modules (fully) bounded are characterized, and it is proved that a ring R is right Artinian if and only if R_R has Krull dimension, all R-modules are fully bounded and ideals of R are finitely generated as right ideals. For certain fully bounded $\mathcal{L}_2$-Noetherian modules M_R, it is shown that the Krull dimension of M_R is at most equal to the classical Krull dimension of R when both dimensions exist.

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1. Introduction

Throughout rings will have unit elements and modules will be right unitary. We use the notations $N \trianglelefteq M$, $N \leq_e M$ to denote respectively that N is a fully invariant, essential submodule of M. Hence $N \trianglelefteq_e M$ means N is a fully invariant essential submodule of M. Recall from [3] that R is a *right bounded* ring if for every $I \leq_e R_R$ there exists $B \trianglelefteq_e R_R$ with $B \subseteq I$. The ring R is called *right fully bounded* if every prime factor of R is a right bounded ring. Right fully bounded right Noetherian rings (right FBN) and modules over them have been studied extensively and are known to have a number of interesting properties; see, for example, [5–8]. The concept of right bounded rings has been generalized to bounded modules by earlier authors. In [10], modules M_R with the property that R/P is a right fully bounded ring for every $P \in \mathrm{Ass}(M_R)$ were called 'bounded' and they were studied when R is right Noetherian [10, Theorem 2.6]. In [6], modules M_R with $\mathrm{ann}_R(M/N)/\mathrm{ann}_R(M) \leq_e R/\mathrm{ann}_R(M)$, for any $N \leq_e M_R$, were studied and called bounded. To avoid confusion, we shall say that the latter modules are *L-bounded*, while we define a *bounded module* M_R by the condition $\mathrm{ann}_R(M/N) \leq_e R_R$

for any $N \leq_e M_R$. Our bounded modules form a class bigger than the class of L-bounded modules. Clearly, a module M_R is L-bounded if and only if M is bounded as a module over $R/\mathrm{ann}_R(M)$.

We study bounded modules and show that they are invariant under Morita equivalences (Theorem 3.3). It follows that L-boundedness and right boundedness are Morita invariant properties. It is shown that all R-modules are bounded if and only if $\mathrm{Soc}(R_R) \leq_e R_R$ (Theorem 3.5) and, as a corollary, we show that right semi-Artinian rings are precisely rings with all modules L-bounded. A module M_R is called *fully bounded* if $(M/P)_R$ is L-bounded for any fully invariant $\mathcal{L}_2$-prime submodule $P \lhd M_R$ in the sense of [11]. In [12] $\mathcal{L}_2$-prime submodules were called 'fully prime submodules'. The class $\mathrm{Spec}_2(M_R)$ of fully invariant $\mathcal{L}_2$-prime submodules of a given module M_R properly lies between the class of 'prime' submodules of M_R in the sense of [2] and the class of prime submodules N of M_R in the sense of $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ for any $0 \neq N \leq M$. A characterization of rings with all modules fully bounded is given in Theorem 3.7 and it is shown that a ring R is right Artinian if and only if R_R has Krull dimension, all R-modules are fully bounded and ideals of R are finitely generated as right ideals (Proposition 3.8). In the last section, we deal with bounded modules with Krull dimension and, for a quasi projective fully bounded $\mathcal{L}_2$-Noetherian module M_R, it is shown that $\mathrm{K.dim}(M) \leq \mathrm{Cl.K.dim}(R)$ when the dimensions exist. A well-known result on right FBN rings R stating that $\mathrm{K.dim}(R_R) = \mathrm{Cl.K.dim}(R)$ may then be obtained by Theorem 4.1; see also [3, Theorem 15.13]. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1, 3].

