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A NOTE ON HOMEOMORPHIC MEASURES ON TOPOLOGICAL GROUPS

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1. Introduction

The classical von Neumann–Oxtoby–Ulam Theorem states the following:

Given non-atomic Borel probability measures μ, λ on I^n such that

(1) $\mu(A)>0$, $\lambda(A)>0$ for all open $A \subset I^n$
(2) $\mu(\partial I^n)=\lambda(\partial I^n)=0$,

there exists a homeomorphism h of I^n onto itself fixing the boundary pointwise such that for any λ-measurable set S

$$\mu(h(S))=\lambda(S).$$

It is known that the above theorem remains valid if I^n is replaced by any compact finite dimensional manifold [**2**], [**4**] or with I^∞, the Hilbert cube, [**8**].

We shall say that a space X has the *homeomorphic measure property* if the above theorem remains valid with I^n replaced by X.

In this note we characterise those compact connected abelian metric groups having the homeomorphic measure property as precisely those which are locally connected. These are T^a, where T is the circle group and a is a non-negative integer or possibly $\aleph_0$.

Our proof depends on the following key result:

Theorem A. [**6**] *A countable product of finite dimensional compact manifolds has the homeomorphic measure property.*

2. Compact connected metrisable abelian groups

We shall regard T, the circle group, as the unit circle on the complex plane with complex multiplication as the group operation.

Lemma 1. *If f is a continuous homomorphism of $G \cong T^m$, onto $H \cong T^n$, there exist subgroups A and B of G such that $G = A \times B$, $A \cong T^n$ and $B \cong T^{m-n}$, $f(\{e\} \times B) = \{e\}$ and $f(A \times \{e\}) = H$.*

Proof. Let C be the kernel of f. Then $C=B\times F$ where $B\cong T^{m-n}$ and F is a finite group. As $B\subseteq G$ and $B\cong T^{m-n}$, Section 25.31 in [3] implies $G=A\times B$ where $A\cong T^n$.

Lemma 2. *If f is a continuous homomorphism of $G\cong T^n$ onto $H=H_1\times\cdots\times H_n$ where each $H_i\cong T$, then there exist subgroups $G_1,G_2,\ldots,G_n$ such that $G=G_1\times\cdots\times G_n$, $G_i\cong T$, and $f(G_i)=H_i$ for $i=1,2,\ldots,n$.*

Proof. Now $f^{-1}(H_1)$ is a subgroup of T^n and so is $G_1\times F$ where $G_1\cong T^r$, $r\geqq 1$, and F is a finite group. But then $f(G_1)$ is a connected subgroup of H_1 and so $f(G_1)=H_1$. As $G_1\cong T^r$, we must have $G=B\times G_1$ where $B=T^{n-r}$ and $f(B)=\{e\}\times H_2\times\cdots\times H_n\cong T^{n-1}$. So $n-r\geqq n-1$, which implies $r=1$.

So $G=B_1\times G_1$, $G_1\cong T$, $f(G_1)=H_1$ and $f(B_1)=\{e\}\times H_2\times\cdots\times H_n$. As above, there is a subgroup $G_2\subseteq B_1$ such that $f(G_2)=H_2$ and $G_2\cong T$. Continue inductively choosing $G_1,\ldots,G_n$.

So we have subgroups $G_1,\ldots,G_n$ of G each isomorphic to T and $G_i\cap G_j=\{e\}$ for $i\neq j$. Thus the group generated by $G_1,G_2,\ldots,G_n$ is $G_1\times\cdots\times G_n$ and is isomorphic to T^n. As $G_1\times\cdots\times G_n\subseteq G\cong T^n$ we must have $G=G_1\times\cdots\times G_n$ and the lemma is proved.

Observe that any continuous homomorphism f of T into T is $f(e^{2\pi i\theta})=e^{2\pi ik\theta}$ where k is a non-negative integer. We have demonstrated that any continuous homomorphism f of T^m onto T^n is isomorphic to a canonical homomorphism q which is a product of n homomorphisms of T into T. That is, there are isomorphisms r and r' of T^m and T^n, respectively, such that $f=r'\circ q\circ r$.