2. Preliminaries

We begin by recalling some definitions from [11]. An R-module M is called *$\mathcal{L}_2$-Noetherian* if M finitely generates all of its fully invariant submodules and has ascending chain condition (acc) on them. Some examples of $\mathcal{L}_2$-Noetherian modules are Noetherian self-generator modules and modules without nonzero fully invariant submodules. Note that the module R_R is $\mathcal{L}_2$-Noetherian if and only if every ideal of R is finitely generated as a right ideal. A proper submodule P of a module M_R is called *$\mathcal{L}_2$-prime* if, for every $W_1, W_2 \trianglelefteq M_R$, the condition $W_1 \star W_2 \subseteq P$ implies that $W_1 \subseteq P$ or $W_2 \subseteq P$ where $W_1 \star W_2 = \mathrm{Hom}_R(M, W_1)W_2$. If (0) is an $\mathcal{L}_2$-prime submodule of M_R then M is called an *$\mathcal{L}_2$-prime R-module*. In the following we present some facts on $\mathcal{L}_2$-Noetherian and $\mathcal{L}_2$-prime modules for later use.

Lemma 2.1. *Let M_R be a nonzero module with $MI = 0$ for some $I \lhd R$.*

(i) *If $N \trianglelefteq M_R$ and $Q/N \trianglelefteq M/N$, then $Q \trianglelefteq M$.*
(ii) *If M_R is quasi projective and $K \leq L \trianglelefteq M$, then $L/K \trianglelefteq M/K$.*
(iii) *M_R is $\mathcal{L}_2$-Noetherian if and only if $M_{R/I}$ is $\mathcal{L}_2$-Noetherian.*
(iv) *If M_R is $\mathcal{L}_2$-Noetherian then M/N is an $\mathcal{L}_2$-Noetherian R-module for any $N \trianglelefteq M_R$.*

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(v) *If M_R is quasi projective, $N \lhd M$ and $N \le P$ then $P/N \in \mathrm{Spec}_2(M/N)$ if and only if $P \in \mathrm{Spec}_2(M)$.*
(vi) *If $M_{R/I}$ is bounded then M_R is bounded. Consequently, L-bounded and fully bounded $\mathcal{L}_2$-prime modules are bounded.*
(vii) *$M_{R/I}$ is fully bounded if and only if M_R is fully bounded.*

Proof. (i), (ii) and (iii) follow by routine arguments and (v) follows by part (ii).

To prove (iv), let $N \trianglelefteq M_R$ and notice that M/N has acc on its fully invariant submodules by part (i). On the other hand, if $L/N \trianglelefteq M/N$ then $L \trianglelefteq M$ and so M finitely generates L by our assumption. Hence there exists an R-epimorphism $f : M^{(n)} \to L$ for some positive integer n. Let $\iota_i : M \to M^{(n)}$ be the natural inclusion map for $i = 1, \ldots, n$. Since $N \trianglelefteq M$, $f\iota_i(N) \subseteq N$. This shows that the map $g : (M/N)^{(n)} \to L/N$ with $g(x_1 + N, \ldots, x_n + N) = f(x_1, \ldots, x_n) + N$ is well defined. Clearly g is also an R-epimorphism. Thus M/N finitely generates L/N, proving that M/N is an $\mathcal{L}_2$-Noetherian R-module.

(vi) and (vii) follow by definitions and the fact that if J/I is an essential right ideal of R/I then $J \le_e R$. □

In [4], an R-module M with $S = \mathrm{End}_R(M)$ was called *endoprime* if $\mathrm{l.ann}_S(N)$ is zero for any $0 \ne N \trianglelefteq M_R$. It is easy to verify that retractable endoprime modules are $\mathcal{L}_2$-prime. (A module M_R is retractable if $\mathrm{Hom}_R(M, N) \ne 0$ for any $0 \ne N \le M$.) We will use the following result to characterize rings with all modules fully bounded.

Proposition 2.2. *The following statements are equivalent for a ring R.*

(i) *R is a prime ring.*
(ii) *$\mathrm{End}_R(F)$ is a prime ring for some free R-module F.*
(iii) *$\mathrm{End}_R(F)$ is a prime ring for every free R-module F.*
(iv) *Every free R-module is $\mathcal{L}_2$-prime.*

Proof. For the equivalences (i) ⇔ (ii) ⇔ (iii), note that if $S = \mathbb{CFM}_\Gamma(R)$, a column finite matrix ring over R for some nonempty set Γ, and $(a_{ij})S(b_{ij}) = 0$ for some matrices $(a_{ij}), (b_{ij}) \in S$ then $E_{kk}(a_{ij})(E_{ll}RE_{rr})(b_{ij})E_{tt} = 0$ for any $k, l, r, t \in \Gamma$ where E_{xy} is the matrix with 1 as (x, y)th entry and zero elsewhere. It follows that $a_{kl}Rb_{rt} = 0$ for any $k, l, r, t \in \Gamma$. Thus if $(b_{ij}) \ne 0$ then we can deduce that $(a_{ij}) = 0$ provided that R is prime ring.
(iii) ⇒ (iv). Apply [4, Proposition 1.3(3)] for a free R-module F. Thus F_R is endoprime and hence an $\mathcal{L}_2$-prime module.
(iv) ⇒ (i). This is clear. □

3. The class of (fully) bounded modules

We study the class of bounded modules and show that bounded, L-bounded and right bounded modules are Morita invariant properties. Several characterizations of rings R with essential socles are given. In particular, semi-Artinian (respectively Artinian) rings are characterized in terms of L-bounded (respectively fully bounded) R-modules.

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Proposition 3.1.

(i) *The class of bounded modules is closed under taking submodules, factor modules and finite direct sums.*

(ii) *If M_R is a quasi projective fully bounded module, then M/N is fully bounded for any $N \trianglelefteq M$.*

Proof. (i) Let M_R be bounded and $N \le_e K \le M_R$. There exists $L \le M$ such that $K \oplus L \le_e M_R$. Thus $N \oplus L \le_e M_R$. Since M_R is bounded, there exists $I \le_e R_R$ such that $MI \subseteq N \oplus L$. It follows that $KI \subseteq N$, proving that K_R is bounded.

To prove that M/K is a bounded R-module, let $N/K \le_e M/K$. Then $N \le_e M_R$ and so there exists $J \le_e R_R$ such that $MJ \subseteq N$ by our assumption. This shows that $(M/K)J \subseteq N/K$.

Finally, assume that $V = \bigoplus_{i=1}^{k} M_i$ where each M_i is a bounded R-module. Suppose that $N \le_e V_R$. It is easy to verify that $N \cap M_i \le_e M_i$ for each i. Thus by our assumption on M_i, there exists $I_i \le_e R$ such that $M_i I_i \subseteq N$. Let $I = \bigcap_{i=1}^{k} I_i$. Then $I \le_e R$ and $VI \subseteq N$, as desired.
(ii) This follows from Lemma 2.1(v). □

Corollary 3.2. *A ring R is right (fully) bounded if and only if any finitely generated R-module is (fully) bounded.*

Proof. We only prove the fully bounded case. Assume that R is a right fully bounded ring, M_R is finitely generated, $P \in \mathrm{Spec}_2(M)$ and $I = \mathrm{ann}_R(M/P)$. By [11, Proposition 2.1(ii)], I is a prime ideal of R. Thus R/I is a right bounded ring by our assumption. Since now $(M/P)_{(R/I)}$ is finitely generated, M/P is bounded as an R/I-module by Proposition 3.1(i), proving that M_R is fully bounded. The other direction is clear. □