Let f be a canonical continuous homomorphism of T^m onto T^n:

$$f(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n},\ldots,e^{2\pi i\theta_m})=(e^{2\theta ik_1\theta_1},\ldots,e^{2\pi ik_n\theta_n}).$$

If $m=n$, it is easily seen that f is a $\prod_{i=1}^n k_i$ to one covering. In general, f has connected kernel if and only if $k_i=1$ for each i and if A is a connected set of small enough diameter, then $f^{-1}(A)$ consists of $\prod_{i=1}^n k_i$ disjoint connected sets which are translates in T^m of each other.

Suppose G is a compact group and g is a continuous homomorphism of G onto T^n. Then g preserves normalised Haar measure. This readily follows from the fact that T^n is dyadically decomposable, that is, expressible as a disjoint union of sets which are translates of each other.

Suppose G is a compact connected metrisable abelian group, then G is an inverse limit space $\{G_i,p_i^{i+1}\}_{i\in\mathbb{N}}$ where the factor spaces $G_i=T^{n_i}$ for some positive integer n_i and where the bonding maps p_i^{i+1} are continuous surjective homomorphisms. (This well-known result follows easily for example from Corollary 1 of Theorem 14 of [5].) We shall let π_i denote the projection of G onto the factor G_i, so that $\pi_i=p_i^{i+1}\circ\pi_{i+1}$. If only a finite number of the bonding maps have disconnected kernel, then G can be expressed as an inverse limit space $\{G_i',p_i'^{i+1}\}_{i\in\mathbb{N}}$ where all the bonding maps have connected kernels. Thus by Theorem 4.3 of [1], G is locally connected.

Lemma 3. *Let A be a closed arc contained in G. Let λ and λ_i be normalised Haar* *measures on G and G_i, respectively. If p_i^{i+1} has disconnected kernel and $\pi_i(A) \neq G_i$ then*

$$\lambda(A) \leqq \lambda_{i+1}(\pi_{i+1}(A)) \leqq \tfrac{1}{2}\lambda_i(\pi_i(A)).$$

Proof. Let $\{A_k\}_k$ be a family of non-overlapping closed subarcs of A such that $A = \bigcup_k A_k$. Suppose the lemma holds for each A_k separately, then the lemma holds for A. Therefore, we may assume without loss of generality that A is of as small a diameter as we choose, so that $q_i^{-1}(\pi_i(A))$, where $q_i = p_i^{i+1}$, is a disjoint union of closed connected sets $S_1, S_2, \ldots, S_m$, $m \geqq 2$, where each S_j is a translate of S_1. Thus $\lambda_{i+1}(S_1) = \lambda_{i+1}(S_j)$ for each j and since p_i^{i+1} preserves Haar measure $\lambda_{i+1}(S_1) = (1/m)\lambda_j(\pi_i(A))$. By connectivity $\pi_{i+1}(A) \subseteq S_j$ for some j. This proves the lemma.

Theorem 1. *Let G be a compact connected metrisable abelian group. Then G has the homeomorphic measure property if and only if G is locally connected.*

Proof. Sufficiency follows from Theorem A. Suppose G is not locally connected. As G contains a one parameter subgroup, G contains a closed arc S. Let f be a homeomorphism of I onto S and define a Borel measure μ on G *by* $\mu(A) = m(f^{-1}(A \cap S))$ where m is linear Lebesgue measure. Define a Borel measure α on G by $\alpha = \frac{1}{2}(\mu + \lambda)$. Then α is a locally positive non-atomic Borel probability measure on G but cannot be homeomorphic to λ because $\alpha(S) = \frac{1}{2}$ and by Lemma 3 $\lambda(h(S)) = 0$ for every homeomorphism h of G onto itself.

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