Theorem 3.3. *Boundedness is a Morita invariant property.*

Proof. Let R and S be Morita equivalent rings with category equivalence $\alpha : \text{Mod-}R \to \text{Mod-}S$. We first claim that if M is an R-module such that $\mathrm{ann}_R(M) \le_e R_R$ then $\mathrm{ann}_S(\alpha(M)) \le_e S_S$. Let $I = \mathrm{ann}_R(M)$ and $B = \mathrm{ann}_S(\alpha(R/I))$. By [1, Proposition 21.11], R/I is Morita equivalent to S/B. Since M is faithful as an R/I-module, $B = \mathrm{ann}_S(\alpha(M))$ by [1, Proposition 21.6]. Then also $\alpha(R/I)$ is a projective generator in $\text{Mod-}S/B$. Thus there exist $L \in \text{Mod-}S/B$ and $n \ge 1$ such that $(S/B) \oplus L = [\alpha(R/I)]^n$ in $\text{Mod-}S/B$ as well as in $\text{Mod-}S$. Now consider the exact sequence $0 \to I \to R \to R/I$. Since Morita equivalences preserve co-kernels and essential monomorphisms, $\alpha(R/I)$ is a co-kernel of an essential monomorphism in $\text{Mod-}S$. Hence it is a singular S-module. Therefore, $(S/B)_S$ is singular and so $B \le_e S_S$, as claimed.

Suppose now that M is a bounded R-module. It is easy to verify that a module X_R is bounded if $\mathrm{ann}_R(\text{co-ker} f) \le_e R_R$, for every essential monomorphism $f : N_R \to X_R$. Thus $(\alpha(M))_S$ is bounded by the first part, and the proof is complete. □

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Corollary 3.4. *Both L-boundedness and right boundedness are Morita invariant properties.*

Proof. Note that a module M_R is L-bounded if and only if it is bounded as a module over $R/\mathrm{ann}_R(M)$. Now let R and S be Morita equivalent rings with category equivalence $\alpha : \text{Mod-}R \to \text{Mod-}S$. Suppose that M_R is an L-bounded module, $I = \mathrm{ann}_R(M)$ and $B = \mathrm{ann}_S(\alpha(M))$. As observed in the proof of Theorem 3.3, $(R/I) \overset{\alpha}{\approx} (S/B)$. Also S is isomorphic to a direct summand of a finite direct sum of copies $\alpha(R)$ in Mod-S. Therefore, the result is obtained by Theorem 3.3 and Proposition 3.1(i). □

Theorem 3.5. *The following statements are equivalent for a ring R.*

(i) $\mathrm{Soc}(R_R) \leq_e R_R$.
(ii) *R is a right bounded ring and the class of bounded R-modules is closed under direct sums.*
(iii) *Every free R-module is bounded.*
(iv) *Every R-module is bounded.*

Proof. (i) ⇒ (ii). Let $B = \mathrm{Soc}(R_R)$. Since B lies in all the essential right ideals of R, the ring R is right bounded and in R, any intersection of essential right ideals is essential. Therefore, as observed in the proof of Proposition 3.1(i), any direct sum of bounded modules is a bounded module.
(ii) ⇒ (iii). This is clear.
(iii) ⇒ (iv). Apply Proposition 3.1(i) and the fact that any R-module is a homomorphic image of a free R-module.
(iv) ⇒ (i). Let $\{I_\alpha\}_{\alpha\in A}$ be the family of all essential right ideals of R. Then we have $\bigoplus_{\alpha\in A} I_\alpha \leq_e R^{(A)}$. By our assumption, the R-module $R^{(A)}$ is bounded. Thus there exists $I \trianglelefteq_e R_R$ such that $R^{(A)}I \subseteq \bigoplus_{\alpha\in A} I_\alpha$. This shows that $I \subseteq \bigcap_{\alpha\in A} I_\alpha = \mathrm{Soc}(R_R)$. Thus (i) holds. □

A ring R is said to be *right semi-Artinian* if for every $I \triangleleft R$, the right socle of the ring R/I is nonzero (or equivalently, every nonzero R-module has a nonzero socle).

Corollary 3.6. *A ring R is right semi-Artinian if and only if all R-modules are L-bounded.*

Proof. (⇒) Apply Theorem 3.5.
(⇐). Let $I \trianglelefteq R$, M be an R/I-module and $B = \mathrm{ann}_R(M)$. By our assumption, M_R is L-bounded and so $M_{R/B}$ is bounded. It follows that $M_{R/I}$ is bounded by Lemma 2.1(vi). Now apply Theorem 3.5 to deduce that the right socle of the ring R/I is an essential right ideal. This shows that R is a right semi-Artinian ring. □

It is known that in prime rings the right and left socles coincide. We say that a ring R is *pre semi-Artinian* if, for every prime ideal P of R, the socle of the ring R/P is nonzero (or equivalently, since R/P is a prime ring, an essential (left) right ideal). Note that for every infinite set Λ and every field F, the von Neumann regular ring $R = F^{\Lambda}/F^{(\Lambda)}$ is pre semi-Artinian but not semi-Artinian because $\mathrm{Soc}(R_R) = 0$.

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THEOREM 3.7. *All nonzero R-modules are fully bounded if and only if R is a pre semi-Artinian ring.*

PROOF. (⇒) Let P be a prime ideal of R and $T = R/P$. In view of Theorem 3.5, we shall show that every free T-module is bounded. Let F be a free T-module. By our assumption and Lemma 2.1(vii), F_T is fully bounded. On the other hand, F_T is $\mathcal{L}_2$-prime by Proposition 2.2. Hence F_T is bounded, as desired.
(⇐). This is clear by the definition and Theorem 3.3. □

PROPOSITION 3.8. *A ring R is right Artinian if and only if R_R has Krull dimension, ideals in R are finitely generated as right ideals and all R-modules are fully bounded.*

PROOF. (⇒) This follows by Theorem 3.5 and the well-known result that a right Artinian ring is right Noetherian with zero Krull dimension.
(⇐) By [11, Proposition 3.2(ii) and Theorem 3.1] and Theorem 3.5, R is right Artinian. □

REMARK 3.9. Proposition 3.8 does not hold if we replace 'fully bounded' by 'bounded'. For example, if $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ then $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix} = \mathrm{Soc}(R_R) \leq_e R_R$ and hence all R-modules are bounded by Theorem 3.5. But R is not even a right semi-Artinian ring. This example shows also that there exist bounded R-modules which are not L-bounded by Corollary 3.6 and Theorem 3.5.

4. Fully bounded modules with Krull dimension

Let R be a ring, M an R-module. Then according to [9, Corollary 6.2.18], $\mathrm{K.dim}(M) \leq \mathrm{K.dim}(R_R)$, provided that both dimensions exist. Also it is known that when R is right Noetherian, $\mathrm{Cl.K.dim}(R) \leq \mathrm{K.dim}(R_R)$, and that the inequality $\mathrm{K.dim}(R_R) \leq \mathrm{Cl.K.dim}(R)$ holds for right FBN rings R; see [3, Theorem 15.13]. In this section we prove a theorem which gives classical Krull dimension as an upper bound for the Krull dimensions of certain modules, even if the base ring does not have Krull dimension. In particular, for certain modules M over rings R with $\mathrm{Cl.K.dim}(R) < \mathrm{K.dim}(R_R)$ (for example, if $R = \mathbb{Z} \oplus A$ where A is the nth Weyl algebra over $\mathbb{C}$), the inequality $\mathrm{K.dim}(M) \leq \mathrm{K.dim}(R_R)$ is improved by $\mathrm{K.dim}(M) \leq \mathrm{Cl.K.dim}(R)$. The crucial inequality $\mathrm{K.dim}(R_R) \leq \mathrm{Cl.K.dim}(R)$ for right FBN rings R is also a corollary of our Theorem 4.1.

THEOREM 4.1. *Let R be a ring and M_R be a nonzero quasi projective fully bounded $\mathcal{L}_2$-Noetherian module. Assume that* $\mathrm{Cl.K.dim}(R)$ *and* $\mathrm{K.dim}(M)$ *exist. Then* $\mathrm{K.dim}(M) \leq \mathrm{Cl.K.dim}(R)$.

PROOF. We prove the theorem by induction on $\mathrm{Cl.K.dim}(R)$. Since M_R is $\mathcal{L}_2$-Noetherian, by [11, Theorem 3.1] there exists an $\mathcal{L}_2$-prime submodule $P \trianglelefteq M_R$ such that $\mathrm{K.dim}(M) = \mathrm{K.dim}[(M/P)_R]$. Let $I = \mathrm{ann}_R(M/P)$, which is a prime ideal of R. Suppose first that $\mathrm{Cl.K.dim}(R) = 0$. Hence by hypothesis, M/P is a bounded module

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over the simple ring R/I. Thus M/P is a semisimple R/I-module with Krull dimension. This shows that $\text{K.dim}(M) = 0$.

Now assume that $\text{Cl.K.dim}(R) = \alpha$ and the theorem holds for any ring with classical Krull dimension less than α. Let $L = M/P$ and $T = R/I$. We shall show that $\text{K.dim}(L_T) \le \alpha$. Notice that L_T is quasi projective, $\mathcal{L}_2$-Noetherian and fully bounded by Lemma 2.1 and Proposition 3.1(ii). In view of [9, Lemma 6.2.8], it is enough to prove that for all $N \le_e L_T$, $\text{K.dim}[(L/N)_T] < \alpha$. Therefore suppose that $N \le_e L_T$. Since M_R is fully bounded, L_T is bounded and so there exists $B \trianglelefteq_e T_T$ such that $LB \subseteq N$. Again $(L/LB)_T$ is $\mathcal{L}_2$-Noetherian and so $\text{K.dim}[(L/LB)_T] = \text{K.dim}[(L/Q)_T]$ for some $\mathcal{L}_2$-prime submodule $Q/LB \trianglelefteq L/LB$. Now $0 \ne B \subseteq \text{ann}_T(L/Q) := C$ and so $\text{Cl.K.dim}(T/C) < \text{Cl.K.dim}(T)$ because C is a nonzero prime ideal in the prime ring T. Thus by the induction assumption, $\text{K.dim}[(L/Q)_{T/C}] \le \text{Cl.K.dim}(T/C) < \alpha$. This shows that $\text{K.dim}[(L/N)_T] \le \text{K.dim}[(L/LB)_T] < \alpha$, and the proof is complete. □

Corollary 4.2. *Over a simple ring R, every quasi-projective Noetherian self-generator fully bounded R-module is either Artinian or does not have Krull dimension.*

Proof. Apply Theorem 4.1. □

Let A, B be rings and M be a left B- right A-bimodule. It is easy to verify that every prime ideal in the formal triangular matrix ring $T = \left[\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right]$ has the form $\left[\begin{smallmatrix} P & 0 \\ M & B \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} A & 0 \\ M & Q \end{smallmatrix}\right]$ for some prime ideals $P \trianglelefteq A$ and $Q \trianglelefteq B$. It follows that T is a pre semi-Artinian ring if and only if A and B are so. Thus, using formal triangular matrix rings, we are able to construct examples of noncommutative pre semi-Artinian rings R which do not have Krull dimensions; for example, suppose that $R = \left[\begin{smallmatrix} F & 0 \\ M & F \end{smallmatrix}\right]$, F is a field and M_F is nonfinitely generated free. Now by a slight modification of the proof of Theorem 4.1, the following result can be proved which gives an upper bound on the Krull dimension of $\mathcal{L}_2$-Noetherian modules over pre semi-Artinian rings (even if the ring may not have Krull dimension).

Theorem 4.3. *Let R be a pre semi-Artinian ring and M_R be $\mathcal{L}_2$-Noetherian. Then* $\text{K.dim}(M) \le \text{Cl.K.dim}(R)$ *provided that both dimensions exist.*

Corollary 4.4. *Let R be a pre semi-Artinian ring with classical Krull dimension α. Then* $\text{K.dim}(M) \le \alpha$ *for any Noetherian self-generator module M_R. In particular, if* $\alpha = 0$ *then every Noetherian self-generator R-module is Artinian.*

Proof. This follows by Theorem 4.3 and the fact that self-generator Noetherian modules are $\mathcal{L}_2$-Noetherian modules with Krull dimensions. □

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